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Study on the stability and its simulation algorithm of a nonlinear impulsive ABC-fractional coupled system with a Laplacian operator via F-contractive mapping

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Abstract

In this paper, we study the solvability and generalized Ulam–Hyers (UH) stability of a nonlinear Atangana–Baleanu–Caputo (ABC) fractional coupled system with a Laplacian operator and impulses. First, this system becomes a nonimpulsive system by applying an appropriate transformation. Secondly, the existence and uniqueness of the solution are obtained by an F-contractive operator and a fixed-point theorem on metric space. Simultaneously, the generalized UH-stability is established based on nonlinear analysis methods. Thirdly, a novel numerical simulation algorithm is provided. Finally, an example is used to illustrate the correctness and availability of the main results. Our study is a beneficial exploration of the dynamic properties of viscoelastic turbulence problems.

Mathematics Subject Classification: 34A08; 34A37; 34D20

Keywords: Coupled ABC-fractional system; Laplacian operator; Solvability and stability; F-contractive mapping; Simulation algorithm

1 Introduction

In 2016, Atangana and Baleanu [8] first put forward a new fractional derivative in the Caputo sense. It is referred to as an ABC-fractional derivative. Compared to both Riemann–Liouville and Caputo fractional derivatives, ABC-fractional derivatives employ a special Mittag–Leffler function as the integral kernel to avoid singularity, which can be explained by the analysis below. Let the order of the derivatives be $0 < \delta < 1$, then the integral kernel $\mathscr{E}_\delta[-\frac{\delta}{1-\delta}(t-s)] = \sum_{k=0}^{\infty} \frac{[-\frac{\delta}{1-\delta}(t-s)]^k}{\Gamma(\delta k+1)}$ of the ABC-fractional derivatives satisfies $\mathscr{E}_\delta[-\frac{\delta}{1-\delta}(t-s)] \to 1$ (nonsingular), as $s \to t$. However, the integral kernel $(t-s)^{-\delta}$ of both Riemann–Liouville and Caputo fractional derivatives agrees with $(t-s)^{-\delta} \to \infty$ (singular), as $s \to t$. Therefore, the study of ABC-fractional differential systems has become one of the hot topics in recent years. For example, some scholars have studied their theoretical problems such as research methods [18, 27], important inequalities [19], qualitative analysis [5], chaos analysis [9], and numerical approximations [49]. Other researchers applied ABC-fractional calculus theory to explore some application problems [1, 3, 10–12, 17, 20, 25, 28, 36, 38, 44]. Specifically, Zhao et al. conducted a series of studies [21, 53–55, 60, 61] on the solvability and stability of some ABC-fractional differential systems in the past two years.

In 1983, Leibenson [31] first proposed the p-Laplacian differential equation model to describe the turbulence problem in porous media. The most basic form of p-Laplacian differential equation is as follows:

$$\Phi_p\big(\mathcal{X}'(t)\big)' = f\big(t, \mathcal{X}(t)\big), \quad t \in (0, T), p > 1,$$

where $T > 0$, $\Phi_p : z \to |z|^{p-2}z$ is called the p-Laplacian operator. Its inverse is $\Phi_p^{-1} = \Phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Due to its strong physical background and application, the p-Laplacian differential equation has become one of the most famous and important second-order nonlinear ordinary differential equations, and has been extensively and deeply studied. In recent years, the nonlinear p-Laplacian fractional differential system has been favored by some scholars. For example, Alsaedi et al. [7] discussed the multiplicity of positive solutions for a nonlinear high-order Riemann–Liouville fractional integral boundary value problem with p-Laplacian. Zhao [62] studied the existence and generalized UH-stability of solution for a nonlinear Caputo–Fabrizio fractional coupled Laplacian equation. Rao and Ahmadini [39] applied the Guo–Krasnosel'skii fixed-point theorem to obtain the multiplicity of positive solutions for a system of mixed Hadamard fractional boundary value problems with a (p_1, p_2)-Laplacian operator. Actually, there have been some papers dealing with various boundary value problems (BVP) of a p-Laplacian system involving Riemann–Liouville or Caputo or Hadamard fractional derivatives, for instance, integral BVP [2, 6], multipoint BVP [32, 40], infinite BVP [43], singular BVP [24], periodic BVP [63].

As is well known, many evolutionary processes cannot maintain permanent stability, and their development process always experiences brief and drastic changes. For example, in population dynamics systems, the number of species can sharply decrease or species may even become extinct due to factors such as earthquakes, tsunamis, epidemics, and short-term overhunting. This type of situation is called an impulsive phenomenon. The impulsive differential equations are one of the powerful tools for describing impulsive phenomena. The theory and application of impulsive differential equations have flourished. In recent years, fractional impulsive differential equations have remained a hot topic of interest for scholars. For example, Benkerrouche et al. [13] applied two fixed-point theorems to study the existence, uniqueness, and UH-stability of solutions to the multiterm impulsive Caputo–Hadamard-type differential equations. Priya and Kaliraj [37] utilized the Rothe's fixed-point technique to discuss the controllability of neutral nonlinear fractional-ordered impulsive systems. Xiao and Li [48] probed into the exponential stability of impulsive nonlinear conformable fractional delayed systems by the principle of comparison and the Lyapunov function method. Phu and Hoa [35] investigated the Mittag–Leffler stability of nonlinear uncertain dynamic systems with impulse effects with the random-order fractional derivative. Sivalingam and Govindaraj [42] provided a new numerical algorithm for the time-varying impulsive fractional differential equation. Although the impulsive phenomenon can cause drastic changes to the system in a short period of time, we expect the long-term behavior of the system to be stable. Therefore, many concepts of system stability have been proposed. For example, the UH-stability was first proposed by Hyers and Ulam

[22, 45] in the 1940s. Later, a series of generalizations were made on UH-stability, such as generalized UH-stability, Ulam–Hyers–Rassias (UHR) stability, and generalized UHR-stability. Recently, some scholars have made major achievements in the study of UH-type stability of fractional-order differential systems. For example, Zada et al. [52] discussed the stability of an impulsive coupled system of fractional integrodifferential equations. Yu [51] established the β-UH-stability of a fractional differential equation with noninstantaneous impulses. Chen and Lin [14] investigated the Ulam-type stability of impulsive and delayed fractional differential systems. Mehmood et al. [33] dealt with the UH-type stability of coupled ABC-fractional differential systems. Yaghoubi et al. [50] adopted the frequency-based method to analyze the UH-type stability of polynomial fractional differential equations. Zhao [56] explored the UH- and UHR-stability of nonsingular exponential kernel fractional Langevin systems. Some important achievements on the stability of fractional differential equations can also be found in the literature [4, 15, 16, 26, 29, 30]. However, it is rare to study the UH-type stability of ABC-fractional differential equations with impulses because the structure of the differential equations is more complex than that of a single differential equation. Additionally, there are no studies combining ABC-fractional derivative with a coupled Laplacian system. Consequently, it is novel and interesting to probe these problems.

Inspired by the aforementioned, we mainly consider the following nonlinear impulsive ABC-fractional coupled system with a (p_1, p_2)-Laplacian:

$$\begin{cases} {}^{\mathrm{ABC}}\mathscr{D}^{\nu_1}_{t_k^+}[\Phi_{p_1}({}^{\mathrm{ABC}}\mathscr{D}^{\mu_1}_{t_k^+}\mathfrak{X}_1(t))] = f_1(t, \mathfrak{X}_1(t), \mathfrak{X}_2(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ {}^{\mathrm{ABC}}\mathscr{D}^{\nu_2}_{t_k^+}[\Phi_{p_2}({}^{\mathrm{ABC}}\mathscr{D}^{\mu_2}_{t_k^+}\mathfrak{X}_2(t))] = f_2(t, \mathfrak{X}_1(t), \mathfrak{X}_2(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ \mathfrak{X}_1(t_k^+) = (1+\xi_{1k})\mathfrak{X}_1(t_k^-), \qquad {}^{\mathrm{ABC}}\mathscr{D}^{\mu_1}_{t_k^+}\mathfrak{X}_1(t_k^+) = (1+\zeta_{1k}){}^{\mathrm{ABC}}\mathscr{D}^{\mu_1}_{t_{k-1}^+}\mathfrak{X}_1(t_k^-), \\ \mathfrak{X}_2(t_k^+) = (1+\xi_{2k})\mathfrak{X}_2(t_k^-), \qquad {}^{\mathrm{ABC}}\mathscr{D}^{\mu_2}_{t_k^+}\mathfrak{X}_2(t_k^+) = (1+\zeta_{2k}){}^{\mathrm{ABC}}\mathscr{D}^{\mu_2}_{t_{k-1}^+}\mathfrak{X}_2(t_k^-), \\ \mathfrak{X}_1(0) = w_1, \qquad \mathfrak{X}_2(0) = w_2, \qquad {}^{\mathrm{ABC}}\mathscr{D}^{\mu_1}_{0^+}\mathfrak{X}_1(0) = v_1, \qquad {}^{\mathrm{ABC}}\mathscr{D}^{\mu_2}_{0^+}\mathfrak{X}_2(0) = v_2, \end{cases} \tag{1.1}$$

where $I = [0, T]$, $\{t_k\}_{k=1}^n$ is an impulsive point sequence satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$; $w_1, w_2, v_1, v_2 \in \mathbb{R}$, $\xi_{1k}, \xi_{2k}, \zeta_{1k}, \zeta_{2k} \neq -1$, $0 < \mu_1, \mu_2, \nu_1, \nu_2 \leq 1$ and $p_1, p_2 > 1$ are some constants; ${}^{\mathrm{ABC}}\mathscr{D}^*$ is the $*$-order ABC-fractional derivative; $\Phi_{p_i}(z) = |z|^{p_i-2}z$ $(i = 1, 2)$, and its inverse $\Phi_{p_i}^{-1} = \Phi_{q_i}$, provided that $\frac{1}{p_i} + \frac{1}{q_i} = 1$; $f_i \in C(I \times \mathbb{R}^2, \mathbb{R})$ $(i = 1, 2)$ is nonlinear; $\mathfrak{X}_i(t_k^+)$ and ${}^{\mathrm{ABC}}\mathscr{D}^{\mu_i}_{t_k^+}\mathfrak{X}_i(t_k^+)$ represent the right limit; $\mathfrak{X}_i(t_k^-)$ and ${}^{\mathrm{ABC}}\mathscr{D}^{\mu_i}_{t_{k-1}^+}\mathfrak{X}_i(t_k^-)$ express the left limit satisfying $\mathfrak{X}_i(t_k^-) = \mathfrak{X}_i(t_k)$ and ${}^{\mathrm{ABC}}\mathscr{D}^{\mu_i}_{t_{k-1}^+}\mathfrak{X}_i(t_k^-) = {}^{\mathrm{ABC}}\mathscr{D}^{\mu_i}_{t_{k-1}^+}\mathfrak{X}_i(t_k)$, $i = 1, 2$.

Remark 1.1 Compared to previous papers such as [23, 41, 56], our system (1.1) considers a coupled system of equations involving impulsive effects and multiple fractional derivatives μ_1, μ_2, ν_1, and ν_2, which includes a single equation and is more complex and difficult to study.

The objective of this manuscript is to investigate the existence and generalized UH-stability of solutions for system (1.1). Our main contributions include the following aspects. (i) As no papers have been found yet to address the nonlinear ABC-fractional differential coupled Laplacian system with impulses, we first consider the system (1.1) to fill this gap. (ii) The research method for impulsive differential equations is usually carried out piecewise based on impulsive intervals. This method is relatively complex in constructing the existence space of the solution and conducting prior estimation. We overcome this disadvantage by applying an appropriate transformation to convert the impulsive system (1.1) into a nonimpulsive system. (iii) By constructing a F-contractive mapping and a complete metric space, we apply a new fixed-point theorem on metric space to obtain the existence and uniqueness of the solution of system (1.1). As the F-contraction mapping is an important extension of contraction mapping, it expands the scope of application of contraction mapping methods in the study of operator equation solutions defined on complete metric spaces. In addition, the generalized UH-stability of system (1.1) is also established by nonlinear analysis methods. (iv) We propose a novel numerical simulation algorithm for system (1.1).

The remaining framework of the paper is as follows. Section 2 reviews some necessary content about ABC-fractional calculus. Based on the F-contractive mapping and a new fixed-point theorem on a metric space, we obtain some sufficient conditions to ensure that system (1.1) has a unique solution in Sect. 3. Section 4 further builds the generalized UH-stability of system (1.1). In Sect. 5, we first provide a novel numerical simulation algorithm. Then, an example is applied to verify the correctness of our theoretical results and the effectiveness of the algorithm. A concise conclusion is made in Sect. 6.

2 Preliminaries

Definition 2.1 ([23]) For $0 < \gamma \leq 1$, $b > a$ and $\mathscr{W} : [a,b] \to \mathbb{R}$, the left-sided γ-order ABC-fractional integral of $\mathscr{W}$ is defined by

$$^{\mathrm{ABC}}\mathcal{J}_a^{\gamma}\mathscr{W}(t) = \frac{1-\gamma}{\mathfrak{N}(\gamma)}\mathscr{W}(t) + \frac{\gamma}{\mathfrak{N}(\gamma)\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\mathscr{W}(s)\,ds,$$

where $\mathfrak{N}(\alpha)$ is a normalization constant with $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$.

Definition 2.2 ([8]) For $0 < \gamma \leq 1$, $b > a$ and $\mathscr{W} \in C^1(a,b)$, the left-sided γ-order ABC-fractional derivative of $\mathscr{W}$ is defined by

$$^{\mathrm{ABC}}\mathscr{D}_{a^+}^{\gamma}\mathscr{W}(t) = \frac{\mathfrak{N}(\gamma)}{1-\gamma}\int_a^t \mathscr{E}\left[-\frac{\alpha}{1-\alpha}(t-s)\right]\mathscr{W}'(s)\,ds,$$

where $\mathscr{E}_{\gamma}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\gamma n+1)}$ is the Mittag–Leffer special function with parameter γ.

Lemma 2.1 ([41]) *If* $\mathscr{H} \in C[a,b]$. *then the unique solution of the following IVP*

$$\begin{cases} ^{\mathrm{ABC}}\mathscr{D}_{a^+}^{\gamma}\mathscr{W}(t) = \mathscr{H}(t), & t \geq a, 0 < \gamma \leq 1, \\ \mathscr{W}(a) = \mathscr{W}_a, \end{cases}$$

is given by

$$\mathscr{W}(t) = \mathscr{W}_a + \frac{1-\gamma}{\mathfrak{N}(\gamma)}\big[\mathscr{H}(t) - \mathscr{H}(a)\big] + \frac{\gamma}{\mathfrak{N}(\gamma)\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\mathscr{H}(s)\,ds.$$

Lemma 2.2 *Let* $p > 1$. *The* $\mathfrak{p}$*-Laplacian operator* $\Phi_{\mathfrak{p}}(z) = |z|^{\mathfrak{p}-2}z$ *has the following features*:

(i) *If* $z \geq 0$, *then* $\Phi_{\mathfrak{p}}(z) = z^{\mathfrak{p}-1}$, *and* $\Phi_{\mathfrak{p}}(z)$ *is increasing with respect to* z;

(ii) *For all* $z, w \in \mathbb{R}$, $\Phi_p(zw) = \Phi_p(z)\Phi_p(w)$;
(iii) *If* $\frac{1}{p} + \frac{1}{q} = 1$, *then* $\Phi_q[\Phi_p(z)] = \Phi_p[\Phi_q(z)] = z$, *for all* $z \in \mathbb{R}$;
(iv) *For all* $z, w \geq 0$, $z \leq w \Leftrightarrow \Phi_q(z) \leq \Phi_q(w)$;
(v) $0 \leq z \leq \Phi_q^{-1}(w) \Leftrightarrow 0 \leq \Phi_q(z) \leq w$;
(vi) $\left|\Phi_q(z) - \Phi_q(w)\right| \leq \begin{cases} (q-1)\overline{M}^{q-2}|z-w|, & q \geq 2, 0 \leq z, w \leq \overline{M}; \\ (q-1)\underline{M}^{q-2}|z-w|, & 1 < q < 2, z, w \geq \underline{M} \geq 0. \end{cases}$

The fixed-point theory introduced below is an important means to solve our problem.

Definition 2.3 ([46]) A function $F : (0, +\infty) \to \mathbb{R}$ is called the Wardowski function if F satisfies the following:
(f1) For all $x, y > 0$, $x < y \Rightarrow F(x) < F(y)$, namely, F is strictly monotonically increasing;
(f2) $\lim_{n\to\infty} F(x_n) = -\infty \Leftrightarrow \lim_{n\to\infty} x_n = 0$, $\forall x_n \geq 0$;
(f3) It has $\alpha \in (0, 1)$ such that $\lim_{x\to 0^+} x^\alpha F(x) = 0$.

All Wardowski functions are marked as $\mathfrak{F}$. Wardowski [47] replaced (f2) with (f2)′: for any sequence $\{x_n\} \subset (0, \infty)$, $\lim_{n\to\infty} F(x_n) = -\infty \Rightarrow \lim_{n\to\infty} x_n = 0$, and called F satisfying (f1) and (f2)′ the semi-Wardowski function.

Definition 2.4 ([46]) Let $(\mathbb{X}, \rho)$ be a complete metric space, and $\mathcal{T} : \mathbb{X} \to \mathbb{X}$ be an operator. If there exist $\lambda > 0$ and $F \in \mathfrak{F}$ such that

$$\rho(\mathcal{T}x, \mathcal{T}y) > 0 \quad \Rightarrow \quad \lambda + F\big(\rho(\mathcal{T}x, \mathcal{T}y)\big) \leq F\big(\rho(x, y)\big), \quad \forall x, y \in \mathbb{X},$$

then $\mathcal{T}$ is called a F-contraction.

Remark 2.1 It is easy to verify that $F(z) = \log z$ meets the conditions (f1)–(f3), i.e., $F(z) = \log z \in \mathfrak{F}$. Concurrently, $\lambda + F(\rho(\mathcal{T}x, \mathcal{T}y)) \leq F(\rho(x, y))$ implies that $\rho(\mathcal{T}x, \mathcal{T}y) < e^{-\lambda}\rho(x, y)$, that is, $\mathcal{T}$ is a Banach contraction. In other words, F-contraction is a generalization of Banach contraction.

Lemma 2.3 ([34]) *Let* $\mathcal{T} : \mathbb{X} \to \mathbb{X}$ *be an operator defined on the complete metric space* $(\mathbb{X}, \rho)$. *Assume that the following are true*:
(a1) *There exist* $\lambda > 0$ *and* $F \in \mathfrak{F}$ *such that*

$$\rho(\mathcal{T}x, \mathcal{T}y) > 0 \quad \Rightarrow \quad \lambda + F\big(\rho(\mathcal{T}x, \mathcal{T}y)\big) \leq F\big(\mathcal{S}(x, y)\big), \quad \forall x, y \in \mathbb{X},$$

where

$$\mathcal{S}(x, y) = \max\left\{\rho(x, y), \rho(x, \mathcal{T}x), \rho(y, \mathcal{T}y), \frac{\rho(x, \mathcal{T}y) + \rho(\mathcal{T}x, y)}{2}\right\};$$

(a2) *One of* F *and* $\mathcal{T}$ *is continuous.*
Then, there exists a unique $x^* \in \mathbb{X}$ *such that* $\mathcal{T}x^* = x^*$.

3 Existence and uniqueness of solution

This section is devoted to proving the existence and uniqueness of the solution to system (1.1). We first transform the impulsive system (1.1) into a nonimpulsive system. In what follows, let $\Phi_{p_1}({}^{ABC}\mathscr{D}^{\mu_1}_{t_k^+}\mathscr{X}_1(t)) = \mathscr{Y}_1(t)$, $\Phi_{p_2}({}^{ABC}\mathscr{D}^{\mu_2}_{t_k^+}\mathscr{X}_2(t)) = \mathscr{Y}_2(t)$, and apply Lemma 2.2, then system (1.1) becomes the following system:

$$\begin{cases} {}^{ABC}\mathscr{D}^{\mu_1}_{t_k^+}\mathscr{X}_1(t) = \Phi_{q_1}(\mathscr{Y}_1(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ {}^{ABC}\mathscr{D}^{\nu_1}_{t_k^+}\mathscr{Y}_1(t) = f_1(t, \mathscr{X}_1(t), \mathscr{X}_2(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ {}^{ABC}\mathscr{D}^{\mu_2}_{t_k^+}\mathscr{X}_2(t) = \Phi_{q_2}(\mathscr{Y}_2(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ {}^{ABC}\mathscr{D}^{\nu_2}_{t_k^+}\mathscr{Y}_2(t) = f_2(t, \mathscr{X}_1(t), \mathscr{X}_2(t)), \quad t \in (t_k, t_{k+1}] \subset I, \\ \mathscr{X}_1(t_k^+) = (1+\xi_{1k})\mathscr{X}_1(t_k^-), \qquad \mathscr{Y}_1(t_k^+) = \Phi_{p_1}(1+\zeta_{1k})\mathscr{Y}_1(t_k^-), \\ \mathscr{X}_2(t_k^+) = (1+\xi_{2k})\mathscr{X}_2(t_k^-), \qquad \mathscr{Y}_2(t_k^+) = \Phi_{p_2}(1+\zeta_{2k})\mathscr{Y}_2(t_k^-), \\ \mathscr{X}_1(0) = w_1, \qquad \mathscr{X}_2(0) = w_2, \qquad \mathscr{Y}_1(0) = \Phi_{p_1}(v_1), \qquad \mathscr{Y}_2(0) = \Phi_{p_2}(v_2). \end{cases} \tag{3.1}$$

Obviously, the solvability of system (1.1) and system (3.1) are completely equivalent. Hence, it suffices to discuss the existence and uniqueness of the solution to system (3.1). To this end, consider the following nonimpulsive ABC-fractional differential system:

$$\begin{cases} {}^{ABC}\mathscr{D}^{\mu_1}_{0^+}\mathscr{W}_1(t) = \prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\mathscr{V}_1(t)), \\ {}^{ABC}\mathscr{D}^{\nu_1}_{0^+}\mathscr{V}_1(t) \\ \quad = \Phi_{p_1}(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{1k}})f_1(t, \prod_{0\le t_k<t}(1+\xi_{1k})\mathscr{W}_1(t), \prod_{0\le t_k<t}(1+\xi_{2k})\mathscr{W}_2(t)), \\ {}^{ABC}\mathscr{D}^{\mu_2}_{0^+}\mathscr{W}_2(t) = \prod_{0\le t_k<t}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\mathscr{V}_2(t)), \\ {}^{ABC}\mathscr{D}^{\nu_2}_{0^+}\mathscr{V}_2(t) \\ \quad = \Phi_{p_2}(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{2k}})f_2(t, \prod_{0\le t_k<t}(1+\xi_{1k})\mathscr{W}_1(t), \prod_{0\le t_k<t}(1+\xi_{2k})\mathscr{W}_2(t)), \\ \mathscr{W}_1(0) = w_1, \qquad \mathscr{W}_2(0) = w_2, \qquad \mathscr{V}_1(0) = \Phi_{p_1}(v_1), \qquad \mathscr{V}_2(0) = \Phi_{p_2}(v_2). \end{cases} \tag{3.2}$$

Lemma 3.1 *For systems* (3.1) *and* (3.2), *the following assertions hold*:

(b1) *If* $\mathscr{W}_i(t)$ *and* $\mathscr{V}_i(t)$ $(i = 1, 2)$ *satisfy system* (3.2), *then* $\mathscr{X}_i(t) = \prod_{0\le t_k<t}(1+\xi_{ik})\mathscr{W}_i(t)$ *and* $\mathscr{Y}_i(t) = \prod_{0\le t_k<t}\Phi_{p_i}(1+\zeta_{ik})\mathscr{V}_i(t)$ *satisfy system* (3.1);

(b2) *If* $\mathscr{X}_i(t)$ *and* $\mathscr{Y}_i(t)$ $(i = 1, 2)$ *satisfy system* (3.1), *then* $\mathscr{W}_i(t) = \prod_{0\le t_k<t}(1+\xi_{ik})^{-1}\mathscr{X}_i(t)$ *and* $\mathscr{V}_i(t) = \prod_{0\le t_k<t}[\Phi_{p_i}(1+\zeta_{ik})]^{-1}\mathscr{Y}_i(t)$ *satisfy system* (3.2).

Proof Obviously, (3.1) and (3.2) have the same initial conditions. Assume that $\mathscr{W}_i(t)$ and $\mathscr{V}_i(t)$ $(i = 1, 2)$ satisfy system (3.2), when $t \in (t_k, t_{k+1})$ $(k = 0, 1, 2, \dots, n)$, we substitute $\mathscr{X}_i(t) = \prod_{0\le t_k<t}(1+\xi_{ik})\mathscr{W}_i(t)$ and $\mathscr{Y}_i(t) = \prod_{0\le t_k<t}\Phi_{p_i}(1+\zeta_{ik})\mathscr{V}_i(t)$ into the first four equations of (3.2) to obtain the first four equations of (3.1), which means that $\mathscr{X}_i(t) = \prod_{0\le t_k<t}(1+\xi_{ik})\mathscr{W}_i(t)$ and $\mathscr{Y}_i(t) = \prod_{0\le t_k<t}\Phi_{p_i}(1+\zeta_{ik})\mathscr{V}_i(t)$ satisfy system (3.1). When $t = t_k$ $(k = 1, 2, \dots, n)$, we have

$$\mathscr{X}_i(t_k^+) = \prod_{j=1}^{k}(1+\xi_{ij})\mathscr{W}_i(t_k), \qquad \mathscr{Y}_i(t_k^+) = \prod_{j=1}^{k}\Phi_{p_i}(1+\zeta_{ij})\mathscr{V}_i(t_k) \tag{3.3}$$

and

$$\mathscr{X}_i(t_k^-) = \prod_{j=1}^{k-1}(1+\xi_{ij})\mathscr{W}_i(t_k), \qquad \mathscr{Y}_i(t_k^-) = \prod_{j=1}^{k-1}\Phi_{p_i}(1+\zeta_{ij})\mathscr{V}_i(t_k). \tag{3.4}$$

Together with (3.3) and (3.4), we obtain

$$\mathfrak{X}_i(t_k^+) = (1+\xi_{ik})\mathfrak{X}_i(t_k^-), \qquad \mathcal{Y}_i(t_k^+) = \Phi_{p_i}(1+\zeta_{ik})\mathcal{Y}_i(t_k^-). \tag{3.5}$$

Equation (3.5) is the impulsive conditions of (3.1). Thus, the assertion (b1) is true.

Next, we show that the assertion (b2) holds. In fact, when $t \in (t_k, t_{k+1}]$ ($k = 0,1,2,\dots,n$), it is same manner as the proof of (b1) that $\mathcal{W}_i(t) = \prod_{0\le t_k<t}(1+\xi_{ik})^{-1}\mathfrak{X}_i(t)$ and $\mathcal{V}_i(t) = \prod_{0\le t_k<t}[\Phi_{p_i}(1+\zeta_{ik})]^{-1}\mathcal{Y}_i(t)$ satisfy system (3.2). In the small neighborhood of $t = t_k$ ($k = 1,2,\dots,n$), we derive from (3.5), $\mathfrak{X}_i(t_k^-) = \mathfrak{X}_i(t_k)$ and $\mathcal{Y}_i(t_k^-) = \mathcal{Y}_i(t_k)$ that

$$\mathcal{W}_i(t_k^-) = \prod_{j=1}^{k-1}(1+\xi_{ij})^{-1}\mathfrak{X}_i(t_k^-) = \prod_{j=1}^{k-1}(1+\xi_{ij})^{-1}\mathfrak{X}_i(t_k) = \mathcal{W}_i(t_k), \tag{3.6}$$

$$\mathcal{V}_i(t_k^-) = \prod_{j=1}^{k-1}\big[\Phi_{p_i}(1+\zeta_{ij})\big]^{-1}\mathcal{Y}_i(t_k^-) = \prod_{j=1}^{k-1}\big[\Phi_{p_i}(1+\zeta_{ij})\big]^{-1}\mathcal{Y}_i(t_k) = \mathcal{V}_i(t_k), \tag{3.7}$$

$$\mathcal{W}_i(t_k^+) = \prod_{j=1}^{k}(1+\xi_{ij})^{-1}\mathfrak{X}_i(t_k^+) = \prod_{j=1}^{k-1}(1+\xi_{ij})^{-1}\mathfrak{X}_i(t_k^-), \tag{3.8}$$

$$\mathcal{V}_i(t_k^+) = \prod_{j=1}^{k}\big[\Phi_{p_i}(1+\zeta_{ij})\big]^{-1}\mathcal{Y}_i(t_k) = \prod_{j=1}^{k-1}\big[\Phi_{p_i}(1+\zeta_{ij})\big]^{-1}\mathcal{Y}_i(t_k^-). \tag{3.9}$$

Equations (3.6)–(3.9) mean that $\mathcal{W}_i(t)$ and $\mathcal{V}_i(t)$ are continuous on $[0,T]$. The proof of Lemma 3.1 is completed. □

Lemma 3.2 *Assume that $w_1, w_2, v_1, v_2 \in \mathbb{R}$, $\xi_{1k}, \xi_{2k}, \zeta_{1k}, \zeta_{2k} \neq -1$, $0 < \mu_1, \mu_2, \nu_1, \nu_2 \le 1$ and $p_1, p_2 > 1$ are some constants, $f_1, f_2 \in C(I \times \mathbb{R}^2, \mathbb{R})$, $\mathfrak{X}_i(t_k^-) = \mathfrak{X}_i(t_k)$ and ${}^{\mathrm{ABC}}\mathcal{D}^{\mu_i}_{t_{k-1}^+}\mathfrak{X}_i(t_k^-) = {}^{\mathrm{ABC}}\mathcal{D}^{\mu_i}_{t_{k-1}^+}\mathfrak{X}_i(t_k)$, $i = 1,2$. Then, the nonimpulsive* ABC*-fractional differential system (3.2) is equivalent to the following integral system*:

$$\begin{cases} \mathcal{W}_1(t) = w_1 + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}[\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\hbar_1(t,\mathcal{W}_1(t),\mathcal{W}_2(t))) - v_1] \\ \qquad + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1} \\ \qquad \times \prod_{0\le t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\hbar_1(s,\mathcal{W}_1(s),\mathcal{W}_2(s)))\,ds, \\ \mathcal{W}_2(t) = w_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}[\prod_{0\le t_k<t}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\hbar_2(t,\mathcal{W}_1(t),\mathcal{W}_2(t))) - v_2] \\ \qquad + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1} \\ \qquad \times \prod_{0\le t_k<s}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\hbar_2(s,\mathcal{W}_1(s),\mathcal{W}_2(s)))\,ds, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} &\hbar_1(s,\mathcal{W}_1(s),\mathcal{W}_2(s)) \\ &\quad = \Phi_{p_1}(v_1) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\bigg[\Phi_{p_1}\bigg(\prod_{0\le t_k<s}\frac{1}{1+\zeta_{1k}}\bigg) \\ &\qquad \times f_1\bigg(s, \prod_{0\le t_k<s}(1+\xi_{1k})\mathcal{W}_1(s), \prod_{0\le t_k<s}(1+\xi_{2k})\mathcal{W}_2(s)\bigg) - f_1(0,w_1,w_2)\bigg] \end{aligned}$$

$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^s (s-\tau)^{\nu_1-1} \Phi_{p_1}\left(\prod_{0\le t_k<\tau} \frac{1}{1+\zeta_{1k}} \right)$$

$$\times f_1\left(\tau, \prod_{0\le t_k<\tau} (1+\xi_{1k})\mathscr{W}_1(\tau), \prod_{0\le t_k<\tau} (1+\xi_{2k})\mathscr{W}_2(\tau) \right) d\tau, \tag{3.11}$$

$$\hbar_2\big(s, \mathscr{W}_1(s), \mathscr{W}_2(s)\big)$$

$$= \Phi_{p_2}(\upsilon_2) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)} \Bigg[\Phi_{p_2}\left(\prod_{0\le t_k<s} \frac{1}{1+\zeta_{2k}} \right)$$

$$\times f_2\left(s, \prod_{0\le t_k<s} (1+\xi_{1k})\mathscr{W}_1(s), \prod_{0\le t_k<s} (1+\xi_{2k})\mathscr{W}_2(s) \right) - f_2(0, w_1, w_2) \Bigg]$$

$$+ \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)} \int_0^s (s-\tau)^{\nu_2-1} \Phi_{p_2}\left(\prod_{0\le t_k<\tau} \frac{1}{1+\zeta_{2k}} \right)$$

$$\times f_2\left(\tau, \prod_{0\le t_k<\tau} (1+\xi_{1k})\mathscr{W}_1(\tau), \prod_{0\le t_k<\tau} (1+\xi_{2k})\mathscr{W}_2(\tau) \right) d\tau. \tag{3.12}$$

Proof For simplicity, we denote

$$g_1\big(s, \boldsymbol{\mathscr{W}}(s)\big) = \prod_{0\le t_k<s} (1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}\big(\mathscr{V}_1(s)\big),$$

$$g_2\big(s, \boldsymbol{\mathscr{W}}(s)\big) = \Phi_{p_1}\left(\prod_{0\le t_k<s} \frac{1}{1+\zeta_{1k}} \right) f_1\left(s, \prod_{0\le t_k<s} (1+\xi_{1k})\mathscr{W}_1(s), \prod_{0\le t_k<s} (1+\xi_{2k})\mathscr{W}_2(s) \right),$$

$$g_3\big(s, \boldsymbol{\mathscr{W}}(s)\big) = \prod_{0\le t_k<s} (1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}\big(\mathscr{V}_2(s)\big),$$

$$g_4\big(s, \boldsymbol{\mathscr{W}}(s)\big) = \Phi_{p_2}\left(\prod_{0\le t_k<s} \frac{1}{1+\zeta_{2k}} \right) f_2\left(s, \prod_{0\le t_k<s} (1+\xi_{1k})\mathscr{W}_1(s), \prod_{0\le t_k<s} (1+\xi_{2k})\mathscr{W}_2(s) \right).$$

If $\boldsymbol{\mathscr{W}}(t) = (\mathscr{W}_1(t), \mathscr{V}_1(t), \mathscr{W}_2(t), \mathscr{V}_2(t)) \in [C([0,T],\mathbb{R})]^4$ is a solution of system (3.2), then it follows from Lemma 2.1 that

$$\mathscr{W}_1(t) = \mathscr{W}_1(0) + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} \big[g_1\big(t, \boldsymbol{\mathscr{W}}(t)\big) - \hbar_1\big(0, \boldsymbol{\mathscr{W}}(0)\big) \big]$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-1} g_1\big(s, \boldsymbol{\mathscr{W}}(s)\big)\, ds, \tag{3.13}$$

$$\mathscr{V}_1(t) = \mathscr{V}_1(0) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big[\hbar_2\big(t, \boldsymbol{\mathscr{W}}(t)\big) - g_2\big(0, \boldsymbol{\mathscr{W}}(0)\big) \big]$$

$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-s)^{\nu_1-1} g_2\big(s, \boldsymbol{\mathscr{W}}(s)\big)\, ds, \tag{3.14}$$

$$\mathscr{W}_2(t) = \mathscr{W}_2(0) + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)} \big[g_3\big(t, \boldsymbol{\mathscr{W}}(t)\big) - \hbar_3\big(0, \boldsymbol{\mathscr{W}}(0)\big) \big]$$

$$+ \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)} \int_0^t (t-s)^{\mu_2-1} g_3\big(s, \boldsymbol{\mathscr{W}}(s)\big)\, ds, \tag{3.15}$$

$$\mathcal{V}_2(t) = \mathcal{V}_2(0) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\big[g_4\big(t, \boldsymbol{\mathcal{W}}(t)\big) - \hbar_4\big(0, \boldsymbol{\mathcal{W}}(0)\big)\big] + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^t (t-s)^{\nu_2-1} g_4\big(s, \boldsymbol{\mathcal{W}}(s)\big)\,ds. \tag{3.16}$$

Noting that

$$g_1\big(0, \boldsymbol{\mathcal{W}}(0)\big) = \Phi_{q_1}\big(\mathcal{V}_1(0)\big), \qquad g_2\big(0, \boldsymbol{\mathcal{W}}(0)\big) = f_1\big(0, \mathcal{W}_1(0), \mathcal{W}_2(0)\big),$$
$$g_3\big(0, \boldsymbol{\mathcal{W}}(0)\big) = \Phi_{q_2}\big(\mathcal{V}_2(0)\big), \qquad g_4\big(0, \boldsymbol{\mathcal{W}}(0)\big) = f_2\big(0, \mathcal{W}_1(0), \mathcal{W}_2(0)\big)$$

and the initial conditions $\mathcal{W}_1(0) = \omega_1$, $\mathcal{W}_2(0) = \omega_2$, $\mathcal{V}_1(0) = \Phi_{p_1}(\upsilon_1)$, $\mathcal{V}_2(0) = \Phi_{p_2}(\upsilon_2)$, we derive from (3.13)–(3.16) that

$$\begin{cases} \mathcal{W}_1(t) = \omega_1 + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}[g_1(t, \boldsymbol{\mathcal{W}}(t)) - \upsilon_1] \\ \qquad + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1} g_1(s, \boldsymbol{\mathcal{W}}(s))\,ds, \\ \mathcal{V}_1(t) = \Phi_{p_1}(\upsilon_1) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}[g_2(t, \boldsymbol{\mathcal{W}}(t)) - f_1(0, \omega_1, \omega_2)] \\ \qquad + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-s)^{\nu_1-1} g_2(s, \boldsymbol{\mathcal{W}}(s))\,ds, \\ \mathcal{W}_2(t) = \omega_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}[g_3(t, \boldsymbol{\mathcal{W}}(t)) - \upsilon_2] \\ \qquad + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1} g_3(s, \boldsymbol{\mathcal{W}}(s))\,ds, \\ \mathcal{V}_2(t) = \Phi_{p_2}(\upsilon_2) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}[g_4(t, \boldsymbol{\mathcal{W}}(t)) - f_2(0, \omega_1, \omega_2)] \\ \qquad + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^t (t-s)^{\nu_2-1} g_4(s, \boldsymbol{\mathcal{W}}(s))\,ds. \end{cases} \tag{3.17}$$

Conversely, if $\boldsymbol{\mathcal{W}}(s) = (\mathcal{W}_1(s), \mathcal{V}_1(s), \mathcal{W}_2(s), \mathcal{V}_2(s)) \in [C([0,T], \mathbb{R})]^4$ is a solution of (3.17), then it is also a solution of (3.2) because the above derivation is completely reversible. In (3.17), substituting $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ into the first and second equations, respectively, we obtain the integral system (3.10). The proof is completed. □

According to Lemma 3.2, let $\mathbb{X} = [C([0,T], \mathbb{R})]^2$, a metric $\rho : \mathbb{X} \to \mathbb{X}$ is defined by

$$\rho\big(\mathcal{W}(t), \overline{\mathcal{W}}(t)\big) = \max\Big\{\sup_{0\le t\le T}\big|\mathcal{W}_1(t) - \overline{\mathcal{W}}_1(t)\big|, \sup_{0\le t\le T}\big|\mathcal{W}_2(t) - \overline{\mathcal{W}}_2(t)\big|\Big\}, \tag{3.18}$$

for all $\mathcal{W}(t), \overline{\mathcal{W}}(t) \in \mathbb{X}$, where $\mathcal{W}(t) = (\mathcal{W}_1(t), \mathcal{W}_2(t))$, $\overline{\mathcal{W}}(t) = (\overline{\mathcal{W}}_1(t), \overline{\mathcal{W}}_2(t))$. It is easy to prove that $(\mathbb{X}, \rho)$ is a complete metric space.

Remark 3.1 In view of Lemmas 3.1 and 3.2, if $\mathcal{W}^*(t) = (\mathcal{W}_1^*(t), \mathcal{W}_2^*(t)) \in \mathbb{X}$ is a solution of (3.10), then $\mathcal{X}^*(t) = (\mathcal{X}_1^*(t), \mathcal{X}_2^*(t))$ is a solution of (1.1), where

$$\mathcal{X}_1^*(t) = \prod_{0\le t_k<t}(1+\xi_{1k})\mathcal{W}_1^*(t), \qquad \mathcal{X}_2^*(t) = \prod_{0\le t_k<t}(1+\xi_{2k})\mathcal{W}_2^*(t).$$

Based on Remark 3.1, to discuss the existence and uniqueness of the solution to system (1.1), it suffices to conduct the same discussion on system (3.10). We first need the following underlying assumptions:

(H_1) $\omega_1 \neq 0$ or $\omega_2 \neq 0$, $T, \upsilon_i > 0$, $0 < \mu_i, \nu_i \le 1$, $p_i > 1$, $\xi_{1k}, \xi_{2k}, \zeta_{1k}, \zeta_{2k} > -1$, $f_i \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$, $i = 1, 2$, $k = 1, 2, \ldots, n$;

(H$_2$) For all $t \in [0,T]$, $u,v \in \mathbb{R}$, there exist some constants $m_i, M_i > 0$ such that

$$m_i \le f_i(t,u,v) \le M_i, \quad i = 1,2;$$

(H$_3$) For all $t \in [0,T]$, $u,\overline{u},v,\overline{v} \in \mathbb{R}$, there exist a constant $\lambda > 0$ and some continuous functions $\mathscr{L}_{i1}(t), \mathscr{L}_{i2}(t) \ge 0$ such that

$$\left|f_i(t,u,v) - f_i(t,\overline{u},\overline{v})\right| \le e^{-\lambda}\left[\mathscr{L}_{i1}(t)|u-\overline{u}| + \mathscr{L}_{i2}(t)|v-\overline{v}|\right].$$

For the sake of brevity and fluency in the subsequent text, we introduce some symbols below:

$$\begin{aligned}
&\|\mathscr{L}_{i1}\|_T = \max_{0\le t\le T}\mathscr{L}_{i1}(t), \qquad \|\mathscr{L}_{i2}\|_T = \max_{0\le t\le T}\mathscr{L}_{i2}(t),\\
&\underline{\mathscr{M}_i} = \upsilon_i^{p_i-1} + \frac{1-\nu_i}{\mathfrak{N}(\nu_i)}\left[m_i\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}} - M_i\right],\\
&\overline{\mathscr{M}_i} = \upsilon_i^{p_i-1} + \frac{1-\nu_i}{\mathfrak{N}(\nu_i)}\left[M_i\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}} - m_i\right] + \frac{M_iT^{\nu_i}}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}},\\
&\Theta_i = \frac{1}{\mathfrak{N}(\mu_i)\mathfrak{N}(\nu_i)}\left[(1-\mu_i)(1-\nu_i) + \frac{(1-\mu_i)T^{\nu_i}}{\Gamma(\nu_i)} + \frac{(1-\nu_i)T^{\mu_i}}{\Gamma(\mu_i)} + \frac{\mu_i\nu_iT^{\mu_i+\nu_i}}{\Gamma(\mu_i)\Gamma(\nu_i)}\right],\\
&\Delta_i = \prod_{0\le t_k<t}\frac{1+\zeta_{ik}}{1+\xi_{ik}}\left[\|\mathscr{L}_{i1}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{ik})^{p_i-1}} + \|\mathscr{L}_{i2}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{ik})^{p_i-1}}\right],\\
&\overline{\vartheta_i} = (q_i-1)\Theta_i\Delta_i\overline{\mathscr{M}_i}^{q_i-2}, \qquad \underline{\vartheta_i} = (q_i-1)\Theta_i\Delta_i\underline{\mathscr{M}_i}^{q_i-2}, \quad i = 1,2.
\end{aligned}$$

(H$_4$) We further assume that one of the following conditions holds: $\overline{\vartheta_1}, \overline{\vartheta_2} < 1$ when $q_1, q_2 \ge 2$; or $\overline{\vartheta_1}, \underline{\vartheta_2} < 1$ when $q_1 \ge 2, 1 < q_2 < 2$; or $\underline{\vartheta_1}, \overline{\vartheta_2} < 1$ when $1 < q_1 < 2, q_2 \ge 2$; or $\underline{\xi_1}, \underline{\xi_2} < 1$ when $1 < q_1, q_2 < 2$;

Theorem 3.1 *Assume that* (H$_1$)–(H$_4$) *hold. If* $\underline{\mathscr{M}_1}, \underline{\mathscr{M}_2} > 0$, *then system* (3.10) *has a unique nonzero solution* $\mathscr{W}^*(t) = (\mathscr{W}_1^*(t), \mathscr{W}_2^*(t)) \in \mathbb{X}$.

Proof $(\mathscr{W}_1(0), \mathscr{W}_2(0)) = (w_1, w_2) \ne (0,0)$ implies that $(\mathscr{W}_1(t), \mathscr{W}_2(t)) \not\equiv (0,0)$, $\forall t \in [0,T]$. We introduce a complete metric space $(\mathbb{X}, \rho)$ defined as (3.18). According to Lemma 3.2, for all $\mathscr{W}(t) = (\mathscr{W}_1(t), \mathscr{W}_2(t)) \in \mathbb{X}$, we define a vector operator $\mathscr{F} : \mathbb{X} \to \mathbb{X}$ as follows:

$$\mathscr{F}\left(\mathscr{W}(t)\right) = \left(\mathscr{F}_1\left(\mathscr{W}(t)\right), \mathscr{F}_2\left(\mathscr{W}(t)\right)\right), \quad t \in [0,T], \tag{3.19}$$

where

$$\begin{aligned}
\mathscr{F}_1\left(\mathscr{W}(t)\right) = w_1 &+ \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\left[\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}\left(\hbar_1\left(t, \mathscr{W}_1(t), \mathscr{W}_2(t)\right)\right) - \upsilon_1\right]\\
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1}\\
&\times \prod_{0\le t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}\left(\hbar_1\left(s, \mathscr{W}_1(s), \mathscr{W}_2(s)\right)\right)ds,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{F}_2\big(\mathcal{W}(t)\big) &= w_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}\Bigg[\prod_{0\le t_k<t}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}\big(\hbar_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\big) - v_2\Bigg] \\ &\quad + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1} \\ &\quad \times \prod_{0\le t_k<s}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}\big(\hbar_2\big(s,\mathcal{W}_1(s),\mathcal{W}_2(s)\big)\big)\,ds, \end{aligned} \tag{3.21}$$

$\hbar_1(t,\mathcal{W}_1(t),\mathcal{W}_2(t))$ and $\hbar_2(t,\mathcal{W}_1(t),\mathcal{W}_2(t))$ are defined as (3.11) and (3.12), respectively.

For all $\mathcal{W}(t)=(\mathcal{W}_1(t),\mathcal{W}_2(t))$, $t\in[0,T]$, we derive from (3.11), (H_1), and (H_2) that

$$\begin{aligned}\hbar_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\le v_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\Bigg[M_1\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} - m_1\Bigg] \\ &\quad + \frac{\nu_1 M_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}}\int_0^t (t-\tau)^{\nu_1-1}\,d\tau \\ &\le v_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\Bigg[M_1\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} - m_1\Bigg] \\ &\quad + \frac{M_1 T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} = \overline{\mathcal{M}_1} \end{aligned} \tag{3.22}$$

and

$$\hbar_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) \ge v_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\Bigg[m_1\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} - M_1\Bigg] = \underline{\mathcal{M}_1}. \tag{3.23}$$

Similarly, it follows from (3.12), (H_1), and (H_2) that

$$\begin{aligned}\hbar_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\le v_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\Bigg[M_2\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} - m_2\Bigg] \\ &\quad + \frac{M_2 T^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} = \overline{\mathcal{M}_2} \end{aligned} \tag{3.24}$$

and

$$\hbar_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) \ge v_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\Bigg[m_2\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} - M_2\Bigg] = \underline{\mathcal{M}_2}. \tag{3.25}$$

Obviously, $\underline{\mathcal{M}_1}\le\overline{\mathcal{M}_1}$, $\underline{\mathcal{M}_2}\le\overline{\mathcal{M}_2}$. For all $\mathcal{W}=(\mathcal{W}_1,\mathcal{W}_2)$, $\overline{\mathcal{W}}=(\overline{\mathcal{W}}_1,\overline{\mathcal{W}}_2)\in\mathbb{X}$, $t\in[0,T]$, we apply the condition (H_3) to obtain the following estimate

$$\begin{aligned}&\big|\hbar_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - \hbar_1\big(t,\overline{\mathcal{W}}_1(t),\overline{\mathcal{W}}_2(t)\big)\big| \\ &\quad\le \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\Phi_{p_1}\Bigg(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{1k}}\Bigg)\Bigg|f_1\Bigg(t,\prod_{0\le t_k<t}(1+\xi_{1k})\mathcal{W}_1(t),\prod_{0\le t_k<t}(1+\xi_{2k})\mathcal{W}_2(t)\Bigg) \\ &\qquad - f_1\Bigg(t,\prod_{0\le t_k<t}(1+\xi_{1k})\overline{\mathcal{W}}_1(t),\prod_{0\le t_k<t}(1+\xi_{2k})\overline{\mathcal{W}}_2(t)\Bigg)\Bigg|\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1}\Phi_{p_1}\Bigg(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{1k}}\Bigg)\bigg|f_1\bigg(\tau,\prod_{0\le t_k<\tau}(1+\xi_{1k})\mathscr{W}_1(\tau),\\
&\prod_{0\le t_k<\tau}(1+\xi_{2k})\mathscr{W}_2(\tau)\bigg)-f_1\bigg(\tau,\prod_{0\le t_k<\tau}(1+\xi_{1k})\overline{\mathscr{W}}_1(\tau),\prod_{0\le t_k<\tau}(1+\xi_{2k})\overline{\mathscr{W}}_2(\tau)\bigg)\bigg|\,d\tau\\
\le{}&\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}}\bigg[\mathscr{L}_{11}(t)\prod_{0\le t_k<t}(1+\xi_{1k})\big|\mathscr{W}_1(t)-\overline{\mathscr{W}}_1(t)\big|\\
&+\mathscr{L}_{12}(t)\prod_{0\le t_k<t}(1+\xi_{2k})\big|\mathscr{W}_2(t)-\overline{\mathscr{W}}_2(t)\big|\bigg]e^{-\lambda}+\frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}}\\
&\times\int_0^t(t-\tau)^{\nu_1-1}\bigg[\mathscr{L}_{11}(\tau)\prod_{0\le t_k<\tau}(1+\xi_{1k})\big|\mathscr{W}_1(\tau)-\mathscr{W}_1(\tau)\big|\\
&+\mathscr{L}_{12}(\tau)\prod_{0\le t_k<\tau}(1+\xi_{2k})\big|\mathscr{W}_2(\tau)-\mathscr{W}_2(\tau)\big|\bigg]e^{-\lambda}\,d\tau\\
\le{}&\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}}\bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<t}(1+\xi_{1k})\times\rho(\mathscr{W},\overline{\mathscr{W}})\\
&+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<t}(1+\xi_{2k})\times\rho(\mathscr{W},\overline{\mathscr{W}})\bigg]e^{-\lambda}+\frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}}\\
&\times\int_0^t(t-\tau)^{\nu_1-1}\bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<\tau}(1+\xi_{1k})\times\rho(\mathscr{W},\mathscr{W})\\
&+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<\tau}(1+\xi_{2k})\times\rho(\mathscr{W},\mathscr{W})\bigg]e^{-\lambda}\,d\tau\\
\le{}&\bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{1k})^{p_1-1}}+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{1k})^{p_1-1}}\bigg]\\
&\times\bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}+\frac{T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\bigg]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W}).
\end{aligned}
\tag{3.26}
$$

Similar to (3.26), we have

$$
\begin{aligned}
&\big|\hbar_2\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)-\hbar_2\big(t,\overline{\mathscr{W}}_1(t),\overline{\mathscr{W}}_2(t)\big)\big|\\
&\le\bigg[\|\mathscr{L}_{21}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{2k})^{p_2-1}}+\|\mathscr{L}_{22}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{2k})^{p_2-1}}\bigg]\\
&\quad\times\bigg[\frac{1-\nu_2}{\mathfrak{N}(\nu_2)}+\frac{T^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\bigg]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W}).
\end{aligned}
\tag{3.27}
$$

According to (3.20), we obtain

$$
\begin{aligned}
&\big|\mathscr{F}_1\big(\mathscr{W}(t)\big)-\mathscr{F}_1\big(\overline{\mathscr{W}}(t)\big)\big|\\
&=\bigg|\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\big[\Phi_{q_1}\big(\hbar_1\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\big)\\
&\quad-\Phi_{q_1}\big(\hbar_1\big(t,\overline{\mathscr{W}}_1(t),\overline{\mathscr{W}}_2(t)\big)\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1}\prod_{0\le t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\\
&\times\Big[\Phi_{q_1}\big(\hbar_1\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)\big)-\Phi_{q_1}\big(\hbar_1\big(s,\overline{\mathscr{W}}_1(s),\overline{\mathscr{W}}_2(s)\big)\big)\Big]\,ds\Big|\\
&\le\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\big|\Phi_{q_1}\big(\hbar_1\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\big)\\
&-\Phi_{q_1}\big(\hbar_1\big(t,\overline{\mathscr{W}}_1(t),\overline{\mathscr{W}}_2(t)\big)\big)\big|\\
&+\frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1}\prod_{0\le t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\big|\Phi_{q_1}\big(\hbar_1\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)\big)\\
&-\Phi_{q_1}\big(\hbar_1\big(s,\overline{\mathscr{W}}_1(s),\overline{\mathscr{W}}_2(s)\big)\big)\big|\,ds.
\end{aligned}
\tag{3.28}
$$

When $q_1\ge 2$, from Lemma 2.2 (vi), (3.22), (3.26), and (3.28), we yield

$$
\begin{aligned}
&\big|\mathscr{F}_1\big(\mathscr{W}(t)\big)-\mathscr{F}_1\big(\overline{\mathscr{W}}(t)\big)\big|\\
&\le\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})(q_1-1)\overline{\mathscr{M}_1}^{q_1-2}\\
&\times\big|\hbar_1\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)-\hbar_1\big(t,\overline{\mathscr{W}}_1(t),\overline{\mathscr{W}}_2(t)\big)\big|\\
&+\frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\\
&\times(q_1-1)\overline{\mathscr{M}_1}^{q_1-2}\int_0^t (t-s)^{\mu_1-1}\big|\hbar_1\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)-\hbar_1\big(s,\overline{\mathscr{W}}_1(s),\overline{\mathscr{W}}_2(s)\big)\big|\,ds\\
&\le\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})(q_1-1)\overline{\mathscr{M}_1}^{q_1-2}\Bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{1k})^{p_1-1}}\\
&+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{1k})^{p_1-1}}\Bigg]\times\left[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}+\frac{T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\right]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W})\\
&+\frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})(q_1-1)\overline{\mathscr{M}_1}^{q_1-2}\int_0^t (t-s)^{\mu_1-1}\\
&\times\Bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{1k})^{p_1-1}}+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{1k})^{p_1-1}}\Bigg]\\
&\times\left[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}+\frac{T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\right]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W})\,ds\\
&\le\frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\left[(1-\mu_1)(1-\nu_1)+\frac{(1-\mu_1)T^{\nu_1}}{\Gamma(\nu_1)}+\frac{(1-\nu_1)T^{\mu_1}}{\Gamma(\mu_1)}+\frac{\mu_1\nu_1T^{\mu_1+\nu_1}}{\Gamma(\mu_1)\Gamma(\nu_1)}\right]\\
&\times(q_1-1)\overline{\mathscr{M}_1}^{q_1-2}\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Bigg[\|\mathscr{L}_{11}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{1k})^{p_1-1}}\\
&+\|\mathscr{L}_{12}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{1k})^{p_1-1}}\Bigg]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W})\\
&=(q_1-1)\Theta_1\Delta_1\overline{\mathscr{M}_1}^{q_1-2}e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W})=\overline{\vartheta}_1e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W}).
\end{aligned}
\tag{3.29}
$$

When $1 < q_1 < 2$, by applying Lemma 2.2 (vi), (3.23), (3.26), and (3.28), we obtain

$$\begin{aligned}
&\left|\mathcal{F}_1\big(\mathcal{W}(t)\big) - \mathcal{F}_1\big(\overline{\mathcal{W}}(t)\big)\right| \\
&\leq \frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\bigg[(1-\mu_1)(1-\nu_1) + \frac{(1-\mu_1)T^{\nu_1}}{\Gamma(\nu_1)} \\
&\quad + \frac{(1-\nu_1)T^{\mu_1}}{\Gamma(\mu_1)} + \frac{\mu_1\nu_1T^{\mu_1+\nu_1}}{\Gamma(\mu_1)\Gamma(\nu_1)}\bigg](q_1-1)\underline{\mathcal{M}_1}^{q_1-2}\prod_{0\leq t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k}) \\
&\quad \times \bigg[\|\mathcal{L}_{11}\|_T\prod_{0\leq t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{1k})^{p_1-1}} + \|\mathcal{L}_{12}\|_T\prod_{0\leq t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{1k})^{p_1-1}}\bigg]e^{-\lambda}\times\rho(\mathcal{W},\mathcal{W}) \\
&= (q_1-1)\Theta_1\Delta_1\underline{\mathcal{M}_1}^{q_1-2}e^{-\lambda} = \underline{\vartheta_1}e^{-\lambda}\times\rho(\mathcal{W},\mathcal{W}).
\end{aligned} \tag{3.30}$$

Similar to (3.28)–(3.30), we have

$$\left|\mathcal{F}_2\big(\mathcal{W}(t)\big) - \mathcal{F}_2\big(\overline{\mathcal{W}}(t)\big)\right| \leq \overline{\vartheta_2}e^{-\lambda}\times\rho(\mathcal{W},\mathcal{W}), \quad q_2 \geq 2 \tag{3.31}$$

and

$$\left|\mathcal{F}_2\big(\mathcal{W}(t)\big) - \mathcal{F}_2\big(\overline{\mathcal{W}}(t)\big)\right| \leq \underline{\vartheta_2}e^{-\lambda}\times\rho(\mathcal{W},\mathcal{W}), \quad 1 < q_2 < 2. \tag{3.32}$$

Thus, we derive from (3.29)–(3.32) that

$$\rho\big(\mathcal{F}(\mathcal{W}),\mathcal{F}(\overline{\mathcal{W}})\big) \leq \begin{cases}\max\{\overline{\vartheta_1},\overline{\vartheta_2}\}e^{-\lambda}\times\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1,q_2\geq 2,\\ \max\{\overline{\vartheta_1},\underline{\vartheta_2}\}e^{-\lambda}\times\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1\geq 2, 1<q_2<2,\\ \max\{\underline{\vartheta_1},\overline{\vartheta_2}\}e^{-\lambda}\times\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1<2, q_2\geq 2,\\ \max\{\underline{\vartheta_1},\underline{\vartheta_2}\}e^{-\lambda}\times\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1,q_2<2.\end{cases} \tag{3.33}$$

By (H_4), we know that

$$-\infty < \log\max\{\overline{\vartheta_1},\overline{\vartheta_2}\}, \log\max\{\overline{\vartheta_1},\underline{\vartheta_2}\}, \log\max\{\underline{\vartheta_1},\overline{\vartheta_2}\}, \log\max\{\underline{\vartheta_1},\underline{\vartheta_2}\} < 0. \tag{3.34}$$

Taking the logarithm on both sides of (3.33) and applying (3.34), we have

$$\begin{aligned}
\log\rho\big(\mathcal{F}(\mathcal{W}),\mathcal{F}(\overline{\mathcal{W}})\big) &\leq \begin{cases}\log\max\{\overline{\vartheta_1},\overline{\vartheta_2}\} - \lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1,q_2\geq 2,\\ \log\max\{\overline{\vartheta_1},\underline{\vartheta_2}\} - \lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1\geq 2, 1<q_2<2,\\ \log\max\{\underline{\vartheta_1},\overline{\vartheta_2}\} - \lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1<2, q_2\geq 2,\\ \log\max\{\underline{\vartheta_1},\underline{\vartheta_2}\} - \lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1,q_2<2,\end{cases} \\
&< \begin{cases}-\lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1,q_2\geq 2,\\ -\lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & q_1\geq 2, 1<q_2<2,\\ -\lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1<2, q_2\geq 2,\\ -\lambda + \log\rho(\mathcal{W},\overline{\mathcal{W}}), & 1<q_1,q_2<2,\end{cases}
\end{aligned}$$

which implies that

$$\log\rho\big(\mathcal{F}(\mathcal{W}),\mathcal{F}(\overline{\mathcal{W}})\big)+\lambda<\log\rho(\mathcal{W},\overline{\mathcal{W}}). \tag{3.35}$$

In addition, it is clear that

$$\begin{aligned}\rho(\mathcal{W},\overline{\mathcal{W}})\leq\max\bigg\{&\rho(\mathcal{W},\overline{\mathcal{W}}),\rho\big(\mathcal{W},\mathcal{F}(\mathcal{W})\big),\rho\big(\overline{\mathcal{W}},\mathcal{F}(\overline{\mathcal{W}})\big),\\&\frac{\rho(\mathcal{W},\mathcal{F}(\overline{\mathcal{W}}))+\rho(\overline{\mathcal{W}},\mathcal{F}(\overline{\mathcal{W}}))}{2}\bigg\}=\mathcal{S}(\mathcal{W},\overline{\mathcal{W}}).\end{aligned} \tag{3.36}$$

From (3.35) and (3.36), we obtain

$$\log\rho\big(\mathcal{F}(\mathcal{W}),\mathcal{F}(\overline{\mathcal{W}})\big)+\lambda\leq\log\mathcal{S}(\mathcal{W},\overline{\mathcal{W}}). \tag{3.37}$$

We choose an F-contraction mapping as $F(z)=\log z$, then (3.37) can be rewritten as

$$F\big(\rho\big(\mathcal{F}(\mathcal{W}),\mathcal{F}(\overline{\mathcal{W}})\big)\big)+\lambda\leq F\big(\mathcal{S}(\mathcal{W},\overline{\mathcal{W}})\big). \tag{3.38}$$

Equation (3.38) indicates that the condition (a1) in Lemma 2.3 holds. Obviously, $F(z)$ is continuous on $(0,\infty)$, which means that the condition (a2) in Lemma 2.3 is true. Thus, it follows from Lemma 2.3 that $\mathcal{F}$ exists and a unique fixed point $\mathcal{W}^*(t)=(\mathcal{W}_1^*(t),\mathcal{W}_2^*(t))\in\mathbb{X}$, which is the unique solution of (3.10). The proof is completed. □

4 Generalized UH-stability

This section focuses on the generalized UH-stability of problem (1.1). Therefore, for all $\delta>0$, we consider the impulsive fractional differential inequalities below:

$$\begin{cases}{}^{\mathrm{ABC}}\mathcal{D}^{\nu_1}_{t_k^+}[\Phi_{p_1}({}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_k^+}\mathcal{X}_1(t))]-f_1(t,\mathcal{X}_1(t),\mathcal{X}_2(t))\leq\delta, & t\in(t_k,t_{k+1}]\subset I,\\ {}^{\mathrm{ABC}}\mathcal{D}^{\nu_2}_{t_k^+}[\Phi_{p_2}({}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_k^+}\mathcal{X}_2(t))]-f_2(t,\mathcal{X}_1(t),\mathcal{X}_2(t))\leq\delta, & t\in(t_k,t_{k+1}]\subset I,\\ \mathcal{X}_1(t_k^+)=(1+\xi_{1k})\mathcal{X}_1(t_k^-), \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_k^+}\mathcal{X}_1(t_k^+)=(1+\zeta_{1k}){}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_{k-1}^+}\mathcal{X}_1(t_k^-),\\ \mathcal{X}_2(t_k^+)=(1+\xi_{2k})\mathcal{X}_2(t_k^-), \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_k^+}\mathcal{X}_2(t_k^+)=(1+\zeta_{2k}){}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_{k-1}^+}\mathcal{X}_2(t_k^-),\\ \mathcal{X}_1(0)=w_1, \quad \mathcal{X}_2(0)=w_2, \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{0^+}\mathcal{X}_1(0)=v_1, \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{0^+}\mathcal{X}_2(0)=v_2.\end{cases} \tag{4.1}$$

Remark 4.1 $\mathcal{X}(t)=(\mathcal{X}_1(t),\mathcal{X}_2(t))$ is a solution of inequalities (4.1) iff there has a continuous function $\varphi(t)=(\varphi_1(t),\varphi_2(t))$ such that

$$\begin{cases}|\varphi_1(t)|\leq\delta, \quad |\varphi_2(t)|\leq\delta, \quad t\in(t_k,t_{k+1}]\subset I,\\ {}^{\mathrm{ABC}}\mathcal{D}^{\nu_1}_{t_k^+}[\Phi_{p_1}({}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_k^+}\mathcal{X}_1(t))]=f_1(t,\mathcal{X}_1(t),\mathcal{X}_2(t))+\varphi_{1k}(t), & t\in(t_k,t_{k+1}]\subset I,\\ {}^{\mathrm{ABC}}\mathcal{D}^{\nu_2}_{t_k^+}[\Phi_{p_2}({}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_k^+}\mathcal{X}_2(t))]=f_2(t,\mathcal{X}_1(t),\mathcal{X}_2(t))+\varphi_{2k}(t), & t\in(t_k,t_{k+1}]\subset I,\\ \mathcal{X}_1(t_k^+)=(1+\xi_{1k})\mathcal{X}_1(t_k^-), \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_k^+}\mathcal{X}_1(t_k^+)=(1+\zeta_{1k}){}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{t_{k-1}^+}\mathcal{X}_1(t_k^-),\\ \mathcal{X}_2(t_k^+)=(1+\xi_{2k})\mathcal{X}_2(t_k^-), \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_k^+}\mathcal{X}_2(t_k^+)=(1+\zeta_{2k}){}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{t_{k-1}^+}\mathcal{X}_2(t_k^-),\\ \mathcal{X}_1(0)=w_1, \quad \mathcal{X}_2(0)=w_2, \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_1}_{0^+}\mathcal{X}_1(0)=v_1, \quad {}^{\mathrm{ABC}}\mathcal{D}^{\mu_2}_{0^+}\mathcal{X}_2(0)=v_2.\end{cases} \tag{4.2}$$

Consider the integral system below:

$$\begin{cases} \mathscr{W}_1(t) = w_1 + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}[\prod_{0\le t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\hbar_1^{\varphi}(t,\mathscr{W}_1(t),\mathscr{W}_2(t))) - \upsilon_1] \\ \qquad + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1} \\ \qquad \times \prod_{0\le t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\hbar_1^{\varphi}(s,\mathscr{W}_1(s),\mathscr{W}_2(s)))\,ds, \\ \mathscr{W}_2(t) = w_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}[\prod_{0\le t_k<t}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\hbar_2^{\varphi}(t,\mathscr{W}_1(t),\mathscr{W}_2(t))) - \upsilon_2] \\ \qquad + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1} \\ \qquad \times \prod_{0\le t_k<s}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\hbar_2^{\varphi}(s,\mathscr{W}_1(s),\mathscr{W}_2(s)))\,ds, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &\hbar_1^{\varphi}\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big) \\ &\quad = \Phi_{p_1}(\upsilon_1) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\Bigg[\Phi_{p_1}\Bigg(\prod_{0\le t_k<s}\frac{1}{1+\zeta_{1k}}\Bigg)\Bigg(\varphi_1(s) \\ &\qquad + f_1\Bigg(s,\prod_{0\le t_k<s}(1+\xi_{1k})\mathscr{W}_1(s),\prod_{0\le t_k<s}(1+\xi_{2k})\mathscr{W}_2(s)\Bigg)\Bigg) - f_1(0,w_1,w_2) - \varphi_1(0)\Bigg] \\ &\qquad + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^s (s-\tau)^{\nu_1-1}\Phi_{p_1}\Bigg(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{1k}}\Bigg)\Bigg(\varphi_1(\tau) \\ &\qquad + f_1\Bigg(\tau,\prod_{0\le t_k<\tau}(1+\xi_{1k})\mathscr{W}_1(\tau),\prod_{0\le t_k<\tau}(1+\xi_{2k})\mathscr{W}_2(\tau)\Bigg)\Bigg)\,d\tau, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\hbar_2^{\varphi}\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big) \\ &\quad = \Phi_{p_2}(\upsilon_2) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\Bigg[\Phi_{p_2}\Bigg(\prod_{0\le t_k<s}\frac{1}{1+\zeta_{2k}}\Bigg)\Bigg(\varphi_2(s) \\ &\qquad + f_2\Bigg(s,\prod_{0\le t_k<s}(1+\xi_{1k})\mathscr{W}_1(s),\prod_{0\le t_k<s}(1+\xi_{2k})\mathscr{W}_2(s)\Bigg)\Bigg) - f_2(0,w_1,w_2) - \varphi_2(0)\Bigg] \\ &\qquad + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^s (s-\tau)^{\nu_2-1}\Phi_{p_2}\Bigg(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{2k}}\Bigg)\Bigg(\varphi_2(\tau) \\ &\qquad + f_2\Bigg(\tau,\prod_{0\le t_k<\tau}(1+\xi_{1k})\mathscr{W}_1(\tau),\prod_{0\le t_k<\tau}(1+\xi_{2k})\mathscr{W}_2(\tau)\Bigg)\Bigg)\,d\tau. \end{aligned} \tag{4.5}$$

According to Lemmas 3.1 and 3.2, Remark 3.1, and Remark 4.1, we have the following result.

Lemma 4.1 *If* $\mathscr{W}(t) = (\mathscr{W}_1(t),\mathscr{W}_2(t)) \in \mathbb{X}$ *is a solution of* (4.3), *then* $\mathscr{X}(t) = (\mathscr{X}_1(t),\mathscr{X}_2(t))$ *is the solution of* (4.2), *which is also the solution of* (4.1), *where*

$$\mathscr{X}_1(t) = \prod_{0\le t_k<t}(1+\xi_{1k})\mathscr{W}_1(t), \qquad \mathscr{X}_2(t) = \prod_{0\le t_k<t}(1+\xi_{2k})\mathscr{W}_2(t).$$

Therefore, it follows from Lemma 4.1 that the existence of the solutions to the inequalities (4.1) and the integral system (4.3) is equivalent. Additionally, from Lemma 3.1, we know that the generalized UH-stability of systems (1.1) and (3.10) is equivalent. Next, we will only discuss the generalized UH-stability for system (3.10).

Definition 4.1 System (3.10) is generalized UH-stable on the metric space $(\mathbb{X},\rho)$ iff, for all $\delta > 0$ and any solution $\mathcal{W} = (\mathcal{W}_1,\mathcal{W}_2) \in \mathbb{X}$ of (4.3), there exists an $\omega \in C(\mathbb{R},\mathbb{R}^+)$ with $\omega(0) = 0$ and a unique solution $\mathcal{W}^* = (\mathcal{W}_1^*,\mathcal{W}_2^*) \in \mathbb{X}$ of (3.10) such that

$$\rho(\mathcal{W},\mathcal{W}^*) \le \omega(\delta).$$

Theorem 4.1 *Provided that* (H_1)–(H_4) *hold, then system* (3.10) *is generalized UH-stable.*

Proof By Theorem 3.1, we know that system (3.10) has a unique solution $\mathcal{W}^*(t) = (\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)) \in \mathbb{X}$. For all $\delta > 0$ (δ small enough), similar to (3.22)–(3.25), we derive from (H_1), (H_2), Remark 4.1, (4.4), and (4.5) that

$$\begin{aligned}\hbar_1^\varphi\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\le \upsilon_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\bigg[(M_1+\delta)\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} + \delta - m_1\bigg] \\ &\quad + \frac{(M_1+\delta)T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} \triangleq \overline{\mathcal{M}_1}(\delta), \end{aligned} \tag{4.6}$$

$$\begin{aligned}\hbar_1^\varphi\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\ge \upsilon_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\bigg[(m_1-\delta)\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{1k})^{p_1-1}} - \delta - M_1\bigg] \\ &\triangleq \underline{\mathcal{M}_1}(\delta) > 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned}\hbar_2^\varphi\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\le \upsilon_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\bigg[(M_2+\delta)\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} + \delta - m_2\bigg] \\ &\quad + \frac{(M_2+\delta)T^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} \triangleq \overline{\mathcal{M}_2}(\delta) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\hbar_2^\varphi\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) &\ge \upsilon_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\bigg[(m_2-\delta)\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{2k})^{p_2-1}} - \delta - M_2\bigg] \\ &\triangleq \underline{\mathcal{M}_2}(\delta) > 0. \end{aligned} \tag{4.9}$$

From (3.22)–(3.25) and (4.6)–(4.9), one has

$$0 < \underline{\mathcal{M}_1}(\delta) < \underline{\mathcal{M}_1} < \overline{\mathcal{M}_1} < \overline{\mathcal{M}_1}(\delta), \qquad 0 < \underline{\mathcal{M}_2}(\delta) < \underline{\mathcal{M}_2} < \overline{\mathcal{M}_2} < \overline{\mathcal{M}_2}(\delta). \tag{4.10}$$

For any solution $\mathcal{W}(t) = (\mathcal{W}_1(t),\mathcal{W}_2(t)) \in \mathbb{X}$ to system (4.3) and the unique solution $\mathcal{W}^*(t) = (\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)) \in \mathbb{X}$ to system (3.10), by (3.11), (3.12), (4.4), (4.5), and (4.2), we have

$$\begin{aligned}&\big|\hbar_i^\varphi\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - \hbar_i\big(t,\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)\big)\big| \\ &\quad = \Big|\big[\hbar_i\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - \hbar_i\big(t,\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)\big)\big]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\nu_i}{\mathfrak{N}(\nu_i)}\left[\Phi_{p_i}\left(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{ik}}\right)\varphi_i(t)-\varphi_i(0)\right]+\frac{\nu_i}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\int_0^t(t-\tau)^{\nu_i-1}\\
&\times\Phi_{p_i}\left(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{ik}}\right)\varphi_i(\tau)\,d\tau\Bigg|\\
&\le\left|\hbar_i\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)-\hbar_i\big(t,\mathscr{W}_1^*(t),\mathscr{W}_2^*(t)\big)\right|\\
&+\frac{1-\nu_i}{\mathfrak{N}(\nu_i)}\left[\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}}+1\right]\delta+\frac{\delta T^{\nu_i}}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}}\\
&=\left|\hbar_i\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)-\hbar_i\big(t,\mathscr{W}_1^*(t),\mathscr{W}_2^*(t)\big)\right|+\Xi_i(\delta),\quad i=1,2,
\end{aligned}
\tag{4.11}
$$

where $\Xi_i(\delta)=\frac{1-\nu_i}{\mathfrak{N}(\nu_i)}[\prod_{0\le t_k<t}\frac{1}{(1+\zeta_{ik})^{p_i-1}}+1]\delta+\frac{\delta T^{\nu_i}}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\prod_{0\le t_k<\tau}\frac{1}{(1+\zeta_{ik})^{p_i-1}}$.

When $q_i\ge 2$ ($i=1,2$), similar to (3.28), we derive from (3.10), (3.29), (3.31), (4.3), (4.10), (4.11), and (vi) in Lemma 2.2 that

$$
\begin{aligned}
&\left|\mathscr{W}_i(t)-\mathscr{W}_i^*(t)\right|\\
&\le\frac{1-\mu_i}{\mathfrak{N}(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})\Big|\Phi_{q_i}\big(\hbar_i^{\varphi}\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\big)\\
&-\Phi_{q_i}\big(\hbar_i\big(t,\mathscr{W}_1^*(t),\mathscr{W}_2^*(t)\big)\big)\Big|+\frac{\mu_i}{\mathfrak{N}(\mu_i)\Gamma(\mu_i)}\int_0^t(t-s)^{\mu_i-1}\prod_{0\le t_k<s}(1+\xi_{ik})^{-1}\\
&\times(1+\zeta_{ik})\Big|\Phi_{q_i}\big(\hbar_i^{\varphi}\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)\big)-\Phi_{q_i}\big(\hbar_i\big(s,\mathscr{W}_1^*(s),\mathscr{W}_2^*(s)\big)\big)\Big|\,ds\\
&\le\frac{1-\mu_i}{\mathfrak{N}(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\overline{\mathscr{M}_i}(\delta)^{q_i-2}\Big|\hbar_i^{\varphi}\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\\
&-\hbar_i\big(t,\mathscr{W}_1^*(t),\mathscr{W}_2^*(t)\big)\Big|+\frac{\mu_i}{\mathfrak{N}(\mu_i)\Gamma(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\\
&\times\overline{\mathscr{M}_i}(\delta)^{q_i-2}\int_0^t(t-s)^{\mu_i-1}\Big|\hbar_i^{\varphi}\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)-\hbar_i\big(s,\mathscr{W}_1^*(s),\mathscr{W}_2^*(s)\big)\Big|\,ds\\
&=\frac{1-\mu_i}{\mathfrak{N}(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\overline{\mathscr{M}_i}(\delta)^{q_i-2}\Big(\Big|\hbar_i\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\\
&-\hbar_i\big(t,\mathscr{W}_1^*(t),\mathscr{W}_2^*(t)\big)\Big|+\Xi_i(\delta)\Big)+\frac{\mu_i}{\mathfrak{N}(\mu_i)\Gamma(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\\
&\times\overline{\mathscr{M}_i}(\delta)^{q_i-2}\int_0^t(t-s)^{\mu_i-1}\Big(\Big|\hbar_i\big(s,\mathscr{W}_1(s),\mathscr{W}_2(s)\big)-\hbar_i\big(s,\mathscr{W}_1^*(s),\mathscr{W}_2^*(s)\big)\Big|+\Xi_i(\delta)\Big)\,ds\\
&\le\frac{1-\mu_i}{\mathfrak{N}(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\overline{\mathscr{M}_i}(\delta)^{q_i-2}\Bigg(\Bigg[\|\mathscr{L}_{i1}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{ik})^{p_i-1}}\\
&+\|\mathscr{L}_{i2}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{ik})^{p_i-1}}\Bigg]\Bigg[\frac{1-\nu_i}{\mathfrak{N}(\nu_i)}+\frac{T^{\nu_i}}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\Bigg]e^{-\lambda}\times\rho(\mathscr{W},\mathscr{W})+\Xi_i(\delta)\Bigg)\\
&+\frac{\mu_i}{\mathfrak{N}(\mu_i)\Gamma(\mu_i)}\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})(q_i-1)\overline{\mathscr{M}_i}(\delta)^{q_i-2}\int_0^t(t-s)^{\mu_i-1}\\
&\times\Bigg(\Bigg[\|\mathscr{L}_{i1}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{1k}}{(1+\zeta_{ik})^{p_i-1}}+\|\mathscr{L}_{i2}\|_T\prod_{0\le t_k<t}\frac{1+\xi_{2k}}{(1+\zeta_{ik})^{p_i-1}}\Bigg]
\end{aligned}
$$

$$\times \left[\frac{1-\nu_i}{\mathfrak{N}(\nu_i)} + \frac{T^{\nu_i}}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\right] e^{-\lambda} \times \rho(\mathscr{W},\mathscr{W}) + \Xi_i(\delta)\Bigg) ds$$
$$\leq \overline{\theta_i}(\delta) e^{-\lambda} \times \rho(\mathscr{W},\mathscr{W}) + \overline{\kappa_i}(\delta), \tag{4.12}$$

where $\overline{\theta_i}(\delta) = (q_i - 1)\Theta_i \Delta_i \overline{\mathscr{M}_i}(\delta)^{q_i-2}$, $\overline{\kappa_i}(\delta) = (q_i - 1)\Theta_i \Delta_i \overline{\mathscr{M}_i}(\delta)^{q_i-2}\Xi_i(\delta)$.

When $1 < q_i < 2$ ($i = 1,2$), similar to (4.12), we have

$$\left|\mathscr{W}_i(t) - \mathscr{W}_i^*(t)\right| \leq \underline{\theta_i}(\delta) e^{-\lambda} \times \rho(\mathscr{W},\mathscr{W}) + \underline{\kappa_i}(\delta), \tag{4.13}$$

where $\underline{\theta_i}(\delta) = (q_i - 1)\Theta_i \Delta_i \underline{\mathscr{M}_i}(\delta)^{q_i-2}$, $\underline{\kappa_i}(\delta) = (q_i - 1)\Theta_i \Delta_i \underline{\mathscr{M}_i}(\delta)^{q_i-2}\Xi_i(\delta)$.

For all $\delta > 0$ (δ small enough), we know that $0 < \overline{\theta_1}(\delta), \underline{\theta_1}(\delta), \overline{\theta_2}(\delta), \underline{\theta_2}(\delta) < 1$. Thus, from (4.12) and (4.13), we obtain

$$\rho(\mathscr{W},\mathscr{W}) \leq \begin{cases} \frac{\max\{\overline{\kappa_1}(\delta),\overline{\kappa_2}(\delta)\}}{1-\max\{\overline{\theta_1}(\delta),\overline{\theta_2}(\delta)\}e^{-\lambda}}, & q_1, q_2 \geq 2, \\ \frac{\max\{\overline{\kappa_1}(\delta),\underline{\kappa_2}(\delta)\}}{1-\max\{\overline{\theta_1}(\delta),\underline{\theta_2}(\delta)\}e^{-\lambda}}, & q_1 \geq 2, 1 < q_2 < 2, \\ \frac{\max\{\underline{\kappa_1}(\delta),\overline{\kappa_2}(\delta)\}}{1-\max\{\underline{\theta_1}(\delta),\overline{\theta_2}(\delta)\}e^{-\lambda}}, & 1 < q_1 < 2, q_2 \geq 2, \\ \frac{\max\{\underline{\kappa_1}(\delta),\underline{\kappa_2}(\delta)\}}{1-\max\{\underline{\theta_1}(\delta),\underline{\theta_2}(\delta)\}e^{-\lambda}}, & 1 < q_1, q_2 < 2. \end{cases} \tag{4.14}$$

According to Definition 4.1 and (4.14), we conclude that system (3.10) is generalized UH-stable. The proof is completed. □

5 Simulation algorithms and examples

In this section, we will provide a numerical simulation algorithm for system (1.1) and apply an example to checkout the correctness and effectiveness of our theoretical results and simulation algorithm.

5.1 Simulation algorithms

Based on assumptions (H_1)–(H_4), our simulation algorithm further requires the following assumption:

(H_5) For all $t, u, v \in \mathbb{R}$, $f_i(t,u,v)$ $(i = 1,2)$ has the first-order partial derivative at (t,u,v), that is, $\frac{\partial f_i}{\partial t}$, $\frac{\partial f_i}{\partial u}$, and $\frac{\partial f_i}{\partial v}$ all exist.

By (3.1), (3.2), and Lemmas 3.1 and 3.2, we give a simulation algorithm for system (1.1) as follows:

Step 1: Let $\Phi_{p_1}({}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\mu_1}\mathfrak{X}_1(t)) = \mathscr{Y}_1(t)$, $\Phi_{p_2}({}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\mu_2}\mathfrak{X}_2(t)) = \mathscr{Y}_2(t)$, we will convert system (1.1) to system (3.1).

Step 2: Based on system (3.1), we can obtain system (3.2).

Step 3: According to Lemma 3.2, we can obtain an integral system (3.10).

Step 4: In view of Definition 2.2 and (H_5), we know that $\mathscr{W}_1'(t)$ and $\mathscr{W}_2'(t)$ all exist. Therefore, taking the derivative on both sides of (3.10), we obtain

$$\begin{cases} \mathscr{W}_1'(t) = \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} \prod_{0\leq t_k<t}(1+\xi_{1k})^{-1}(1+\zeta_{1k}) \frac{d[\Phi_{q_1}(\hbar_1(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))]}{dt} \\ \qquad + \frac{\mu_1(\mu_1-1)}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-2} \\ \qquad \times \prod_{0\leq t_k<s}(1+\xi_{1k})^{-1}(1+\zeta_{1k})\Phi_{q_1}(\hbar_1(s,\mathscr{W}_1(s),\mathscr{W}_2(s)))\, ds, \\ \mathscr{W}_2'(t) = \frac{1-\mu_2}{\mathfrak{N}(\mu_2)} \prod_{0\leq t_k<t}(1+\xi_{2k})^{-1}(1+\zeta_{2k}) \frac{d[\Phi_{q_2}(\hbar_2(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))]}{dt} \\ \qquad + \frac{\mu_2(\mu_2-1)}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)} \int_0^t (t-s)^{\mu_2-2} \\ \qquad \times \prod_{0\leq t_k<s}(1+\xi_{2k})^{-1}(1+\zeta_{2k})\Phi_{q_2}(\hbar_2(s,\mathscr{W}_1(s),\mathscr{W}_2(s)))\, ds. \end{cases} \tag{5.1}$$

It follows from the definition of Φ_{q_i} $(i = 1,2)$ that

$$\begin{aligned}
&\frac{d[\Phi_{q_i}(\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))]}{dt}\\
&\quad= \frac{d[|\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))|^{q_i-2}\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))]}{dt}\\
&\quad= \frac{d[(\operatorname{sgn}(\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))))^{q_1-2}\times(\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))^{q_i-1}]}{dt}\\
&\quad= (q_i-1)\big|\hbar_i\big(t,\mathscr{W}_1(t),\mathscr{W}_2(t)\big)\big|^{q_1-2}\times\frac{d[\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))]}{dt}\\
&\quad= \frac{(q_i-1)\Phi_{q_i}(\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))}{\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))}\times\frac{d[\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))]}{dt}.
\end{aligned}\tag{5.2}$$

Let $U = \prod_{0\le t_k<t}(1+\xi_{1k})\mathscr{W}_1(t)$, $V = \prod_{0\le t_k<t}(1+\xi_{2k})\mathscr{W}_2(t)$, we derive from (3.11) and (3.12) that

$$\begin{aligned}
&\frac{d[\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))]}{dt}\\
&\quad= \frac{1-\nu_i}{\mathfrak{N}(\nu_i)}\Phi_{p_i}\bigg(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{ik}}\bigg)\bigg[\frac{\partial f_i}{\partial t}+\frac{\partial f_i}{\partial U}\times\prod_{0\le t_k<t}(1+\xi_{1k})\mathscr{W}_1'(t)\\
&\qquad+\frac{\partial f_i}{\partial V}\times\prod_{0\le t_k<t}(1+\xi_{2k})\mathscr{W}_2'(t)\bigg]+\frac{\nu_i(\nu_i-1)}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\int_0^t(t-\tau)^{\nu_i-2}\Phi_{p_i}\bigg(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{ik}}\bigg)\\
&\qquad\times f_i\bigg(\tau,\prod_{0\le t_k<\tau}(1+\xi_{1k})\mathscr{W}_1(\tau),\prod_{0\le t_k<\tau}(1+\xi_{2k})\mathscr{W}_2(\tau)\bigg)\,d\tau,\quad i=1,2.
\end{aligned}\tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1), we can simplify it to obtain

$$\begin{cases}\mathscr{W}_1'(t) = A_1[B_1\mathscr{W}_1'(t)+C_1\mathscr{W}_2'(t)+D_1]+E_1,\\ \mathscr{W}_2'(t) = A_2[B_2\mathscr{W}_1'(t)+C_2\mathscr{W}_2'(t)+D_2]+E_2,\end{cases}\tag{5.4}$$

where

$$\begin{aligned}
A_i &= \frac{(1-\mu_i)(1-\nu_i)}{\mathfrak{N}(\mu_i)\mathfrak{N}(\nu_i)}\times\prod_{0\le t_k<t}(1+\xi_{ik})^{-1}(1+\zeta_{ik})\times\Phi_{p_i}\bigg(\prod_{0\le t_k<t}\frac{1}{1+\zeta_{ik}}\bigg)\\
&\quad\times\frac{(q_i-1)\Phi_{q_i}(\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t)))}{\hbar_i(t,\mathscr{W}_1(t),\mathscr{W}_2(t))},\\
B_i &= \frac{\partial f_i}{\partial U}\times\prod_{0\le t_k<t}(1+\xi_{1k}),\qquad C_i = \frac{\partial f_i}{\partial V}\times\prod_{0\le t_k<t}(1+\xi_{2k}),\\
D_i &= \frac{\nu_i(\nu_i-1)}{\mathfrak{N}(\nu_i)\Gamma(\nu_i)}\int_0^t(t-s)^{\nu_i-2}\Phi_{p_i}\bigg(\prod_{0\le t_k<\tau}\frac{1}{1+\zeta_{ik}}\bigg)\\
&\quad\times f_i\bigg(s,\prod_{0\le t_k<s}(1+\xi_{1k})\mathscr{W}_1(s),\prod_{0\le t_k<s}(1+\xi_{2k})\mathscr{W}_2(s)\bigg)\,ds
\end{aligned}$$

and

$$E_i = \frac{\mu_i(\mu_i - 1)}{\mathfrak{N}(\mu_i)\Gamma(\mu_i)} \int_0^t (t-s)^{\mu_i - 2} \prod_{0 \le t_k < s} (1+\xi_{ik})^{-1}(1+\zeta_{ik})\Phi_{q_i}\bigl(\hbar_i\bigl(s, \mathscr{W}_1(s), \mathscr{W}_2(s)\bigr)\bigr)\, ds.$$

Applying Cramer's rule, (5.4) is rewritten as

$$\begin{cases} \mathscr{W}_1'(t) = \frac{J_1}{J}, \\ \mathscr{W}_2'(t) = \frac{J_2}{J}, \end{cases} \tag{5.5}$$

where

$$J = \begin{vmatrix} 1 - A_1 B_1 & -A_1 C_1 \\ -A_2 B_2 & 1 - A_2 C_2 \end{vmatrix},$$

$$J_1 = \begin{vmatrix} A_1 D_1 + E_1 & -A_1 C_1 \\ A_2 D_2 + E_2 & 1 - A_2 C_2 \end{vmatrix}$$

and

$$J_2 = \begin{vmatrix} 1 - A_1 B_1 & A_1 D_1 + E_1 \\ -A_2 B_2 & A_2 D_2 + E_2 \end{vmatrix}.$$

Step 5: We use the ode45 toolbox of MATLAB to solve equation (5.5).

Step 6: By applying the relationship $\mathfrak{X}_i(t) = \prod_{0 \le t_k < t}(1+\xi_{ik})\mathscr{W}_i(t)$ $(i = 1,2)$ in Lemma 3.1, we can perform the numerical simulation on system (1.1).

5.2 An example

Consider the following nonlinear fractional coupled (p_1, p_2)-Laplacian systems with non-singular Mittag–Leffler kernel and a single impulsive point:

$$\begin{cases} {}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\nu_1}[\Phi_{p_1}({}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\mu_1}\mathfrak{X}_1(t))] = f_1(t, \mathfrak{X}_1(t), \mathfrak{X}_2(t)), & t \in (0, t_1] \cup (t_1, T], \\ {}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\nu_2}[\Phi_{p_2}({}^{\mathrm{ABC}}\mathscr{D}_{t_k^+}^{\mu_2}\mathfrak{X}_2(t))] = f_2(t, \mathfrak{X}_1(t), \mathfrak{X}_2(t)), & t \in (0, t_1] \cup (t_1, T], \\ \mathfrak{X}_1(t_1^+) = (1+\xi_{11})\mathfrak{X}_1(t_1^-), \quad {}^{\mathrm{ABC}}\mathscr{D}_{t_1^+}^{\mu_1}\mathfrak{X}_1(t_1^+) = (1+\zeta_{11}){}^{\mathrm{ABC}}\mathscr{D}_{0^+}^{\mu_1}\mathfrak{X}_1(t_1^-), \\ \mathfrak{X}_2(t_1^+) = (1+\xi_{21})\mathfrak{X}_2(t_1^-), \quad {}^{\mathrm{ABC}}\mathscr{D}_{t_1^+}^{\mu_2}\mathfrak{X}_2(t_1^+) = (1+\zeta_{21}){}^{\mathrm{ABC}}\mathscr{D}_{0^+}^{\mu_2}\mathfrak{X}_2(t_1^-), \\ \mathfrak{X}_1(0) = w_1, \quad \mathfrak{X}_2(0) = w_2, \quad {}^{\mathrm{ABC}}\mathscr{D}_{0^+}^{\mu_1}\mathfrak{X}_1(0) = v_1, \quad {}^{\mathrm{ABC}}\mathscr{D}_{0^+}^{\mu_2}\mathfrak{X}_2(0) = v_2, \end{cases} \tag{5.6}$$

where $T = \sqrt{2}$, $t_1 = \frac{1}{\sqrt{2}}$, $p_1 = \frac{3}{2}$, $p_2 = \frac{5}{4}$, $\mu_1 = 0.7$, $\nu_1 = 0.6$, $\mu_2 = 0.2$, $\nu_2 = 0.4$, $w_1 = 1$, $w_2 = 5$, $v_1 = 2$, $v_2 = 3$, $\xi_{11} = \frac{1}{2}$, $\xi_{21} = \frac{1}{3}$, $\zeta_{11} = 3$, $\zeta_{21} = 4$, $f_1(t,u,v) = \frac{2+\cos(u)}{100e} + \frac{1}{50e}|\sin(t)|\frac{v}{1+v^2}$, $f_2(t,u,v) = \frac{2+\sin(3t)}{1000e}[\frac{3\pi}{4} + \arctan(u+v)]$.

Take $\mathfrak{N}(x) = 1 - x + \frac{x}{\Gamma(x)}$, $0 < x \le 1$, then $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$. A simple computation yields that $q_1 = 3 > 2$, $q_2 = 5 > 2$, and

$$\frac{1}{100} \le f_1(t,u,v) \le \frac{4}{100e}, \qquad \frac{\pi}{4000e} \le f_2(t,u,v) \le \frac{15\pi}{4000e},$$
$$\bigl|f_1(t,u,v) - f_1(t,\overline{u},\overline{v})\bigr| \le e^{-1}\left[\frac{1}{100}|u - \overline{u}| + \frac{|\sin(t)|}{100}|v - \overline{v}|\right],$$

$$\left|f_2(t,u,v) - f_2(t,\overline{u},\overline{v})\right| \le \frac{2+\sin(3t)}{1000}\left[|u-\overline{u}| + |v-\overline{v}|\right]e^{-1}.$$

Consequently, the conditions (H$_1$)–(H$_3$) are fulfilled. Additionally, $m_1 = \frac{1}{100e}$, $M_1 = \frac{4}{100e}$, $m_2 = \frac{\pi}{4000e}$, $M_2 = \frac{15\pi}{4000e}$, $\mathscr{L}_{11}(t) = \frac{1}{100}$, $\mathscr{L}_{12}(t) = \frac{|\sin(t)|}{100}$, $\mathscr{L}_{21}(t) = \mathscr{L}_{22}(t) = \frac{2+\sin(3t)}{200}$, $\|\mathscr{L}_{11}\|_T = \frac{1}{100}$, $\|\mathscr{L}_{12}\|_T = \frac{\sin(\sqrt{2})}{100}$, $\|\mathscr{L}_{21}\|_T = \|\mathscr{L}_{22}\|_T = \frac{3}{1000}$, and

$$\begin{aligned}
\Theta_1 &= \frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\left[(1-\mu_1)(1-\nu_1) + \frac{(1-\mu_1)T^{\nu_1}}{\Gamma(\nu_1)} + \frac{(1-\nu_1)T^{\mu_1}}{\Gamma(\mu_1)} + \frac{\mu_1\nu_1 T^{\mu_1+\nu_1}}{\Gamma(\mu_1)\Gamma(\nu_1)}\right] \\
&\approx 1.6350, \\
\Theta_2 &= \frac{1}{\mathfrak{N}(\mu_2)\mathfrak{N}(\nu_2)}\left[(1-\mu_2)(1-\nu_2) + \frac{(1-\mu_2)T^{\nu_2}}{\Gamma(\nu_2)} + \frac{(1-\nu_2)T^{\mu_2}}{\Gamma(\mu_2)} + \frac{\mu_2\nu_2 T^{\mu_2+\nu_2}}{\Gamma(\mu_2)\Gamma(\nu_2)}\right] \\
&\approx 1.5861, \\
\Delta_1 &= \frac{1+\zeta_{11}}{1+\xi_{11}}\left[\|\mathscr{L}_{11}\|_T \frac{1+\xi_{11}}{(1+\zeta_{11})^{p_1-1}} + \|\mathscr{L}_{12}\|_T \frac{1+\xi_{21}}{(1+\zeta_{11})^{p_1-1}}\right] \approx 0.0376, \\
\Delta_2 &= \frac{1+\zeta_{21}}{1+\xi_{21}}\left[\|\mathscr{L}_{21}\|_T \frac{1+\xi_{11}}{(1+\zeta_{21})^{p_2-1}} + \|\mathscr{L}_{22}\|_T \frac{1+\xi_{21}}{(1+\zeta_{21})^{p_2-1}}\right] \approx 0.0426, \\
\underline{\mathscr{M}_1} &= \upsilon_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\left[\frac{m_1}{(1+\zeta_{11})^{p_1-1}} - M_1\right] \approx 1.4078 > 0, \\
\underline{\mathscr{M}_2} &= \upsilon_2^{p_2-1} - \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\left[\frac{m_2}{(1+\zeta_{21})^{p_2-1}} - M_2\right] \approx 1.3129 > 0, \\
\overline{\mathscr{M}_1} &= \upsilon_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\left[\frac{M_1}{(1+\zeta_{11})^{p_1-1}} - m_1\right] + \frac{M_1 T^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\frac{1}{(1+\zeta_{11})^{p_1-1}} \approx 1.4236, \\
\overline{\mathscr{M}_2} &= \upsilon_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\left[\frac{M_2}{(1+\zeta_{21})^{p_2-1}} - m_2\right] + \frac{M_2 T^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\frac{1}{(1+\zeta_{21})^{p_2-1}} \approx 1.3200, \\
\overline{\vartheta_1} &= (q_1-1)\Theta_1\Delta_1\overline{\mathscr{M}_1}^{q_1-2} \approx 0.1750 < 1, \\
\overline{\vartheta_2} &= (q_2-1)\Theta_2\Delta_2\overline{\mathscr{M}_2}^{q_2-2} \approx 0.6216 < 1.
\end{aligned}$$

Thus, (H$_4$) is true. It follows from Theorem 3.1 and Theorem 4.1, that system (5.6) has a unique solution, which is generalized UH-stable.

By applying the algorithm in Sect. 5.1 and the ODE45 toolbox in MATLAB 2018b, we have provided numerical simulations of the solution for system (5.6), as shown in Figs. 1 and 2. The simulation shows that the solution of system (1.1) is discontinuous at impulsive point $t = \frac{1}{\sqrt{2}}$.

6 Conclusions

The ABC-fractional differential model has achieved better results in describing some problems in the fields of physics and engineering compared to integer-order differential systems. Some scholars have carried out studies on certain types of ABC-fractional differential equations. However, as far as we know there are no papers dealing with the nonlinear ABC-fractional differential coupled system with a Laplacian and impulses. Therefore, we try to fill the gap by studying system (1.1) in this manuscript. By constructing a complete metric space and F-contraction operator, and applying an important fixed-point theorem on metric space, we obtain the existence and uniqueness of the solution

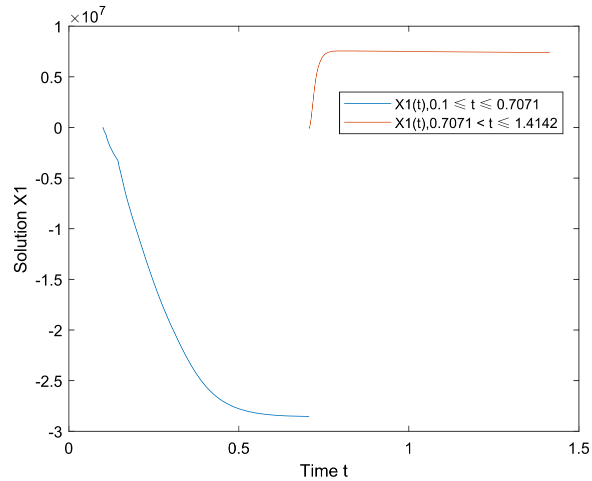

Figure 1 Simulation of solution $\mathfrak{X}_1(t)$

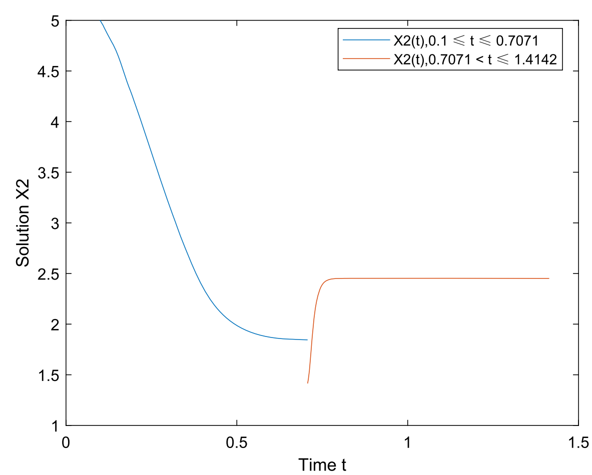

Figure 2 Simulation of solution $\mathfrak{X}_2(t)$

of system (1.1). Meanwhile, the generalized UH-stability is established by using the direct analysis method. In addition, we provide a novel numerical simulation algorithm. We apply an example to validate and demonstrate our theoretical results and algorithms. Our study shows that the existence, uniqueness, and stability of the solution to system (1.1) are closely related to Laplace parameters p_1, p_2, fractional derivative orders μ_i, ν_i ($i = 1,2$), impulse variables ξ_{ik}, ζ_{ik} ($i = 1,2$; $k = 1,2,\ldots,n$), initial values ${}^{ABC}\mathfrak{D}_{0^+}^{\mu_1}\mathfrak{X}_1(0) = \upsilon_1$, ${}^{ABC}\mathfrak{D}_{0^+}^{\mu_2}\mathfrak{X}_2(0) = \upsilon_2$, and $f_i(t,\cdot,\cdot)$ ($i = 1,2$). Moreover, our future academic focus will shift towards reaction–diffusion systems and population-dynamics systems involving fractional derivatives due to some of our preliminary studies [57–59, 64–69].

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Data availability
No data were used to support this study.

Declarations

Competing interests
The author declares that there are no competing interests.

Author contributions
The author read and approved the final manuscript.

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