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Stability analysis of fractional order breast cancer model in chemotherapy patients with cardiotoxicity by applying LADM

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Abstract

Breast cancer is the most common type of cancer in women. Chemotherapy is primarily used for patients with stage 2 to 4 breast cancer. Most chemotherapy drugs are effective at destroying rapidly growing and proliferating cancer cells. However, drugs also damage normal, rapidly growing cells, which can lead to serious side effects. Breast cancer treatment with chemotherapy can affect heart health. Side effects of chemotherapy on the heart are called cardiotoxicity. Therefore, we have constructed a mathematical model from the breast cancer patient population. In this article, we utilize the Caputo–Fabrizio fractional order derivative for mathematical modeling of the breast cancer stages in chemotherapy patients. The use of Caputo–Fabrizio fractional derivative provides a more valuable insight into the complexity of the breast cancer model. The stability of the fractional order model is also proven by the $\mathscr{P}$-stable approach of the fixed point theorem. Also, the numerical simulations are performed via Laplace Adomian decomposition method to establish the dependence of the breast cancer dynamics on the order of the fractional derivatives. Based on the geometric results in the figures, we can conclude that the magnitude of the fractional order has a considerable impact on the days, which the maximum or minimum of the system solutions are reached, with a shift in the time at which this happens as the fractional order decreases from 1. However, it is obvious that the solutions of Caputo–Fabrizio fractional model approach the relevant results of the classical integer order system, when the fractional order approaches to 1.

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1 Introduction

Cancer is one of the leading causes of death in many countries around the world. Cancer development is a gradual process through which normal body cells acquire mutations that allow them to escape their normal function in tissue and survive on their own. Breast cancer is the most common type of cancer in women.

The stages of cancer determine the severity of the cancer. The method used by doctors to describe the stage of the cancer is the TNM (tumor, node, metastasis) system. This

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system uses three criteria to determine the cancer stage, namely tumor size, spread to lymph nodes, and spread to other organs (metastasis). There are different types of cancer treatment. These are surgery, radiotherapy, hormone therapy, targeted therapy, and chemotherapy. This cancer treatment is used to kill cancer cells, remove cancer cells, or prevent cancer cells from receiving cell division signals. Chemotherapy is the most commonly used cancer treatment.

Chemotherapy not only has a positive effect on the patient's recovery, but is also harmful to their health. Treating breast cancer with chemotherapy can cause unwanted side effects on the heart called cardiotoxicity. The cardiotoxicity of chemotherapy has infected patients from children to adults for 35 years [1]. Common cardiotoxic chemotherapy regimens include anthracycline and trastuzumab. Complications of oncological treatment with anthracyclines and trastuzumab have dramatic clinical consequences that can lead to heart failure. Prevention of cardiotoxicity due to chemotherapy remains a challenge for cardiologists and cancer experts to date.

Mathematical modeling can be used to qualify the dynamics of various diseases in nature, such as tumor growth [2], cancer [3, 4], behavior of two-trophic plant-herbivore [5], COVID-19 epidemic [6], phytoplankton-zooplankton model in phytoplankton population [7], etc. However, mathematical models and computer simulations can help monitor tumor growth and cell distribution and observe genetic mutations that lead to aggressive growth and metastasis. On the other hand, finding the analytical and numerical solutions of this models and stability analysis of them has attracted the attention of many researchers [8–10].

Today, fractional calculus describes many complex biological systems by using different definitions of fractional operators, such as the transmission of nerve impulses [11], modeling of two avian influenza epidemic by two fractal-fractional derivatives [12], the behavior of immune and tumor cells in immunogenetic tumor model with nonsingular fractional derivative [13], the outbreak of dengue fever [14], the modeling and optimal control of diabetes and tuberculosis coexistence [15], optimal control of a tumor-immune surveillance with nonsingular derivative operator [16], etc. Although these studies produced better results than the classical integer order models, satisfactory precision may not have been achieved over the entire period due to the appearance of a singularity in the definition of traditional fractional derivatives, the fact that makes these operators useful for describing nonlocal dynamics [17] impractical. To overcome this difficulty, two new nonsingular fractional derivatives in [18] and [19] were proposed.

In this study, we generalize the classical (integer order) system to a novel fractional order system based on the Caputo–Fabrizio ($\mathscr{CF}$) derivative, a new fractional operator with a nonsingular exponential kernel. This fractional order derivative has been used for modeling the anthrax disease in animals [20], the epidemic of childhood diseases [21], etc. To the best of authors' knowledge, the aforementioned fractional model for the breast cancer is introduced for the first time in literature. In contrast to the previous works, this study discusses mathematical models at the population level of people with cancer. We construct the mathematical model analysis of breast cancer stages with side effects on the heart in chemotherapy patients.

Comparative results between this method and the classic method [22] verify the effectiveness of the proposed fractional architecture compared to the pre-existent classical model.

Dynamic analysis performed on systems of equations includes calculation of equilibrium points and stability analysis. Investigation of equilibrium and stability points was used to determine the dynamics of the five populations over time. In analyzing the stability of equilibrium points, we used the Routh–Hurwitz criterion [23].

The Laplace Adomian decomposition method (LADM) is a combination of Laplace transform and the Adomian decomposition method, which is a systematic and powerful tool to achieve the semi-analytical solutions for the fractional order equations involving nonsingular kernel. This approach is employed to obtain the semi-analytical solution of a system of differential equations describing a disease in [24–27] etc. Motivated by this, in this research we solve the fractional order of the breast cancer by LADM.

This paper is structured as follows. Section 2 is devoted to some preliminaries and definitions of Caputo–Fabrizio fractional order operators. In Sect. 3, we formulate the mathematical model of breast cancer in the $\mathscr{CF}$ fractional framework. By using the LADM, the semi-analytical solutions of the proposed fractional model are provided in Sect. 4. In Sect. 5, the stability of the system is analyzed using the Sumudu transform and the iteration method. In Sect. 6, with the help of available data, we perform numerical simulations. Finally, conclusions are summarized in Sect. 7.

2 Preliminaries

In this section we present some definitions and properties of fractional derivatives that will be helpful throughout the article.

Definition 2.1 Assume that $\alpha \in (n - 1, n]$ so that $n = [\alpha] + 1$. For a function $\tilde{w} \in AC_{\mathbb{R}}^n([0, +8))$, the fractional derivative of Caputo type is given by

$$^{\mathscr{C}}\mathscr{D}_t^\alpha\big[\tilde{u}(t)\big] = \int_0^t \tilde{u}^n(\zeta) \frac{(t - \zeta)^{n-\alpha-1}}{\Gamma(n - \alpha)} \, d\zeta,$$

provided that the integral is finite-valued.

Definition 2.2 [18] Let $\tilde{u} \in H^1(a, b)$, $b > a$, and $\alpha \in [0, 1]$. Then the Caputo–Fabrizio derivative is defined as follows:

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha\big[\tilde{u}(t)\big] = \frac{\mathscr{M}(\alpha)}{1 - \alpha} \int_a^t \tilde{u}'(\zeta) \exp\left[-\alpha \frac{t - \zeta}{1 - \alpha}\right] d\zeta, \tag{1}$$

where $\mathscr{M}(\alpha)$ indicates the normalization of the function satisfying the condition $\mathscr{M}(0) = \mathscr{M}(1) = 1$. If $\tilde{u} \notin H^1(a, b)$, then

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha\big[\tilde{u}(t)\big] = \frac{\mathscr{M}(\alpha)}{1 - \alpha} \int_a^t \big(\tilde{u}(t) - \tilde{u}(\zeta)\big) \exp\left[-\alpha \frac{t - \zeta}{1 - \alpha}\right] d\zeta. \tag{2}$$

Definition 2.3 [28] Let $0 < \alpha < 1$. The integral of the fractional order α for a function $\tilde{u}(t)$ is defined as follows:

$$^{\mathscr{CF}}I_t^\alpha\big[\tilde{u}(t)\big] = \frac{2(1 - \alpha)}{(2 - \alpha)\mathscr{M}(\alpha)} \tilde{u}(t) + \frac{2\alpha}{(2 - \alpha)\mathscr{M}(\alpha)} \int_0^t \tilde{u}(\zeta) \, d\zeta, \quad t \geq 0. \tag{3}$$

Considering $\mathscr{M}(\alpha) = \frac{2}{2-\alpha}$, the authors in [28] give the new $\mathscr{CF}$ fractional derivative of order $0 < \alpha < 1$, which is defined as follows:

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha\left[\tilde{u}(t)\right] = \frac{1}{1-\alpha}\int_a^t \tilde{u}'(s)\exp\left[-\alpha\frac{t-\zeta}{1-\alpha}\right]d\zeta. \tag{4}$$

Lemma 2.1 [29] *The Laplace transform of the $\mathscr{CF}$ fractional derivative is given as follows:*

$$\mathcal{L}\left[^{\mathscr{CF}}\mathscr{D}_t^\alpha\tilde{u}(t)\right](\zeta) = \frac{s\tilde{u}(t) - \tilde{u}(0)}{\zeta + \alpha(1-\zeta)}. \tag{5}$$

Definition 2.4 [29] The Sumudu transform of the fractional $\mathscr{CF}$-derivative for $\tilde{u}$ is defined by

$$\mathcal{ST}\left[^{\mathscr{CF}}\mathscr{D}_t^\alpha\right](\zeta) = \mathscr{M}(\alpha)\frac{\mathcal{ST}[\tilde{u}(t)] - \tilde{u}(0)}{\zeta + \alpha(1-\zeta)}. \tag{6}$$

3 Model formulation of breast cancer stages

The patients are classified into four subpopulations of stage 1 to stage 4. Changes in population dynamics from these subpopulations are depicted in a diagram called the compartment diagram, which is drawn in Fig. 1. The model was constructed from five compartments representing subpopulations of the breast cancer patients. Each subpopulation is represented by variables A, B, C, D, and E. Subpopulation A represents patients with Ductal Carcinoma In Situ cancer, stages 1, 2A, and 2B. Subpopulation B represents stage 3A and 3B cancer patients. Subpopulation C expresses the cancer patients of stage 4. Subpopulation D indicates the cancer patients with disease-free conditions after chemotherapy.
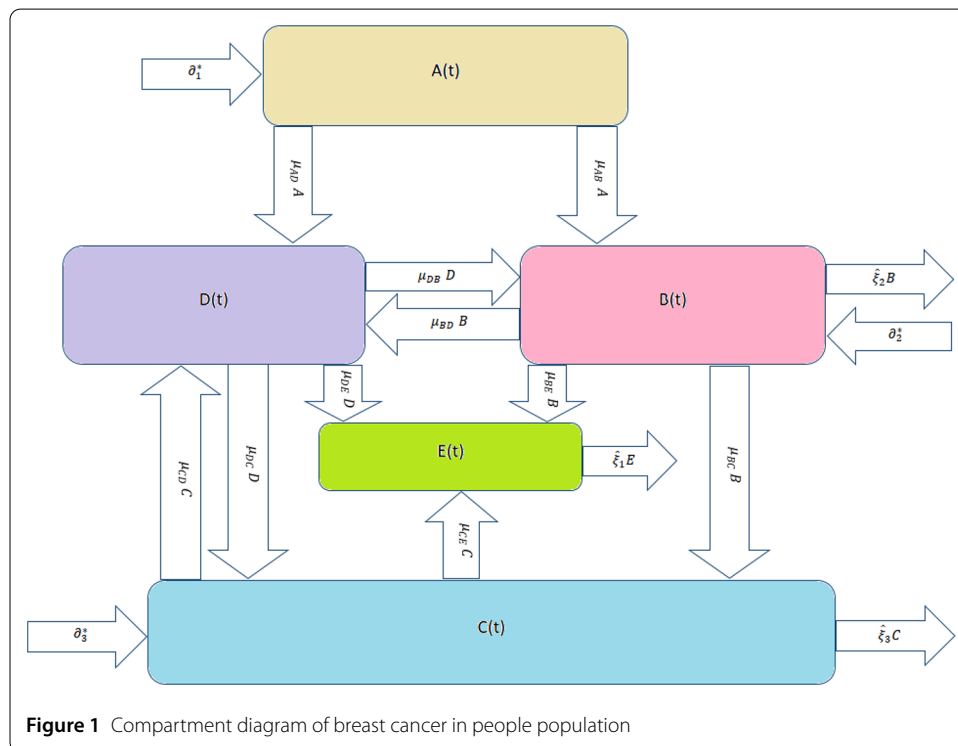

Figure 1 Compartment diagram of breast cancer in people population

In disease-free conditions, cancer is no longer seen by observation. Subpopulation E expresses cancer patients who have cardiotoxicity. The compartment diagram is shown in Fig. 1. The breast cancer stages model is introduced by the following system [22]:

$$\frac{dA}{dt} = \eth_1^* - \mu_{AD}A - \mu_{AB}A,$$

$$\frac{dB}{dt} = \eth_2^* + \mu_{AB}A + \mu_{DB}D - \mu_{BD}B - \mu_{BC}B - \mu_{BE}B - \hat{\xi}_2 B,$$

$$\frac{dC}{dt} = \eth_3^* + \mu_{BC}B + \mu_{DC}D - \mu_{CD}C - \mu_{CE}C - \hat{\xi}_3 C,$$

$$\frac{dD}{dt} = \mu_{AD}A + \mu_{BD}B + \mu_{CD}C - \mu_{DB}D - \mu_{DC}D - \mu_{DE}D,$$

$$\frac{dE}{dt} = \mu_{DE}D + \mu_{CE}C + \mu_{BE}B - \hat{\xi}_1 E. \tag{7}$$

Stages 1 and 2 of the cancer patients were assigned to a subpopulation A with rate θ_1. Patients in subpopulation A who have received chemotherapy have two options, namely recovery 4 (disease-free) at the rate μ_{AD} or worse at the rate μ_{AB}. The patients who were treated in hospital for the first time mostly suffered from stage 3 cancer and were therefore attributed to subpopulation B with rate θ_2. Patients in subpopulation B could die of cancer at a rate of γ_2.

Patients belonging to this subpopulation may become disease free after chemotherapy at the rate μ_{BD} and also deteriorate at the rate μ_{BC}. Subpopulation B with more intensive chemotherapy than subpopulation A may cause patients to experience cardiotoxicity at the μ_{BE} rate.

Patients receiving treatment for the first time may also be included in subpopulation C because the cancer has progressed to metastasis or stage 4. During this disease, it is unlikely that chemotherapy will cure the cancer, so the rate toward a disease-free rate is expected to be μ_{CD}, being the lowest compared to μ_{AD} and μ_{BD}. Conversely, the rate μ_{CE} towards cardio toxicity is considered to be of great value since the patient is undergoing very intensive chemotherapy. In this subpopulation there were also cancer deaths at a rate of γ_3.

Disease freedom in subpopulation D can be increased for patients with subpopulations A, B, and C. This state of freedom from disease can last forever or only for a short time. If it lasts for just a while, patients in subpopulation D can revert to subpopulations B and C, with their respective rates being μ_{DB} and μ_{DC}. Over a longer period of time, subpopulation D may also experience direct cardiotoxicity at the μ_{DE} rate. Patients who experience cardiotoxicity or who belong to subpopulation E may suffer cardiac death at a rate of γ_1. The brief descriptions of the parameters of model (7) are given in Table 1.

We convert the ODE system (7) to the new $\mathscr{CF}$ fractional model of order $\alpha \in (0, 1)$ as follows:

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha A(t) = \eth_1^* - (\mu_{AD} + \mu_{AB})A(t),$$

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha B(t) = \eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t) - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t),$$

$$^{\mathscr{CF}}\mathscr{D}_t^\alpha C(t) = \eth_3^* + \mu_{BC}B(t) + \mu_{DC}D - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t),$$

Table 1 Parameter description

Parameter	Description
$\eth_1^*$	Number of new patients diagnosed to suffer in stage 1 and 2 cancer
$\eth_2^*$	Number of new patients diagnosed to suffer in stage 3 cancer
$\eth_2^*$	Number of new patients diagnosed to suffer in stage 4 cancer
μ_{AD}	Rate of stage 1 or 2 patients who experience a complete response
μ_{AB}	Increased rate from stage 1 or 2 to stage 3 (progressive disease)
μ_{BD}	Rate of stage 3 patients who experience a complete response
μ_{BC}	Increased rate from stage 3 to stage 4 (progressive disease)
μ_{BE}	Rate of stage 3 cancer chemotherapy patients who experience cardiotoxicity
μ_{CD}	Rate of stage 4 patients who experience complete response
μ_{CE}	Rate of stage 4 cancer chemotherapy patients who experience cardiotoxicity
μ_{DB}	Rate of disease-free patients who relapse back to stage 3
μ_{DC}	Rate of disease-free patients who relapse back to stage 4
μ_{DE}	Rate of disease-free patients who experience cardiotoxic
$\hat{\xi}_1$	Death rate of cardiotoxic patients
$\hat{\xi}_2$	Death rate of stage 3 cancer patients
$\hat{\xi}_3$	Death rate of stage 4 cancer patients

$$
{}^{\mathscr{CF}}\mathscr{D}_t^\alpha D(t) = \mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t) - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t),
$$

$$
{}^{\mathscr{CF}}\mathscr{D}_t^\alpha E(t) = \mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t), \tag{8}
$$

subject to the initial conditions

$$
A(0) = A_0 \geq 0, \qquad B(0) = B_0 \geq 0, \qquad C(0) = C_0 \geq 0,
$$

$$
D(0) = D_0 \geq 0, \qquad E(0) = E_0 \geq 0.
$$

4 Solution of model by LADM

In this section, we solve system (8) using the Laplace Adomian decomposition method. LADM combines two powerful techniques: the Adomian decomposition method and the Laplace transform. First, through the Laplace transform we obtain from both sides of equations (8):

$$
\frac{\zeta \mathcal{L}\{A(t)\} - A(0)}{\zeta + \alpha(1 - \zeta)} = \mathcal{L}\big\{\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\big\},
$$

$$
\frac{\zeta \mathcal{L}\{B(t)\} - B(0)}{\zeta + \alpha(1 - \zeta)} = \mathcal{L}\big\{\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t) - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\big\},
$$

$$
\frac{\zeta \mathcal{L}\{C(t)\} - C(0)}{\zeta + \alpha(1 - \zeta)} = \mathcal{L}\big\{\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t) - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\big\},
$$

$$
\frac{\zeta \mathcal{L}\{D(t)\} - D(0)}{\zeta + \alpha(1 - \zeta)} = \mathcal{L}\big\{\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t) - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\big\},
$$

$$
\frac{\zeta \mathcal{L}\{E(t)\} - E(0)}{\zeta + \alpha(1 - \zeta)} = \mathcal{L}\big\{\mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\big\}. \tag{9}
$$

Now, by applying the inverse Laplace transform on the current system, we get:

$$
A(t) = A(0) + \mathcal{L}^{-1}\left\{\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\big\{\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\big\}\right\},
$$

$$
B(t) = B(0) + \mathcal{L}^{-1}\left\{\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\big\{\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t)\right.
$$

$$- (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\Big\},$$

$$C(t) = C(0) + \mathcal{L}^{-1}\left\{\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\big\{\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t)\right.$$

$$\left. - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\big\}\right\},$$

$$D(t) = D(0) + \mathcal{L}^{-1}\left\{\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\big\{\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t)\right.$$

$$\left. - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\big\}\right\},$$

$$E(t) = E(0) + \mathcal{L}^{-1}\left\{\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\big\{\mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\big\}\right\}. \qquad (10)$$

By decomposing the unknown functions of system (8) into the sums of infinite number of components defined by the decomposition method, we have

$$A(t) = \sum_{n=0}^{\infty} A_n(t), \qquad B(t) = \sum_{n=0}^{\infty} B_n(t), \qquad C(t) = \sum_{n=0}^{\infty} C_n(t),$$

$$D(t) = \sum_{n=0}^{\infty} D_n(t), \qquad E(t) = \sum_{n=0}^{\infty} E_n(t). \qquad (11)$$

By substituting (11) in (10), we get

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(t)\right\} = \frac{A(0)}{\zeta} + \frac{\zeta + \alpha(1 - \zeta)}{\zeta^2}\eth_1^* - (\mu_{AD} + \mu_{AB})\frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(t)\right\},$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} B_n(t)\right\} = \frac{B(0)}{\zeta} + \frac{\zeta + \alpha(1 - \zeta)}{\zeta^2}\eth_2^*$$

$$+ \frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\left\{\mu_{AB}\sum_{n=0}^{\infty} A_n(t) + \mu_{DB}\sum_{n=0}^{\infty} D_n(t)\right.$$

$$\left. - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \gamma_2)\sum_{n=0}^{\infty} B_n(t)\right\},$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} C_n(t)\right\} = \frac{C(0)}{\zeta} + \frac{\zeta + \alpha(1 - \zeta)}{\zeta^2}\eth_3^*$$

$$+ \frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\left\{\mu_{BC}\sum_{n=0}^{\infty} B_n(t) + \mu_{DC}\sum_{n=0}^{\infty} D_n(t)\right.$$

$$\left. - (\mu_{CD} + \mu_{CE} - \gamma_3)\sum_{n=0}^{\infty} C_n(t)\right\},$$

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} D_n(t)\right\} = \frac{D(0)}{\zeta} + \frac{\zeta + \alpha(1 - \zeta)}{\zeta}\mathcal{L}\left\{\mu_{AD}\sum_{n=0}^{\infty} A_n(t)\right.$$

$$+ \mu_{BD}\sum_{n=0}^{\infty} B_n(t) + \mu_{CD}\sum_{n=0}^{\infty} C_n(t)$$

$$- (\mu_{DB} + \mu_{DC} + \mu_{DE}) \sum_{n=0}^{\infty} D_n(t) \Bigg\},$$

$$\mathcal{L}\left\{ \sum_{n=0}^{\infty} E_n(t) \right\} = \frac{E(0)}{\zeta} + \frac{\zeta + \alpha(1-\zeta)}{\zeta} \mathcal{L}\Bigg\{ \mu_{DE} \sum_{n=0}^{\infty} D_n(t)$$

$$+ \mu_{CE} \sum_{n=0}^{\infty} C_n(t) + \mu_{BE} \sum_{n=0}^{\infty} B_n(t) - \gamma_1 \sum_{n=0}^{\infty} E_n(t) \Bigg\}, \tag{12}$$

$$\mathcal{L}(A_0) = \frac{A_0}{\zeta},$$

$$\mathcal{L}(A_1) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_1^*}{\zeta} - (\mu_{AD} + \mu_{AD})\mathcal{L}\{A_0\} \right],$$

$$\mathcal{L}(A_2) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_1^*}{\zeta} - (\mu_{AD} + \mu_{AD})\mathcal{L}\{A_1\} \right],$$

$$\vdots$$

$$\mathcal{L}(A_{n+1}) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_1^*}{\zeta} - (\mu_{AD} + \mu_{AD})\mathcal{L}\{A_n\} \right],$$

$$\mathcal{L}(B_0) = \frac{B_0}{\zeta},$$

$$\mathcal{L}(B_1) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_2^*}{\zeta} + \mu_{AB}\mathcal{L}\{A_0\} + \mu_{DB}\mathcal{L}\{D_0\} - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathcal{L}\{B_0\} \right],$$

$$\mathcal{L}(B_2) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_2^*}{\zeta} + \mu_{AB}\mathcal{L}\{A_1\} + \mu_{DB}\mathcal{L}\{D_1\} - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathcal{L}\{B_1\} \right],$$

$$\vdots$$

$$\mathcal{L}(B_{n+1}) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_2^*}{\zeta} + \mu_{AB}\mathcal{L}\{A_n\} + \mu_{DB}\mathcal{L}\{D_n\} \right.$$

$$\left. - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathcal{L}\{B_n\} \right],$$

$$\mathcal{L}(C_0) = \frac{C_0}{\zeta},$$

$$\mathcal{L}(C_1) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_3^*}{\zeta} + \mu_{BC}\mathcal{L}\{B_0\} + \mu_{DC}\mathcal{L}\{D_0\} - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathcal{L}\{C_0\} \right],$$

$$\mathcal{L}(C_2) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_3^*}{\zeta} + \mu_{BC}\mathcal{L}\{B_1\} + \mu_{DC}\mathcal{L}\{D_1\} - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathcal{L}\{C_1\} \right],$$

$$\vdots$$

$$\mathcal{L}(C_{n+1}) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\left[\frac{\eth_3^*}{\zeta} + \mu_{BC}\mathcal{L}\{B_n\} + \mu_{DC}\mathcal{L}\{D_n\} - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathcal{L}\{C_n\} \right],$$

$$\mathcal{L}(D_0) = \frac{D_0}{\zeta},$$

$$\mathcal{L}(D_1) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{AD}\mathcal{L}\{A_0\} + \mu_{BD}\mathcal{L}\{B_0\} + \mu_{CD}\mathcal{L}\{C_0\}$$

$$- (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathcal{L}\{D_0\} \big],$$

$$\mathcal{L}(D_2) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{AD}\mathcal{L}\{A_1\} + \mu_{BD}\mathcal{L}\{B_1\} + \mu_{CD}\mathcal{L}\{C_1\}$$
$$- (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathcal{L}\{D_1\}\big],$$

$$\vdots$$

$$\mathcal{L}(D_{n+1}) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{AD}\mathcal{L}\{A_n\} + \mu_{BD}\mathcal{L}\{B_n\} + \mu_{CD}\mathcal{L}\{C_n\}$$
$$- (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathcal{L}\{D_n\}\big],$$

$$\mathcal{L}(E_0) = \frac{E_0}{\zeta},$$

$$\mathcal{L}(E_1) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{DE}\mathcal{L}\{D_0\} + \mu_{CE}\mathcal{L}\{C_0\} + \mu_{BE}\mathcal{L}\{B_0\} - \hat{\xi}_1\mathcal{L}\{E_0\}\big],$$

$$\mathcal{L}(E_2) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{DE}\mathcal{L}\{D_1\} + \mu_{CE}\mathcal{L}\{C_1\} + \mu_{BE}\mathcal{L}\{B_1\} - \hat{\xi}_1\mathcal{L}\{E_1\}\big],$$

$$\vdots$$

$$\mathcal{L}(E_{n+1}) = \frac{\zeta + \alpha(1-\zeta)}{\zeta}\big[\mu_{DE}\mathcal{L}\{D_n\} + \mu_{CE}\mathcal{L}\{C_n\} + \mu_{BE}\mathcal{L}\{B_n\} - \hat{\xi}_1\mathcal{L}\{E_n\}\big]. \qquad (13)$$

Taking the Laplace inverse on both sides of above equations, we get

$$A_0 = A_0, \qquad B_0 = B_0, \qquad C_0 = C_0, \qquad D_0 = D_0, \qquad E_0 = E_0,$$

$$A_1 = \big[\eth_1^* - (\mu_{AD} + \mu_{AB})A_0\big]\big(1 + \alpha(t-1)\big),$$

$$B_1 = \big[\eth_2^* + \mu_{AB}A_0 + \mu_{DB}D_0 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B_0\big]\big(1 + \alpha(t-1)\big),$$

$$C_1 = \big[\eth_3^* + \mu_{BC}B_0 + \mu_{DC}D_0 - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C_0\big]\big(1 + \alpha(t-1)\big),$$

$$D_1 = \big[\mu_{AD}A_0 + \mu_{BD}B_0 + \mu_{CD}C_0 - (\mu_{DB} + \mu_{DC} + \mu_{DE})D_0\big]\big(1 + \alpha(t-1)\big),$$

$$E_1 = [\mu_{DE}D_0 + \mu_{CE}C_0 + \mu_{BE}B_0 - \hat{\xi}_1 E_0]\big(1 + \alpha(t-1)\big),$$

$$A_2 = \eth_1^*\big(1 + \alpha(t-1)\big)$$
$$- \big(\alpha^2 t^2 - 2\alpha^2 t + 2\alpha t + (\alpha-1)^2\big)\big[(\mu_{AD} + \mu_{AB})\big[\eth_1^* - (\mu_{AD} + \mu_{AB})A_0\big]\big],$$

$$B_2 = \eth_2^*\big(1 + \alpha(t-1)\big) - \big(\alpha^2 t^2 - 2\alpha^2 t + 2\alpha t + (\alpha-1)^2\big)[-\mu_{AB}\big[\eth_1 - (\mu_{AD} + \mu_{AB})A_0\big]$$
$$- \mu_{DB}\big[\mu_{AD}A_0 + \mu_{BD}B_0 + \mu_{CD}C_0 - (\mu_{DB} + \mu_{DC} + \mu_{DE})D_0\big]$$
$$+ (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\big[\eth_2^* + \mu_{AB}A_0 + \mu_{DB}D_0 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B_0\big]$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (14)$$

Similarly, we can get the rest of the terms. So, we get the solution of the model as an infinite series

$$A(t) = \sum_{n=0}^{\infty} A_n(t), \qquad B(t) = \sum_{n=0}^{\infty} B_n(t), \qquad C(t) = \sum_{n=0}^{\infty} C_n(t),$$

$$D(t) = \sum_{n=0}^{\infty} D_n(t), \qquad E(t) = \sum_{n=0}^{\infty} E_n(t). \qquad (15)$$

5 Stability analysis

Special solution by using $\mathcal{ST}$

In this part, we use the Sumudu transform for system (8):

$$\mathcal{ST}\left(^{\mathscr{CF}}\mathscr{D}_t^\alpha A(t)\right) = \mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\right],$$

$$\mathcal{ST}\left(^{\mathscr{CF}}\mathscr{D}_t^\alpha B(t)\right) = \mathcal{ST}\left[\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t) - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\right],$$

$$\mathcal{ST}\left(^{\mathscr{CF}}\mathscr{D}_t^\alpha C(t)\right) = \mathcal{ST}\left[\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t) - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\right],$$

$$\mathcal{ST}\left(^{\mathscr{CF}}\mathscr{D}_t^\alpha D(t)\right) = \mathcal{ST}\left[\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t) - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\right],$$

$$\mathcal{ST}\left(^{\mathscr{CF}}\mathscr{D}_t^\alpha E(t)\right) = \mathcal{ST}\left[\mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\right]. \tag{16}$$

By using Definition 2.4, we get

$$\mathscr{M}(\alpha)\frac{\mathcal{ST}[A(t)] - A(0)}{1 + \alpha(\zeta - 1)} = \mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\right],$$

$$\mathscr{M}(\alpha)\frac{\mathcal{ST}[B(t)] - B(0)}{1 + \alpha(\zeta - 1)} = \mathcal{ST}\left[\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t) - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\right],$$

$$\mathscr{M}(\alpha)\frac{\mathcal{ST}[C(t)] - C(0)}{1 + \alpha(\zeta - 1)} = \mathcal{ST}\left[\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t) - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\right],$$

$$\mathscr{M}(\alpha)\frac{\mathcal{ST}[D(t)] - D(0)}{1 + \alpha(\zeta - 1)}$$

$$= \mathcal{ST}\left[\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t) - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\right],$$

$$\mathscr{M}(\alpha)\frac{\mathcal{ST}[E(t)] - E(0)}{1 + \alpha(\zeta - 1)} = \mathcal{ST}\left[\mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\right]. \tag{17}$$

We obtain

$$\mathcal{ST}\left[A(t)\right] = A(0) + \frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\right],$$

$$\mathcal{ST}\left[B(t)\right] = B(0) + \frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t)\right.$$

$$\left. - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\right],$$

$$\mathcal{ST}\left[C(t)\right] = C(0) + \frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t)\right.$$

$$\left. - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\right],$$

$$\mathcal{ST}\left[D(t)\right] = D(0) + \frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t)\right.$$

$$\left. - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\right],$$

$$\mathcal{ST}\left[E(t)\right] = E(0) + \frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\mu_{DE}D(t) + \mu_{CE}C(t) + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\right]. \tag{18}$$

Taking the inverse $\mathcal{ST}$ on the above system, we obtain the following recursive formula:

$$A_{n+1}(t) = A_n(0) + \mathcal{ST}^{-1}\left\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\right]\right\},$$

$$B_{n+1}(t) = B_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t)\right.\right.$$

$$\left.\left. - (\mu_{BD} + \mu_{BC} + \mu_{BE})B(t) - \hat{\xi}_2 B(t)\right]\right\},$$

$$C_{n+1}(t) = C_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t)\right.\right.$$

$$\left.\left. - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\right]\right\},$$

$$D_{n+1}(t) = D_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t)\right.\right.$$

$$\left.\left. - (\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\right]\right\},$$

$$E_{n+1}(t) = E_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\mu_{DE}D(t) + \mu_{CE}C(t)\right.\right.$$

$$\left.\left. + \mu_{BE}B(t) - \hat{\xi}_1 E(t)\right]\right\}. \tag{19}$$

The approximate solution of the above system is

$$A(t) = \lim_{n\to\infty} A_n(t), \qquad B(t) = \lim_{n\to\infty} B_n(t), \qquad C(t) = \lim_{n\to\infty} C_n(t),$$

$$D(t) = \lim_{n\to\infty} D_n(t), \qquad E(t) = \lim_{n\to\infty} E_n(t). \tag{20}$$

Stability analysis of iteration method

Consider the Banach space $(\mathcal{X}, \|\cdot\|)$, a self-map $\mathscr{P}$ on $\mathcal{X}$, and the recursive method $\mathcal{T}_{n+1} = \Psi(\mathscr{P}, \mathcal{T}_n)$. Denote $\mathscr{M}(\mathscr{P}) \neq \emptyset$ to be a fixed point set of $\mathscr{P}$ and $\lim_{n\to\infty}\mathcal{T}_n = t \in \mathscr{M}(\mathscr{P})$. Let $\{x_n^*\} \subseteq \mathcal{X}$ and define $e_n = \|x_{n+1}^* - \Psi(\mathscr{P}, x_n^*)\|$. If $\lim_{n\to\infty} e_n = 0$ implies that $\lim_{n\to\infty} x_n^* = t$, then the iterative method $\mathcal{T}_{n+1} = \mathcal{T}_{n+1} = \Psi(\mathscr{P}, \mathcal{T}_n)$ is said to be $\mathscr{P}$-stable. Suppose that the sequence $\{x_n^*\}$ is bounded. If Picard iteration $\mathcal{T}_{n+1} = \mathscr{P}\mathcal{T}_n$ is satisfied in all conditions, then $\mathcal{T}_{n+1} = \mathscr{P}\mathcal{T}_n$ is $\mathscr{P}$-stable.

In this order, we need the following.

Theorem 5.1 [23] *Let $\mathscr{P}$ be a self-map on $\mathcal{X}$ such that*

$$\|\mathscr{P}_x - \mathscr{P}_y\| \leq \mathscr{H}_1\|x - \mathscr{P}_x\| + \mathscr{H}_2\|x - y\|$$

for all $x, y \in \mathcal{X}$, where $\mathscr{H}_1 \geq 0$, $\mathscr{H}_2 \in [0, 1)$.

Theorem 5.2 *Let $\mathscr{P}$ be given as follows*:

$$\mathscr{P}[A_n(t)] = A_{n+1}(t) = A_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A(t)\right]\right\},$$

$$\mathscr{P}[B_n(t)] = B_{n+1}(t)$$

$$= B_n(0) + \mathcal{ST}^{-1}\left\{\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\mathcal{ST}\left[\eth_2^* + \mu_{AB}A(t) + \mu_{DB}D(t)\right.\right.$$

$$-(\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B(t)\Big]\bigg\},$$

$$\mathscr{P}\big[C_n(t)\big] = C_{n+1}(t)$$

$$= C_n(0) + \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\eth_3^* + \mu_{BC}B(t) + \mu_{DC}D(t)$$

$$-(\mu_{CD} + \mu_{CE} + \hat{\xi}_3)C(t)\big]\bigg\},$$

$$\mathscr{P}\big[D_n(t)\big] = D_{n+1}(t)$$

$$= D_n(0) + \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\mu_{AD}A(t) + \mu_{BD}B(t) + \mu_{CD}C(t)$$

$$-(\mu_{DB} + \mu_{DC} + \mu_{DE})D(t)\big]\bigg\},$$

$$\mathscr{P}\big[E_n(t)\big] = E_{n+1}(t)$$

$$= E_n(0) + \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\mu_{DE}D(t) + \mu_{CE}C(t)$$

$$+ \mu_{BE}B(t) - \hat{\xi}_1 E(t)\big]\bigg\}. \tag{21}$$

Then $\mathscr{P}$ is $\mathscr{P}$-stable if the following conditions hold:

$$\begin{cases} 1 - (\mu_{AD} + \mu_{AB})\mathscr{F}_1(\nu) < 1, \\ 1 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathscr{F}_2(\nu) + \mu_{AB}\mathscr{F}_3(\nu) + \mu_{DB}\mathscr{F}_4(\nu)) < 1, \\ 1 - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathscr{F}_5(\nu) + \mu_{BC}\mathscr{F}_6(\nu) + \mu_{DC}\mathscr{F}_7(\nu)) < 1, \\ 1 - (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathscr{F}_8(\nu) + \mu_{AD}\mathscr{F}_9(\nu) + \mu_{BD}\mathscr{F}_{10}(\nu) + \mu_{DE}\mathscr{F}_{11}(\nu)) < 1, \\ 1 - \hat{\xi}_1\mathscr{F}_{12}(\nu) + \mu_{DE}\mathscr{F}_{13}(\nu) + \mu_{CE}\mathscr{F}_{14}(\nu) + \mu_{BE}\mathscr{F}_{15}(\nu)) < 1. \end{cases} \tag{22}$$

Proof Computing the following iterations for all $(r, s) \in N$, we prove that $\mathscr{P}$ has a fixed point.

$$\mathscr{P}\big[A_r(t)\big] - \mathscr{P}\big[A_s(t)\big]$$

$$= A_r(t) - A_s(t) + \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\eth_1^* - (\mu_{AD} + \mu_{AB})A_r(t)\big]\bigg\}$$

$$- \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\eth_1^* - (\mu_{AD} + \mu_{AB})A_s(t)\big]\bigg\},$$

$$\mathscr{P}\big[B_r(t)\big] - \mathscr{P}\big[B_s(t)\big]$$

$$= B_r(t) - B_s(t) + \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\eth_2^* + \mu_{AB}A_r(t) + \mu_{DB}D_r(t)$$

$$-(\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B_r(t)\big]\bigg\} - \mathcal{ST}^{-1}\bigg\{\frac{1 + \alpha(\zeta - 1)}{\mathscr{M}(\alpha)}\mathcal{ST}\big[\eth_2^* + \mu_{AB}A_s(t)$$

$$+ \mu_{DB}D_s(t) - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)B_s(t)\big]\bigg\},$$

$$\mathscr{P}\big[C_r(t)\big] - \mathscr{P}\big[C_s(t)\big]$$

$$= C_r(t) - C_s(t) + \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\eth_3^* + \mu_{BC}B_r(t) + \mu_{DC}D_r(t) - (\mu_{CD}\right.\right.$$

$$\left.\left. + \mu_{CE} + \hat{\xi}_3)C_r(t)\right]\right\} - \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\eth_3^* + \mu_{BC}B_s(t) + \mu_{DC}D_s(t)\right.\right.$$

$$\left.\left. - (\mu_{CD} + \mu_{CD} + \mu_{CE} + \hat{\xi}_3)C_s(t)\right]\right\},$$

$$\mathcal{P}\big[D_r(t)\big] - \mathcal{P}\big[D_s(t)\big]$$

$$= D_r(t) - D_s(t) + \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\mu_{AD}A_r(t) + \mu_{BD}B_r(t) + \mu_{CD}C_r(t)\right.\right.$$

$$\left.\left. - (\mu_{DB} + \mu_{DC} + \mu_{DE})D_r(t)\right]\right\} - \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\mu_{AD}A_s(t)\right.\right.$$

$$\left.\left. + \mu_{BD}B_s(t) + \mu_{CD}C_s(t) - (\mu_{DB} + \mu_{DC} + \mu_{DE})D_s(t)\right]\right\},$$

$$\mathcal{P}\big[E_r(t)\big] - \mathcal{P}\big[E_s(t)\big]$$

$$= E_r(t) - E_s(t) + \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\mu_{DE}D_r(t) + \mu_{CE}C_r(t)\right.\right.$$

$$\left.\left. + \mu_{BE}B_r(t) - \hat{\xi}_1 E_r(t)\right]\right\}$$

$$- \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\mu_{DE}D_s(t) + \mu_{CE}C_s(t) + \mu_{BE}B_s(t) - \hat{\xi}_1 E_s(t)\right]\right\}. \tag{23}$$

By taking the norm and applying the triangular inequality on both sides of the above equations, we get

$$\big\| \mathcal{P}\big[A_r(t)\big] - \mathcal{P}\big[A_s(t)\big]\big\|$$

$$\leq \big\| A_r(t) - A_s(t)\big\| + \left\| \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A_r(t)\right]\right\}\right.$$

$$\left. - \mathcal{ST}^{-1}\left\{ \frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)} \mathcal{ST}\left[\eth_1^* - (\mu_{AD} + \mu_{AB})A_s(t)\right]\right\}\right\|$$

$$\leq \big\| A_r(t) - A_s(t)\big\| + \mathcal{ST}^{-1}\left[\mathcal{ST}\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\big[\big\| -(\mu_{AD} + \mu_{AB})\big(A_r(t) - A_s(t)\big)\big\|\big]\right],$$

$$\big\| \mathcal{P}\big[B_r(t)\big] - \mathcal{P}\big[B_s(t)\big]\big\|$$

$$= \big\| B_r(t) - B_s(t)\big\| + \mathcal{ST}^{-1}\left[\mathcal{ST}\frac{1+\alpha(\zeta-1)}{\mathcal{M}(\alpha)}\big[\big\| \mu_{AB}\big(A_r(t) - A_s(t)\big)\big\|\right.$$

$$\left. + \big\| \mu_{DB}\big(D_r(t) - D_s(t)\big)\big\| + \big\| -(\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\big(B_r(t) - B_s(t)\big)\big\|\big]\right]$$

$$\vdots \tag{24}$$

Because the solutions have the same behavior, we have

$$\big\| A_r(t) - A_s(t)\big\| \cong \big\| B_r(t) - B_s(t)\big\| \cong \big\| C_r(t) - C_s(t)\big\|$$

$$\cong \big\| D_r(t) - D_s(t)\big\| \cong \big\| E_r(t) - E_s(t)\big\|. \tag{25}$$

Thus we have

$$\big\| \mathscr{P}\big[A_r(t)\big] - \mathscr{P}\big[A_s(t)\big] \big\| \leq \big[1 - (\mu_{AD} + \mu_{AB})\mathscr{F}_1(v)\big] \big\| A_r(t) - A_s(t) \big\|,$$

$$\big\| \mathscr{P}\big[B_r(t)\big] - \mathscr{P}\big[B_s(t)\big] \big\|$$
$$\leq \big[1 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathscr{F}_2(v) + \mu_{AB}\mathscr{F}_3(v) + \mu_{DB}\mathscr{F}_4(v))\big] \times \big\| B_r(t) - B_s(t) \big\|,$$

$$\big\| \mathscr{P}\big[C_r(t)\big] - \mathscr{P}\big[C_s(t)\big] \big\|$$
$$\leq \big[1 - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathscr{F}_5(v) + \mu_{BC}\mathscr{F}_6(v) + \mu_{DC}\mathscr{F}_7(v))\big] \times \big\| C_r(t) - C_s(t) \big\|,$$

$$\big\| \mathscr{P}\big[D_r(t)\big] - \mathscr{P}\big[D_s(t)\big] \big\|$$
$$\leq \big[1 - (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathscr{F}_8(v) + \mu_{AD}\mathscr{F}_9(v) + \mu_{BD}\mathscr{F}_{10}(v) + \mu_{DE}\mathscr{F}_{11}(v))\big]$$
$$\times \big\| D_r(t) - D_s(t) \big\|,$$

$$\big\| \mathscr{P}\big[E_r(t)\big] - \mathscr{P}\big[E_s(t)\big] \big\|$$
$$\leq \big[1 - \hat{\xi}_1\mathscr{F}_{12}(v) + \mu_{DE}\mathscr{F}_{13}(v) + \mu_{CE}\mathscr{F}_{14}(v) + \mu_{BE}\mathscr{F}_{15}(v))\big] \times \big\| E_r(t) - E_s(t) \big\|, \quad (26)$$

where $\mathscr{F}_i$ for $i = 1, \ldots, 15$ are functions from $\mathcal{ST}^{-1}[\mathcal{ST}\frac{1+\alpha(\zeta-1)}{\mathscr{M}(\alpha)}]$ and

$$1 - (\mu_{AD} + \mu_{AB})\mathscr{F}_1(v) < 1,$$
$$1 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathscr{F}_2(v) + \mu_{AB}\mathscr{F}_3(v) + \mu_{DB}\mathscr{F}_4(v)) < 1,$$
$$1 - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathscr{F}_5(v) + \mu_{BC}\mathscr{F}_6(v) + \mu_{DC}\mathscr{F}_7(v)) < 1,$$
$$1 - (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathscr{F}_8(v) + \mu_{AD}\mathscr{F}_9(v) + \mu_{BD}\mathscr{F}_{10}(v) + \mu_{DE}\mathscr{F}_{11}(v)) < 1,$$
$$1 - \hat{\xi}_1\mathscr{F}_{12}(v) + \mu_{DE}\mathscr{F}_{13}(v) + \mu_{CE}\mathscr{F}_{14}(v) + \mu_{BE}\mathscr{F}_{15}(v)) < 1. \quad (27)$$

Then the self-map $\mathscr{P}$ has a fixed point. Now, we prove that $\mathscr{P}$ satisfies the assumptions of Theorem 5.2. Let (27) hold, then $\mathscr{H}_2 = (0, 0, 0, 0, 0)$ and

$$\mathscr{H}_1 = \begin{cases} 1 - (\mu_{AD} + \mu_{AB})\mathscr{F}_1(v) < 1, \\ 1 - (\mu_{BD} + \mu_{BC} + \mu_{BE} + \hat{\xi}_2)\mathscr{F}_2(v) + \mu_{AB}\mathscr{F}_3(v) + \mu_{DB}\mathscr{F}_4(v)) < 1, \\ 1 - (\mu_{CD} + \mu_{CE} + \hat{\xi}_3)\mathscr{F}_5(v) + \mu_{BC}\mathscr{F}_6(v) + \mu_{DC}\mathscr{F}_7(v)) < 1, \\ 1 - (\mu_{DB} + \mu_{DC} + \mu_{DE})\mathscr{F}_8(v) + \mu_{AD}\mathscr{F}_9(v) \\ \quad + \mu_{BD}\mathscr{F}_{10}(v) + \mu_{DE}\mathscr{F}_{11}(v)) < 1, \\ 1 - \hat{\xi}_1\mathscr{F}_{12}(v) + \mu_{DE}\mathscr{F}_{13}(v) + \mu_{CE}\mathscr{F}_{14}(v) + \mu_{BE}\mathscr{F}_{15}(v)) < 1. \end{cases} \quad (28)$$
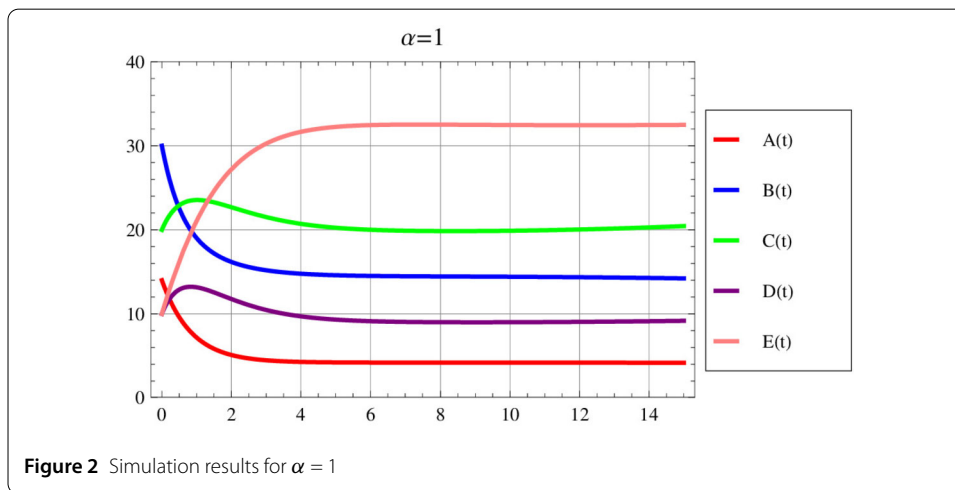
Thus, all the conditions hold for $\mathscr{P}$. Therefore, the proof is complete. □

6 Simulation

In this section, we utilize the procedure introduced in Sect. 4 for solving the fractional system (8). For having a comparison with some other existing method for solving the breast cancer model, we applied the parameter and initial values of [22], which are given in Table 2. All of the results were calculated using the same desktop, Asus DESKTOP-M0F5LBS, Intel(R)Core(TM)i7-6700HQ CPU@ 2.60 GHz, 16 GB memory.
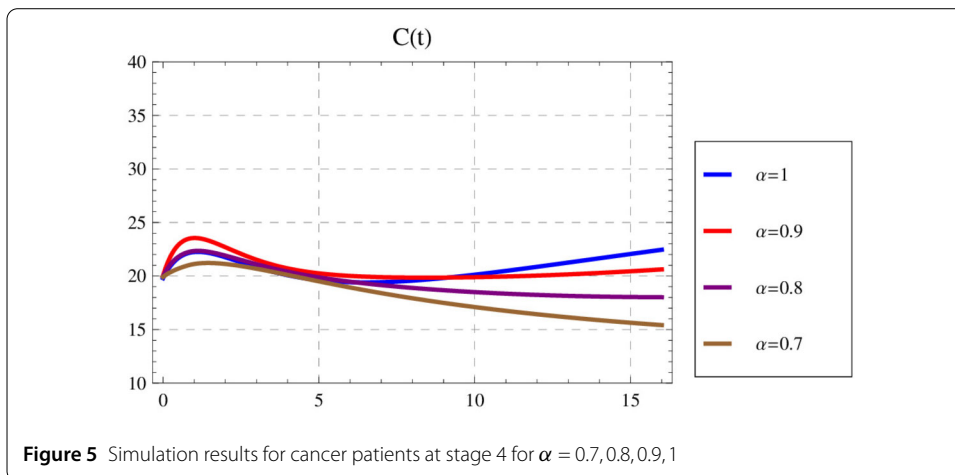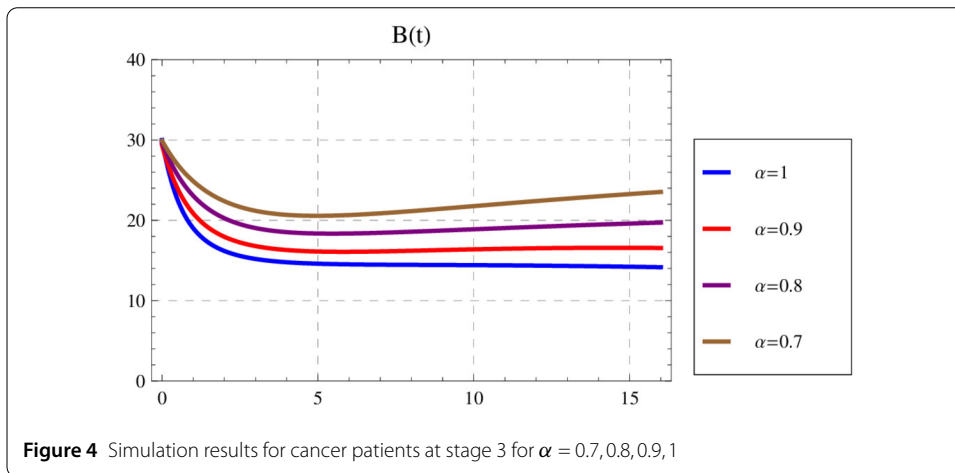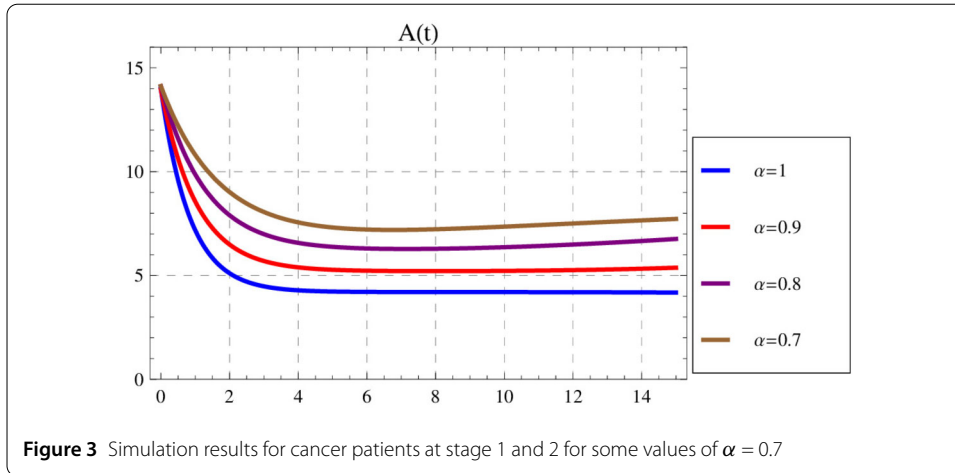
Table 2 Parameter and initial values

Parameter	Value	Parameter	Value
$\eth_1^*$	5	μ_{CD}	0.10
$\eth_2^*$	20	μ_{CE}	0.30
$\eth_2^*$	11	μ_{DB}	0.36
μ_{AD}	0.63	μ_{DC}	0.42
μ_{AB}	0.56	μ_{DE}	0.30
μ_{BD}	0.35	$\hat{\xi}_1$	0.4
μ_{BC}	0.62	$\hat{\xi}_2$	0.5
μ_{BE}	0.30	$\hat{\xi}_3$	0.8
A_0	14	B_0	30
C_0	20	D_0	10
E_0	10		

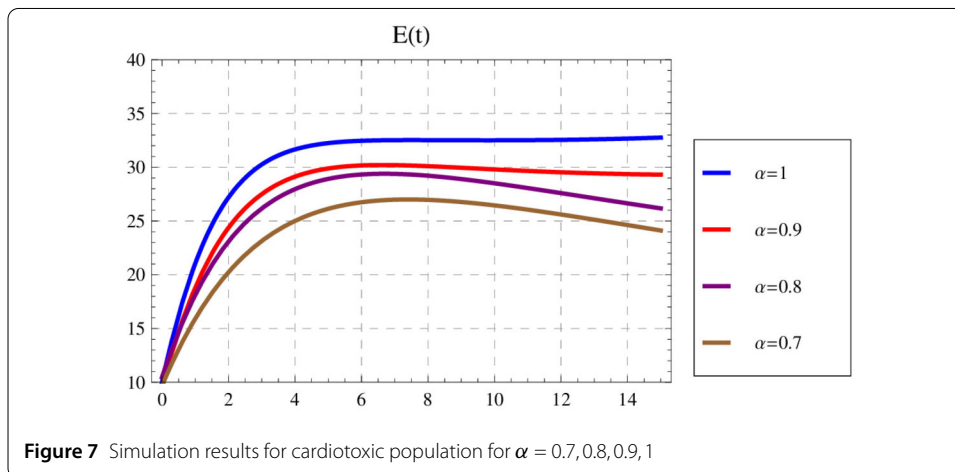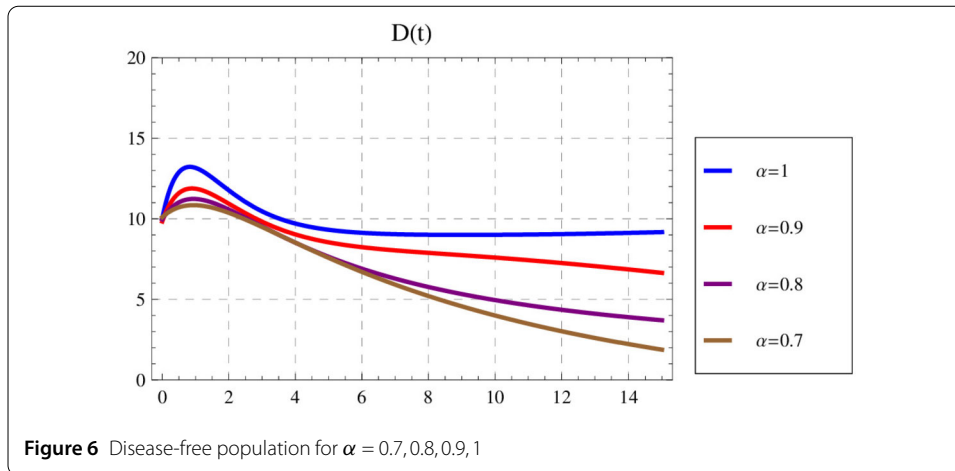**Figure 2** Simulation results for $\alpha = 1$

It is clear that it is impossible to obtain the infinite components of the unknown series using Adom's method, so we truncate the series for $n = 15$. To show the efficiency of the purposive approach, the fractional model (8) was first solved for $\alpha = 1$, and the plots of the numerical solutions are shown in Fig. 2, where the plots are in good agreement with the results of [22].

For studying the impact of the fractional order on the approximated state functions of model (8), we examined some values for α, and the numerical results are reported in Figs. 2–6 in a 15 days period. All the plots in Fig. 2 have decreasing behavior and after some days possess a stable position. As we can see, for the highest fractional order, $\alpha = 1$, the patients in stages 1, 2A, and 2B have reached the minimum number about 4 at the fourth day and showed a stable behavior after 4 days. On the other hand, for the smallest fractional order $\alpha = 0.7$, the population of class A has taken the minimum value about 7 in the sixth day of time period. So, we can conclude that the smoothness of the solution plots has the direct relation with their fractional orders.

The plots of Fig. 4 depict the number of active cases of the breast cancer patients in the class B with initial value 30 after 15 days. Similar to the state function $A(t)$, the relevant plots of $B(t)$ are decreasing, and there is an inverse relation between the magnitude of *alpha* and the number of people in this class. As we can see in this figure, for *alpha* $= 0.9, 1$, the solutions have stable behavior after 4 days.

Figure 3 Simulation results for cancer patients at stage 1 and 2 for some values of $\alpha = 0.7$

Figure 4 Simulation results for cancer patients at stage 3 for $\alpha = 0.7, 0.8, 0.9, 1$

Figure 5 Simulation results for cancer patients at stage 4 for $\alpha = 0.7, 0.8, 0.9, 1$

In Fig. 5, the plots of the approximated solutions for the patients in stage 4 show that the number of people in this class, first, is increasing for $t \in (0, 2)$ and then decreasing until fifteenth day for $\alpha = 0.7, 0.8, 0.9$. For $\alpha = 0.7$, the active cases of this class have reached their minimum value as 15, where the maximum value has been reached for $\alpha = 0.9$, after about

Figure 6 Disease-free population for $\alpha = 0.7, 0.8, 0.9, 1$

Figure 7 Simulation results for cardiotoxic population for $\alpha = 0.7, 0.8, 0.9, 1$

one day. However, this intermittent behavior is justified because the patients of class C are transferred to classes B and D. Consequently, we expect similar behavior for the plots of $D(t)$.

Similar to Fig. 5, in Fig. 6 the cancer patients with disease-free conditions after chemotherapy, first, have reached the maximum number at $(0, 1)$ and then decreased until fifteenth day for $\alpha = 0.7, 0.8, 0.9, 1$. For $\alpha = 0.7$, the active cases of this class have decayed on $(1, 15)$, where the maximum value has been reached for $\alpha = 1$, after about one day.

The plots of Fig. 7 depict the number of the cancer patients who have cardiotoxicity in 15 days, which are achieved by solving the fractional model (8) for $\alpha = 0.7, 0.8, 0.9, 1$. The initial number of this population is 10. All of the plots of this class are increasing and after some time possess stable behavior. It is obviously evident that the there is a direct relation between the fractional order and the number of patients of $E(t)$. Also, the related plots for $\alpha = 0.7, 0.8$ are less smooth than those for $\alpha = 0.9, 1$.

7 Conclusion

In this study, a mathematical model was constructed with five variables and sixteen parameters. Mathematical models were made according to medical phenomena about the chemo cardio toxicity of breast cancer patients. The system was equipped with fractional derivatives in the $\mathscr{CF}$ sense.

The model consists of three subpopulations of breast cancer patients by stage, one disease-free subpopulation, and one cardiotoxic subpopulation. A dynamical analysis is carried out to determine the number of individual sufferers in each subpopulation at any time. The investigated results show a stable equilibrium point. Numerical simulations are made to verify the behavior of solutions around the equilibrium point. Also, stability analysis was performed via the Sumudo transform, and LADM was utilized for obtaining the semi-analytical solutions of the purposed system. LADM has no application restrictions because the perturbation parameter is not required.

Based on the results of simulations, it can be concluded that if all parameters of the model are constant, the state of the population will be stable at a particular time with any initial conditions. This shows that the equilibrium point of the system proved to be stable without conditions. By reducing the relapse rate, an unexpected result is obtained, which is an increase in cardiotoxic subpopulations. Better results are obtained when reducing cardiotoxic rates. Under these conditions, the number of disease-free subpopulations increases, and the number of cardiotoxic subpopulations decreases dramatically.

Also, we analyze the effect of the fractional order on the numerical results by varying the fractional order. As we can see in all diagrams, the subpopulation in class A decreases, while the fractional order increases. For larger values of fractional orders, the rate of decay in the patients of stages 1 and 2 is faster than the same rates for smaller orders. On the contrary, the decreasing rate of other subpopulations is faster for small orders of the fractional order. In the upcoming research works, we will try to model some other cancer diseases by $\mathscr{CF}$ derivative and solve the obtained models by some analytical or numerical approaches. Further research along these lines is under progress and will be reported in due time.

Data availability
Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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