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Random exponential attractor for a stochastic reaction-diffusion equation in $L^{2p}(D)$

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Abstract

In this paper, we establish some sufficient conditions for the existence of a random exponential attractor for a random dynamical system in a Banach space. As an application, we consider a stochastic reaction-diffusion equation with multiplicative noise. We show that the random dynamical system $\phi(t,\omega)$ generated by this stochastic reaction-diffusion equation is uniformly Fréchet differentiable on a positively invariant random set in $L^{2p}(D)$ and satisfies the conditions of the abstract result, then we obtain the existence of a random exponential attractor in $L^{2p}(D)$, where p is the growth of the nonlinearity satisfying $1 < p \leq 3$.

Keywords: Random exponential attractor; Random dynamical system; Stochastic reaction-diffusion equation; Multiplicative noise

1 Introduction

As we know, the random attractor plays a key role in the study of asymptotic behavior of a random dynamical system (RDS). Both the existence and the estimates of Hausdorff and fractal dimensions of random attractors have been studied intensively since Crauel and Flandoli 1994 [1], see, e.g., [2–16] and the references therein. However, a random attractor is possibly infinite dimensional and sometimes attracts orbits at a slow rate, making it unobservable in practical experiments and numerical simulations. The concept of random exponential attractor was introduced by Shrikyan and Zelik in [17]. By definition, a random exponential attractor is a positively invariant finite dimensional set that contains the random attractor and possesses the exponential attraction property.

In [17], Shrikyan and Zelik presented some sufficient conditions for the existence and robustness of random exponential attractors for dissipative RDS. As pointed out there, the main difficulty in constructing a random exponential attractor, in contrast to the deterministic case, is that a typical trajectory of an RDS is unbounded in time. Therefore, some restrictive assumptions were imposed on the global Lipschitz continuity of all nonlinear terms to guarantee the time average of these quantities can be controlled. But the conditions are not easy to verify for some stochastic PDEs. Recently, Zhou [18, 19] established a new criterion for the existence of a random exponential attractor for non-autonomous

RDS. Their conditions are limited to checking the boundedness of some random variables in the mean. Then they applied the abstract result to a non-autonomous stochastic reaction-diffusion equation in $\mathbb{R}^3$ and the first-order stochastic lattice system. However, in concrete applications, the assumptions rely on the orthogonal projections, so they cannot be directly applied to RDS defined in Banach spaces.

In this article, we mainly consider the existence of a random exponential attractor in Banach space. Motivated by [18–21], we show that if the cocycle $\phi(T,\omega)$ is C^1 (in the topology of a Banach space X) on a positively invariant random set $\chi(\omega)$ for a.s. $\omega \in \Omega$ and some large enough time T (independent of ω), and the Fréchet derivative $D_v(\phi(T,\omega))$ at every point inside $\chi(\omega)$ can be split into a compact operator and a contraction (in the mean sense), then we can construct a random exponential attractor for the discrete cocyle $\phi(nT,\theta_{mT}\omega)$ in the Banach space X. Following a similar process as presented in [19], we can get a random exponential attractor for the continuous cocycle $\phi(t,\omega)$.

As an application of the theory developed in the paper, the following problem in a bounded domain $D \subset \mathbb{R}^3$ with smooth boundary ∂D is considered

$$du - \big(\Delta u - |u|^{p-1}u - f(x,u)\big)\,dt = g(x)\,dt + bu \circ dW(t), \quad x \in D, t > 0, \tag{1.1}$$

with the initial-boundary value conditions

$$\begin{cases} u(x,0) = u_0(x), & x \in D, \\ u = 0, & \text{on } \partial D, \end{cases} \tag{1.2}$$

where b is a positive constant, the term $u \circ dW(t)$ in (1.1) is understood in the sense of Stratonovich interation and $W(t)$ is a two-sided real-valued Wiener process on a probability space specified in Sect. 3. The nonlinearity $f \in C^2$ satisfies the following conditions:

$$c_1|u|^{q-2}u - c_0 \le f'(x,u) \le c_2|u|^{q-2}u + c_0, \quad 1 \le q < p; \tag{1.3}$$

$$f(x,u)u \ge \nu|u|^{q+1} - \beta(x); \tag{1.4}$$

$$\big|f''(x,u)\big| \le c_3\big(1 + |u|^{q-2}\big), \tag{1.5}$$

for some $1 < p \le 3$, $c_i > 0$ $(i = 0,1,2,3)$, $\nu > 0$ and for all $u \in \mathbb{R}$. We also assume that $\beta \in L^{3p}(D)$ and $g \in L^{6p}(D)$. A typical function in applications is $f(x,u) = a|u|^{q-1}u + h(x)$ with $a > 0$ and $h(x) \in L^{\frac{3p}{2}}(D)$.

The above equation (1.1) is known as a reaction-diffusion equation [22] perturbed by a white noise $g(x)\,dt + bu \circ dW(t)$. In biology and physics, stochastic equations like (1.1) have been used as models to study the phenomena of stochastic resonance [23–28], where g is an input signal and $W(t)$ is a Wiener process used to test the impact of stochastic fluctuations on g. We choose the equation (1.1) since the long-term behavior of solutions for equations like (1.1) has been studied widely for both deterministic and stochastic cases. They are canonical examples to study the existence of global attractors and random attractors. In this respect, we refer the readers to [1, 4, 8–11, 17, 20, 22, 29–32], among others. Until now, as we know, there is no result concerning the existence of random exponential attractors in Banach space for (1.1). We extend the technique presented in [20, 21] to stochastic case to get the Fréchet differentiability in the Banach space $L^{2p}(D)$, and this is nontrival, since the trajectory of an RDS is unbounded in time. Fortunately, some random variables can be controlled in the mean for a large time T in a certain absorbing set and this is sufficient to construct a random exponential attractor in $L^{2p}(D)$.

Our main tasks in this paper include: (1) Give an abstract result for the existence of a random exponential attractor in general Banach space. (2) Establish the RDS $\phi(t,\omega)$ generated by equation (1.1)–(1.2) and construct the absorbing set $\chi(\omega)$. (3) Prove that the RDS generated by (1.1)–(1.5) is uniformly Fréchet differentiable in the topology of $L^{2p}(D)$. (4) Check the assumptions in the abstract result presented in Sect. 2 for $\phi(t,\omega)$ and prove that $\phi(t,\omega)$ possesses a random exponential attractor in $L^{2p}(D)$. Our main result in this paper is as follows:

Theorem 1.1 *Suppose* (1.3)∼(1.5) *hold. Then the RDS generated by* (1.1)∼(1.2) *possesses a random exponential attractor* $\{\mathcal{E}(\omega)\}_{\omega\in\Omega}$ *in* $L^{2p}(D)$.

This paper is organized as follows. In Sect. 2, we recall some basic concepts and present our main result for the existence of a random exponential attractor in a Banach space. In Sect. 3, we first prove that the RDS is C^1 on a positively invariant absorbing set in $L^{2p}(D)$, then apply the abstract result in Sect. 2 to show that the RDS possesses a random exponential attractor in $L^{2p}(D)$.

Throughout this paper, we denote by $\|\cdot\|_X$ the norm of Banach space X. The inner product and norm of $L^2(D)$ are written as $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. We also use $\|u\|_r$ to denote the norm of $u\in L^r(D)$ $(r\geq 1, r\neq 2)$ and $|u|$ to denote the modular of u. The letters c and $c_i (i=1,2,\dots)$ are generic positive constants and the constant c may change their values from line to line even in the same line.

2 Preliminaries and abstract results

We first recall some basic concepts and results related to random exponential attractors and then establish a result for the existence of a random exponential attractor in Banach space.

Definition 2.1 Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system (MDS) if $\theta_t\colon\mathbb{R}\times\Omega\to\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t}=\theta_s\circ\theta_t$ for all $s,t\in\mathbb{R}$ and $\theta_t\mathbb{P}=\mathbb{P}$ for all $t\in\mathbb{R}$.

Definition 2.2 The RDS on X over an MDS $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is a mapping $\phi:\mathbb{R}^+\times\Omega\times X\to X,(t,\omega,x)\mapsto\phi(t,\omega,x)$, which is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable and satisfies for $\mathbb{P}$-a.s. $\omega\in\Omega$,

(i) $\phi(0,\omega,\cdot)$ is the identity on X;

(ii) $\phi(t+s,\omega,\cdot)=\phi(t,\theta_s\omega,\cdot)\circ\phi(s,\omega,\cdot)$ (cocycle property) on X for all $s,t\in\mathbb{R}^+$.

An RDS is said to be continuous on X if $\phi(t,\omega):X\to X$ is continuous for all $t\in\mathbb{R}^+$ and $\mathbb{P}$-a.s. $\omega\in\Omega$.

Definition 2.3 (1) A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for $\mathbb{P}$-a.s. $\omega\in\Omega$,

$$\lim_{t\to\infty}e^{-\beta t}d\big(B(\theta_{-t}\omega)\big)=0\quad\text{for all }\beta>0,$$

where $d(B)=\sup_{x\in B}\|x\|_X$.

(2) A random variable $r(\omega) \geq 0$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} r(\theta_{-t}\omega) = 0 \quad \text{for all } \beta > 0.$$

In the following, we denote $\mathcal{D}_X$ and $\mathcal{D}_r$ the collections of all tempered family of nonempty subsets of X and $L^r(D)$ respectively with respect to $(\theta_t)_{t\in\mathbb{R}}$.

Definition 2.4 A family $\mathcal{E}(\omega)$ of subsets of X is called a random exponential attractor in $\mathcal{D}_X$ for a continuous RDS $\phi(t,\omega)$ over an MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if $\mathcal{E}(\omega)$ is measurable in ω and there is a set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that for any $\omega \in \tilde{\Omega}$, it holds that

(i) Compactness: $\mathcal{E}(\omega)$ is compact in X;

(ii) Positive invariance: $\phi(t,\theta_{-t}\omega)\mathcal{E}(\theta_{-t}\omega) \subset \mathcal{E}(\omega)$ for all $t \geq 0$;

(iii) Finite-dimensionality: There exists a random variable ζ_ω $(< +\infty)$ such that $\dim_f \mathcal{E}(\omega) \leq \zeta_\omega$, where $\dim_f \mathcal{E}(\omega)$ is the fractal dimension of $\mathcal{E}(\omega)$, defined by $\dim_f \mathcal{E}(\omega) = \limsup_{\varepsilon\to 0^+} \frac{\ln N_\varepsilon(\mathcal{E}(\omega))}{\ln \frac{1}{\varepsilon}}$ and $N_\varepsilon(A)$ denotes the minimal numbers of balls with radius ε covering A in X;

(iv) Exponential attraction: There exist $a > 0$ (independent of ω), $t_{\omega,B} \geq 0$ and $b_{\omega,B} > 0$ such that, for any $B \in \mathcal{D}_X$,

$$d_h\big(\phi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), \mathcal{E}(\omega)\big) \leq b_{\omega,B} e^{-at}, \quad t \geq t_{\omega,B},$$

where $d_h(F_1, F_2)$ denotes the Hausdorff semidistance between F_1 and F_2.

Remark Here we have borrowed the definition of a random exponential attractor from [18, 19]. Note that we do not mention the Hölder continuity condition in [17], since the compactness, positive invariance, finite-dimensionality and exponential attraction are intrinsic qualities for the concept of a random exponential attractor.

We denote $\mathcal{L}(X, Y)$ and $\mathcal{L}(X)$ the bounded linear maps from X to Y and from X into itself, respectively. For a given $\lambda > 0$, we define

$$\mathcal{L}_\lambda(X) = \big\{L \in \mathcal{L}(X) | L = K + C, K \text{ is compact and } \|C\| < \lambda\big\}.$$

Let F be a finite dimensional subspace of X. The quotient map L_F induced by L is defined by: $L_F : X \to X/F$, $x \mapsto Lx + F$ and $\|x\|_{X/F} = \inf\{\|x - f\|_X : \forall f \in F\}$. For the quotient map L_F, we have the following lemma (see Lemma 2.1 in [21]).

Lemma 2.1 *For every $L \in \mathcal{L}_\lambda(X)$ there exist a finite dimensional subspace $F \subset X$ such that if L_F is the linear map induced by L, then $\|L_F\| < 2\lambda$.*

If $L \in \mathcal{L}_\lambda(X)$, we define $\nu_\lambda(L)$ as the minimum integer n such that there exists a subspace $F \subset X$ satisfying $\dim_F = n$ and $\|L_F\| < 2\lambda$. By Lemma 2.1 we see that $\nu_\lambda(L)$ is well-defined and finite. We also need a covering result for a linear bounded mapping acts on the balls in X. We give this result in the following lemma, for more details we refer the readers to [21].

Lemma 2.2 *If $L \in \mathcal{L}(X)$ and $F \subset X$ is a subspace with $\dim_F = n$ and $\|L_F\| < \infty$, then*

$$N_{(1+\varepsilon)\lambda r}\big(L\big(B(0;r)\big)\big) \le n2^n\left(1 + \frac{\|L\| + \lambda}{\lambda\varepsilon}\right)^n,$$

for all $\varepsilon, r > 0$, $\lambda > \|L_F\|$. Moreover, the centers of the balls in the covering can be chosen in F.

For an RDS $\phi(t,\omega)$ and $T > 0$, we denote $D_v\phi(T,\omega)$ the Fréchet derivative of $\phi(T,\omega)$ at the point v. Assume that there exists a random variable $\lambda_\omega > 0$ such that $D_v\phi(T,\omega) \in \mathcal{L}_{\lambda_\omega}(X)$ for all $v \in \chi(\omega)$ and a.s. $\omega \in \Omega$, where $\chi(\omega)$ is a positively invariant random set for $\phi(t,\omega)$. Then for any given $\varepsilon_0 > 0$ we can find $r_{0,\omega} = r_{0,\omega}(\varepsilon_0) > 0$ such that for all $v \in \chi(\omega)$ and $0 < r \le r_{0,\omega}$,

$$\phi(T,\omega)B(v;r) \subset \phi(T,\omega)v + D_v\phi(T,\omega)B(0;r) + B(0;\varepsilon_0 r). \tag{2.1}$$

By Lemma 2.1 and Lemma 2.2 we have the following estimate

$$N_{2(1+\varepsilon_1)\lambda_\omega r}\big(D_v\phi(T,\omega)B(0;r)\big) \le K_\omega, \quad \forall\varepsilon_1 > 0, r > 0, \tag{2.2}$$

where

$$K_\omega = \nu(\omega)2^{\nu(\omega)}\left(1 + \frac{\sup_{v\in\chi(\omega)}\|D_v\phi(T,\omega)\|_X + 2\lambda_\omega}{2\lambda_\omega\varepsilon_1}\right)^{\nu(\omega)}, \tag{2.3}$$

and

$$\nu(\omega) = \sup_{v\in\chi(\omega)} \nu_{\lambda_\omega}\big(D_v\phi(T,\omega)\big). \tag{2.4}$$

Setting $\beta_\omega = 2(2(1+\varepsilon_1)\lambda_\omega + \varepsilon_0)$, then we have, for a.s. $\omega \in \Omega$,

$$N_{\beta_\omega r}\big(\phi(T,\omega)\big[B(v;r) \cap \chi(\omega)\big]\big) \le K_\omega, \quad 0 < r \le r_{0,\omega}. \tag{2.5}$$

Moreover, the covering balls are centered in $\phi(T,\omega)\chi(\omega)$.

Let $\phi(t,\omega)$ be a continuous RDS on a Banach space X over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, and

(**H0**) $\phi(t,\omega)$ possesses a random attractor $\mathcal{A}(\omega)$ in X.

Moreover, we make the following assumptions for a.s. $\omega \in \Omega$,

(**H1**) $\chi(\omega)$ is tempered, closed, positively invariant and absorbing;

(**H2**) $\chi(\omega)$ has a finite covering with radius $r_{0,\omega}$, that is, $N_{r_{0,\omega}}(\chi(\omega)) = N_\omega < \infty$, where $r_{0,\omega}$ is tempered and satisfies (2.1);

(**H3**) There is a positive constant T (independent of ω), and a random variable $L_\omega = L_\omega(T) > 0$ such that

$$\big\|\phi(t,\omega,v_1) - \phi(t,\omega,v_2)\big\|_X \le L_\omega\|v_1 - v_2\|_X, \quad \forall v_1, v_2 \in \chi(\omega), \forall t \le T;$$

(**H4**) $\phi(T,\omega)$ is C^1 on $\chi(\omega)$ and $D_v\phi(T,\omega) \in \mathcal{L}_{\lambda_\omega}(X)$ for some positive random variable λ_ω;

(**H5**) (2.5) are satisfied with $0 \le \mathbb{E}[\ln K_\omega] < \infty$, $0 \le \mathbb{E}[\ln N_\omega] < \infty$, and there exists $\varepsilon_0 \in (0,\frac{1}{2})$, $\varepsilon_1 > 0$ such that $-\infty < \mathbb{E}[\ln\beta_\omega] < 0$, where $\beta_\omega = 2(2(1+\varepsilon_1)\lambda_\omega + \varepsilon_0)$.

Our main result in this section read as:

Theorem 2.1 *Assume that conditions (**H0**) ∼ (**H5**) are satisfied. Then there exists a random exponential attractor $\{\mathcal{E}(\omega)\}_{\omega\in\Omega}$ for the continuous cocycle $\phi(t,\omega)$ with the following properties: for a.s. $\omega\in\Omega$,*

(*i*) $\mathcal{E}(\omega)\subset\chi(\omega)$ *is a compact set of* X;

(*ii*) $\phi(t,\theta_{-t}\omega)\mathcal{E}(\theta_{-t}\omega)\subset\mathcal{E}(\omega)$ *for all* $t\geq 0$;

(*iii*) $\dim_f\mathcal{E}(\omega)\leq -\frac{24\mathbb{E}[\ln(K_\omega N_\omega)]}{\mathbb{E}[\ln\beta_\omega]}<\infty$;

(*iv*) *for any* $B\in\mathcal{D}_X$, *there exist* $T_{\omega,B}\geq 0$, $b_\omega>0$ *such that*

$$d_h\big(\phi(t,\theta_{-t}\omega)B(\theta_{-t}\omega),\mathcal{E}(\omega)\big)\leq b_\omega e^{\frac{\mathbb{E}[\ln\beta_\omega]}{8T}t},\quad t\geq T_{\omega,B}.$$

Proof For any $n\in\mathbb{N}, m\in\mathbb{Z}$, we define the discrete cocycle:

$$\phi(n,m,\omega)=\phi(nT,\theta_{mT}\omega),\qquad \chi(m,\omega)=\chi(\theta_{mT}\omega). \tag{2.6}$$

It is easy to check by cocycle property in definition 2.2 that for any $n,n_1,n_2\in\mathbb{N}, m\in\mathbb{Z}$

$$\begin{aligned}&\phi(n_1,m+n_2,\omega)\phi(n_2,m,\omega)\\&\quad=\phi(n_1+n_2,m,\omega),\phi(n,m-n,\omega)\chi(m-n,\omega)\subset\chi(m,\omega).\end{aligned} \tag{2.7}$$

Firstly, from (2.5), we can get the covering of $\phi(n,m-n,\omega)\chi(m-n,\omega)$ by induction on n. If $n=0$, then by (**H2**) and the identity of $\phi(0,\omega)$, we have

$$\phi(0,m,\omega)\chi(m,\omega)=\chi(m,\omega)\subset\bigcup_{i=1}^{N_{m,\omega}}B(u_{0,m,\omega,i};r_{0,m,\omega})\cap\chi(m,\omega), \tag{2.8}$$

where $r_{0,m,\omega}=r_{0,\theta_{mT}\omega}$, $N_{m,\omega}=N_{\theta_{mT}\omega}$ and $u_{0,m,\omega,i}\in\chi(m,\omega)$, $\forall i\leq N_{m,\omega}$. The first generation of points consists of these centers, defined as

$$\mathcal{B}_{0,m,\omega}=\{u_{0,m,\omega,1},u_{0,m,\omega,2},\dots,u_{0,m,\omega,N_{m,\omega}}\}\subset\chi(m,\omega).$$

For $n=1$, we get from (2.5) and (2.8) that

$$\begin{aligned}\phi(1,m-1,\omega)\chi(m-1,\omega)&=\bigcup_{i=1}^{N_{m-1,\omega}}\phi(1,m-1,\omega)\big[B(u_{0,m-1,\omega,i};r_{0,m-1,\omega})\cap\chi(m-1,\omega)\big]\\&\subset\bigcup_{i=1}^{K_{m-1,\omega}N_{m-1,\omega}}B(u_{1,m-1,\omega,i};\beta_{m-1,\omega}r_{0,m-1,\omega}),\end{aligned} \tag{2.9}$$

where $K_{m-1,\omega}=K_{\theta_{(m-1)T}\omega}$ and $u_{1,m-1,\omega,i}\in\phi(1,m-1,\omega)\chi(m-1,\omega)\subset\chi(m,\omega)$, $\forall i\leq K_{m-1,\omega}N_{m-1,\omega}$. The second generation of points consists of these centers

$$\mathcal{B}_{1,m-1,\omega}=\{u_{1,m-1,\omega,1},\dots,u_{1,m-1,\omega,K_{m-1,\omega}N_{m-1,\omega}}\}\subset\chi(m,\omega).$$

When $n=2$, we get from (2.5), (2.7), and (2.9)

$$\phi(2,m-2,\omega)\chi(m-2,\omega)$$

$$\begin{aligned}
&= \phi(1,m-1,\omega)\phi(1,m-2,\omega)\chi(m-2,\omega)\\
&\subset \bigcup_{i=1}^{K_{m-2,\omega}N_{m-2,\omega}} \phi(1,m-1,\omega)\big[B(u_{1,m-2,\omega,i};\beta_{m-2,\omega}r_{0,m-2,\omega})\cap\chi(m-1,\omega)\big]\\
&\subset \bigcup_{i=1}^{K_{m-1,\omega}K_{m-2,\omega}N_{m-2,\omega}} B(u_{2,m-2,\omega,i};\beta_{m-1,\omega}\beta_{m-2,\omega}r_{0,m-2,\omega}),
\end{aligned}$$

where $u_{2,m-2,\omega,i}\in\phi(2,m-2,\omega)\chi(m-2,\omega)\subset\chi(m,\omega), i\le K_{m-1,\omega}K_{m-2,\omega}N_{m-2,\omega}$. The third generation of points consists of these centers

$$\mathcal{B}_{2,m-2,\omega}=\{u_{2,m-2,\omega,1},\dots,u_{2,m-2,\omega,K_{m-1,\omega}K_{m-2,\omega}N_{m-2,\omega}}\}\subset\chi(m,\omega).$$

For general n, we can induce that

$$\phi(n,m-n,\omega)\chi(m-n,\omega)=\bigcup_{i=1}^{K_{1\sim n,m,\omega}N_{m-n,\omega}} B(u_{n,m-n,\omega,i};\beta_{1\sim n,m,\omega}r_{0,m-n,\omega}),$$

where $K_{1\sim n,m,\omega}=K_{m-1,\omega}\cdot K_{m-2,\omega}\cdots K_{m-n,\omega}$, $\beta_{1\sim n,m,\omega}=\beta_{m-1,\omega}\cdot\beta_{m-2,\omega}\cdots\beta_{m-n,\omega}$ and $u_{n,m-n,\omega,i}\in\phi(n,m-n,\omega)\chi(m-n,\omega)\subset\chi(m,\omega), i\le K_{1\sim n,m,\omega}N_{m-n,\omega}$. The $(n+1)$th generation of points is

$$\mathcal{B}_{n,m-n,\omega}=\{u_{n,m-n,\omega,1},\dots,u_{n,m-n,\omega,K_{1\sim n,m,\omega}N_{m-n,\omega}}\}\subset\chi(m,\omega).$$

Secondly, we construct a random exponential attractor by adding the points chosen in step 1 to the random attractor $\mathcal{A}(\omega)\subset\chi(\omega)$ in X. We define

$$\begin{aligned}
\mathcal{B}(m,\omega)&=\overline{\bigcup_{n=0}^{\infty}\mathcal{B}_{n,m-n,\omega}}^{X}\quad\big(\subset\chi(m,\omega)\big),\\
\mathcal{C}(m,\omega)&=\bigcup_{j=0}^{\infty}\phi(j,m-j,\omega)\mathcal{B}(m-j,\omega),
\end{aligned}$$

and

$$\mathcal{E}(m,\omega)=\mathcal{C}(m,\omega)\cup\mathcal{A}(m,\omega),$$

where $\mathcal{A}(m,\omega)=\mathcal{A}(\theta_{mT}\omega)$ and $\mathcal{A}(\omega)$ is the random attractor for $\phi(t,\omega)$ (see assumption (**H0**)).

Finally, by using a similar process presented in [19], we can show that $\{\mathcal{E}(\omega)\}_{\omega\in\Omega}$ is a random exponential attractor for $\{\phi(t,\omega)\}_{t\ge0,\omega\in\Omega}$ in X, here we omit it. The proof is completed. □

3 Application

3.1 The RDS generated by (1.1)∼(1.2) and some useful results

We consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega,\mathcal{F})$. The Brownian motion $W(t,\omega)$ is identified as $\omega(t)$, i.e.,

$W(t,\omega) = \omega(t), t \in \mathbb{R}$. Define the time shift by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \omega \in \Omega, t \in \mathbb{R}$, then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic MDS.

For our purpose, we need to convert the stochastic equation (1.1)∼(1.2) into a deterministic equation with a random parameter. We introduce an one-dimensional Ornstein–Uhlenbeck process, which is given by $z(\theta_t\omega) := -\int_{-\infty}^0 e^\tau(\theta_t\omega)(\tau)\,d\tau, t \in \mathbb{R}$, and it solves the Itô equation

$$dz + z\,dt = dW(t). \tag{3.1}$$

It is known from [33] that the random variable $z(\omega)$ is tempered, and there is a θ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$ measure such that for every $\omega \in \tilde{\Omega}$, $t \to z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)\,ds = 0. \tag{3.2}$$

We set $\alpha(\omega) = e^{-bz(\omega)}$. From (3.2) we can easily show that $\alpha(\omega)$ and $\alpha^{-1}(\omega)$ are tempered.

Let $v(t) = \alpha(\theta_t\omega)u(t)$, and we can consider the following evolution equation with random coefficients but without white noise:

$$\frac{dv}{dt} - \Delta v + \alpha^{1-p}(\theta_t\omega)|v|^{p-1}v + \alpha(\theta_t\omega)f\big(x, \alpha^{-1}(\theta_t\omega)v\big) = g(x) + bz(\theta_t\omega)v, \tag{3.3}$$

with Dirichlet boundary condition

$$v|_{\partial D} = 0, \tag{3.4}$$

and initial condition

$$v(0) = v_0(\omega) = \alpha(\omega)u_0. \tag{3.5}$$

By the normal Faedo–Galerkin methods (see [22]) or a similar result for the deterministic case in [32], one can show that $v(t,\omega,v_0) \in C([0,\infty); L^2(D)) \bigcap L^2(0,T; H_0^1(D)), \forall T > 0$ and $\forall v_0 \in L^2(D)$. By the embeddings $H_0^1(D) \subset L^6(D) \subset L^{2p}(D)$ $(1 < p \le 3)$, we see that $v(t,\omega,v_0) \in L^{2p}(D)$ for $\forall t \ge 0$ and $\forall v_0 \in L^{2p}(D)$. Let $u(t,\omega,u_0) = \alpha^{-1}(\theta_t\omega)v(t,\omega,\alpha(\omega)u_0)$, then $u(t,\omega,u_0)$ is a solution of (1.1)∼(1.2) with $u_0 = \alpha^{-1}(\omega)v_0$. We now define a mapping $\Phi : \mathbb{R}^+ \times \Omega \times L^{2p}(D) \to L^{2p}(D)$ by $\Phi(t,\omega,u_0) = u(t,\omega,u_0) = \alpha^{-1}(\theta_t\omega)v(t,\omega,\alpha(\omega)u_0)$. Then Φ is an RDS generated by (1.1)∼(1.2) and continuous in $L^{2p}(D)$. To simplify the calculations, we only consider the continuous RDS generated by (3.3)∼(3.5), i.e.,

$$\phi(t,\omega,v_0) = v(t,\omega,v_0), \tag{3.6}$$

and check the conditions presented in Theorem 2.1 for $\phi(t,\omega)$.

With a standard procedure (see [32] for deterministic case and [13, 30]) for stochastic case), one can get the existence of a random attractor in $L^{2p}(D)$ for $\phi(t,\omega)$. In order to avoid the paper being tediously long, we just give the result below.

Theorem 3.1 *Assume that* (1.3)∼(1.5) *hold. Then the RDS* $\phi(t,\omega)$ *defined in* (3.6) *has a unique random attractor* $\hat{A}_{2p} = \{A_{2p}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_{2p}$ *in* $L^{2p}(D)$.

The above theorem implies that $\phi(t,\omega)$, defined in (3.6), satisfies the assumption (**H0**). To prove the Fréchet differentiability and to construct the absorbing subset described in assumptions (**H1**)∼(**H5**), we need the following regularity:

Lemma 3.1 *Assume that* (1.3)–(1.5) *hold. Let* $\hat{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_{2p}$ *and* $v_0 \in D(\omega)$. *Then for* $\mathbb{P}$*-a.s.* $\omega \in \Omega$, *there exists* $T_{\hat{D}}(\omega) > 0$ *and a tempered random variable* $M_0(\omega)$, *such that the solution* $v(t,\omega,v_0(\omega))$ *of* (3.3)–(3.5) *satisfies, for all* $t \geq T_{\hat{D}}(\omega) + 1$,

$$\left\| v\big(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|_{6p} \leq M_0(\omega).$$

Proof Multiplying (3.3) with $|v|^{2p-2}v$ to get

$$\begin{aligned}&\frac{1}{2p}\frac{d}{dt}\left\|v(t)\right\|_{2p}^{2p} + \big(-\Delta v, |v|^{2p-2}v\big) + \alpha^{1-p}(\theta_t\omega)\big(|v|^{p-1}v, |v|^{2p-2}v\big)\\&\quad + \alpha(\theta_t\omega)\big(f\big(x,\alpha^{-1}(\theta_t\omega)v\big), |v|^{2p-2}v\big) = \big(g, |v|^{2p-2}v\big) + bz(\theta_t\omega)\left\|v(t)\right\|_{2p}^{2p}.\end{aligned} \tag{3.7}$$

For the second term on the left-hand side of (3.7), using (3.10) and (3.13) in [20] and the imbedding theorem, we have

$$\big(-\Delta v, |v|^{2p-2}v\big) = \frac{2p-1}{p^2}\big(\nabla v^p, \nabla v^p\big), \tag{3.8}$$

and

$$c\|v\|_{2p}^{2p} + c'\|v\|_{6p}^{2p} \leq \left\|\nabla v^p\right\|^2. \tag{3.9}$$

So we obtain

$$\begin{aligned}&\frac{d}{dt}\left\|v(t)\right\|_{2p}^{2p} + c_4\|v\|_{2p}^{2p} + c_5\|v\|_{6p}^{2p} + 2p\alpha(\theta_t\omega)\big(f\big(x,\alpha^{-1}(\theta_t\omega)v\big), |v|^{2p-2}v\big)\\&\quad \leq 2p\big(g, |v|^{2p-2}v\big) + c_6 z(\theta_t\omega)\|v\|_{2p}^{2p} \leq c\|g\|_{2p}\|v\|_{2p}^{2p-1} + c_6 z(\theta_t\omega)\|v\|_{2p}^{2p}\\&\quad \leq c + \frac{c_4}{4}\|v\|_{2p}^{2p} + c_6 z(\theta_t\omega)\|v\|_{2p}^{2p}.\end{aligned} \tag{3.10}$$

Applying (1.4), we can estimate the fourth term on the left-hand side of (3.10) as

$$\begin{aligned}&-2p\alpha^2(\theta_t\omega)\big(f\big(x,\alpha^{-1}(\theta_t\omega)v\big)\alpha^{-1}(\theta_t\omega)v, |v|^{2p-2}\big)\\&\quad \leq 2p\alpha^2(\theta_t\omega)\big(-\nu\alpha^{-q-1}(\theta_t\omega)|v|^{q+1} + \beta, |v|^{2p-2}\big)\\&\quad \leq 2p\alpha^2(\theta_t\omega)\big(\beta, |v|^{2p-2}\big) \leq c\alpha^{2p}(\theta_t\omega) + \frac{c_4}{4}\|v\|_{2p}^{2p}.\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\frac{d}{dt}\left\|v(t)\right\|_{2p}^{2p} + \left(\frac{c_4}{2} - c_6 z(\theta_t\omega)\right)\|v\|_{2p}^{2p} + c_5\|v\|_{6p}^{2p} \leq c\big(1 + \alpha^{2p}(\theta_t\omega)\big). \tag{3.12}$$

Therefore, applying the Gronwall's inequality, we have

$$\left\|v(t)\right\|_{2p}^{2p} \leq e^{-\int_0^t(\frac{c_4}{2} - c_6 z(\theta_s\omega))\,ds}\left\|v(0)\right\|_{2p}^{2p} + c_7\int_0^t e^{\int_t^s(\frac{c_4}{2} - c_6 z(\theta_l\omega))\,dl}\big(1 + \alpha^{2p}(\theta_s\omega)\big)\,ds, \tag{3.13}$$

and

$$\int_t^{t+1} e^{\int_0^s (\frac{c_4}{2}-c_6 z(\theta_l\omega))\,dl} \|v(s)\|_{6p}^{2p}\,ds \le c e^{\int_0^t (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} \|v(t)\|_{2p}^{2p} + c\int_t^{t+1} e^{\int_0^s (\frac{c_4}{2}-c_6 z(\theta_l\omega))\,dl} \big(1+\alpha^{2p}(\theta_s\omega)\big)\,ds. \tag{3.14}$$

We define

$$M_1(\omega) = c_7 \int_{-\infty}^0 e^{\int_0^s (\frac{c_4}{2}-c_6 z(\theta_l\omega))\,dl} \big(1+\alpha^{2p}(\theta_s\omega)\big)\,ds, \tag{3.15}$$

and $M_1(\omega)$ is a tempered random variable by definition. Thus

$$\|v(t)\|_{2p}^{2p} \le e^{-\int_0^t (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} \|v(0)\|_{2p}^{2p} + M_1(\theta_t\omega). \tag{3.16}$$

(3.14) implies that there exists $t_1 = t_1(\omega) \in (t, t+1)$ such that

$$e^{\int_0^{t_1} (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} \|v(t_1)\|_{6p}^{2p} \le c e^{\int_0^t (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} \|v(t)\|_{2p}^{2p} + c\int_t^{t+1} e^{\int_0^s (\frac{c_4}{2}-c_6 z(\theta_l\omega))\,dl} \big(1+\alpha^{2p}(\theta_s\omega)\big)\,ds.$$

Putting (3.16) into the above inequality we get

$$\|v(t_1)\|_{6p}^{2p} \le c e^{-\int_0^{t_1} (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} \|v(0)\|_{2p}^{2p} + c e^{\int_{t_1}^{t} (\frac{c_4}{2}-c_6 z(\theta_s\omega))\,ds} M_1(\theta_t\omega) + c\int_t^{t+1} e^{\int_{t_1}^s (\frac{c_4}{2}-c_6 z(\theta_l\omega))\,dl} \big(1+\alpha^{2p}(\theta_s\omega)\big)\,ds.$$

Therefore

$$\begin{aligned}
\|v(t_1)\|_{6p}^{6p} &\le c e^{-\int_0^{t_1} (c_8-c_9 z(\theta_s\omega))\,ds} \|v(0)\|_{2p}^{6p} + c e^{\int_{t_1}^{t} (c_8-c_9 z(\theta_s\omega))\,ds} M_1^3(\theta_t\omega) \\
&\quad + c\int_t^{t+1} e^{\int_{t_1}^s (c_8-c_9 z(\theta_l\omega))\,dl}\,ds \int_t^{t+1} \big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds \\
&\le c e^{-\int_0^{t_1} (c_8-c_9 z(\theta_s\omega))\,ds} \|v(0)\|_{2p}^{6p} + c e^{c_9\int_t^{t+1} |z(\theta_s\omega)|\,ds} M_1^3(\theta_t\omega) \\
&\quad + c e^{c_9\int_t^{t+1} |z(\theta_s\omega)|\,ds} \int_t^{t+1} \big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds.
\end{aligned} \tag{3.17}$$

Next, we take the inner product of (3.3) with $|v|^{6p-2}v$ in $L^2(D)$,

$$\frac{1}{6p}\frac{d}{dt}\|v(t)\|_{6p}^{6p} + \big(-\Delta v, |v|^{6p-2}v\big) + \alpha^{1-p}(\theta_t\omega)\big(|v|^{p-1}v, |v|^{6p-2}v\big) + \alpha(\theta_t\omega)\big(f\big(x,\alpha^{-1}(\theta_t\omega)v\big), |v|^{6p-2}v\big) = \big(g, |v|^{6p-2}v\big) + bz(\theta_t\omega)\|v(t)\|_{6p}^{6p}.$$

Using $(-\Delta v, |v|^{6p-2}v) = \frac{6p-1}{9p^2}(\nabla|v|^{3p}, \nabla|v|^{3p}) \ge c\|v\|_{6p}^{6p}$ and a similar procedure of (3.12), we obtain

$$\frac{d}{dt}\|v(t)\|_{6p}^{6p} + \big(c_{10} - c_{11} z(\theta_t\omega)\big)\|v(t)\|_{6p}^{6p} \le c\big(1+\alpha^{6p}(\theta_t\omega)\big). \tag{3.18}$$

We integrate the above inequality on $[t_1, t+1]$ ($t_1 \in (t, t+1)$) to get

$$\begin{aligned}\|v(t+1)\|_{6p}^{6p} &\leq e^{\int_{t+1}^{t_1}(c_{10}-c_{11}z(\theta_l\omega))\,dl}\|v(t_1)\|_{6p}^{6p} + c\int_t^{t+1} e^{\int_{t+1}^{s}(c_{10}-c_{11}z(\theta_l\omega))\,dl}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds\\ &\leq ce^{c_{11}\int_t^{t+1}|z(\theta_s\omega)|\,ds}\|v(t_1)\|_{6p}^{6p}\\ &\quad + ce^{c_{11}\int_t^{t+1}|z(\theta_s\omega)|\,ds}\int_t^{t+1}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds. \end{aligned} \tag{3.19}$$

Putting (3.17) into (3.19) and noting that $t_1 \in (t, t+1)$, we have

$$\begin{aligned}\|v(t+1)\|_{6p}^{6p} &\leq c_{12}e^{c_{11}\int_t^{t+1}|z(\theta_s\omega)|\,ds}e^{-\int_0^{t_1}(c_8-c_9z(\theta_s\omega))\,ds}\|v(0)\|_{2p}^{6p}\\ &\quad + c_{13}e^{c_{15}\int_t^{t+1}|z(\theta_s\omega)|\,ds}M_1^3(\theta_t\omega) + c_{14}e^{c_{16}\int_t^{t+1}|z(\theta_s\omega)|\,ds}\int_t^{t+1}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds\\ &\leq c_{12}e^{c_{11}\int_t^{t+1}|z(\theta_s\omega)|\,ds}e^{-c_8t+c_9\int_0^{t+1}|z(\theta_s\omega)|\,ds}\|v(0)\|_{2p}^{6p}\\ &\quad + c_{13}e^{c_{15}\int_t^{t+1}|z(\theta_s\omega)|\,ds}M_1^3(\theta_t\omega) + c_{14}e^{c_{16}\int_t^{t+1}|z(\theta_s\omega)|\,ds}\int_t^{t+1}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds.\end{aligned}$$

Substituting $\theta_{-t-1}\omega$ for ω in the above inequality, we get

$$\begin{aligned}&\|v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{6p}^{6p}\\ &\quad\leq c_{12}e^{c_{11}\int_t^{t+1}|z(\theta_{s-t-1}\omega)|\,ds}e^{-c_8t+c_9\int_0^{t+1}|z(\theta_{s-t-1}\omega)|\,ds}\|v(0)\|_{2p}^{6p}\\ &\qquad + c_{13}e^{c_{15}\int_t^{t+1}|z(\theta_{s-t-1}\omega)|\,ds}M_1^3(\theta_{-1}\omega)\\ &\qquad + c_{14}e^{c_{16}\int_t^{t+1}|z(\theta_{s-t-1}\omega)|\,ds}\int_t^{t+1}\big(1+\alpha^{6p}(\theta_{s-t-1}\omega)\big)\,ds\\ &\quad\leq c_{12}e^{c_{11}\int_{-1}^{0}|z(\theta_s\omega)|\,ds}e^{-c_8t+c_9\int_{-t-1}^{0}|z(\theta_s\omega)|\,ds}\|v(0)\|_{2p}^{6p}\\ &\qquad + c_{13}e^{c_{15}\int_{-1}^{0}|z(\theta_s\omega)|\,ds}M_1^3(\theta_{-1}\omega) + c_{14}e^{c_{16}\int_{-1}^{0}|z(\theta_s\omega)|\,ds}\int_{-1}^{0}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds.\end{aligned} \tag{3.20}$$

Let

$$\begin{aligned}M_2(\omega) &= 1 + c_{13}e^{c_{15}\int_{-1}^{0}|z(\theta_s\omega)|\,ds}M_1^3(\theta_{-1}\omega)\\ &\quad + c_{14}e^{c_{16}\int_{-1}^{0}|z(\theta_s\omega))|\,ds}\int_{-1}^{0}\big(1+\alpha^{6p}(\theta_s\omega)\big)\,ds.\end{aligned} \tag{3.21}$$

From (3.2), we know that $e^{c\int_{-1}^{0}|z(\theta_s\omega)|\,ds}$ is tempered, thus $M_2(\omega)$ is tempered. Since $v_0 \in D(\omega)$, from (3.20) there exists a $T_{\hat{D}}(\omega) > 0$ such that for all $t \geq T_{\hat{D}}(\omega) + 1$

$$\|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|_{6p}^{6p} \leq M_2(\omega).$$

Then the result holds with $M_0(\omega) = M_2^{\frac{1}{6p}}(\omega)$. The proof is completed. □

Let $\hat{B}_0 = \{B_0(\omega)\}_{\omega\in\Omega} = \{u \in L^{2p}(D) : \|u\|_{6p} \leq M_0(\omega)\}_{\omega\in\Omega}$. Then, by Lemma 3.1, we see $\hat{B}_0$ is a random absorbing set over $\mathcal{D}_{2p}$ in $L^{2p}(D)$ and there exists $T_{\hat{B}_0}(\omega) > 0$ such that

$\phi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega) \subset B_0(\omega)$. We define

$$\chi_1(\omega) = \bigcup_{s>T_{\hat{B}_0}(\omega)} \phi(s,\theta_{-s}\omega)B_0(\theta_{-s}\omega). \tag{3.22}$$

3.2 The Fréchet derivative of $\phi(t,\omega)$

Set $F(x,u) = |u|^{p-1}u + f(x,u)$ and denote $D_u(F(x,u))$ the Fréchet derivative at u. From the assumption (1.3)–(1.5), we have

$$\begin{aligned} &F'(x,u) \geq -l, \left|F'(x,u)\right| \leq c\left(1+|u|^{p-1}\right),\\ &\left|F''(x,u)\right| \leq c\left(1+|u|^{p-2}\right), \left|F(x,u)\right| \leq c|u|^p + \kappa(x), \end{aligned} \tag{3.23}$$

where l, c are positive constants and $\kappa(x) \in L^{6p}(D)$.

Lemma 3.2 *Assume that (1.3)–(1.5) hold. Then for any $x \in D$, $F(x,u)$ is from $L^{6p}(D)$ into $L^{2p}(D)$ and Fréchet differentiable, that is, $D_u(F(x,u)) \in \mathcal{L}(L^{6p}(D), L^{2p}(D))$. Moreover, for any u, u_1, u_2, $h \in L^{6p}(D)$ and any $x \in D$, we have*

(1) $D_u(F(x,u))(h) = F'(x,u)h$;
(2) $\|F'(x,u)\|_{\mathcal{L}(L^{6p}(D),L^{2p}(D))} \leq c(1+\|u\|_{6p}^{p-1})$;
(3) $\|F'(x,u_1) - F'(x,u_2)\|_{\mathcal{L}(L^{6p}(D),L^{2p}(D))} \leq c(1+\|u_1\|_{6p}^{p-2} + \|u_2\|_{6p}^{p-2})\|u_1-u_2\|_{6p}$.

Proof By (3.23),

$$\left\|F(x,u)\right\|_{2p} = \left\{\int_D \left|F(x,u)\right|^{2p} dx\right\}^{\frac{1}{2p}} \leq c\|u\|_{2p^2}^p + c\|\kappa\|_{2p} \leq c\|u\|_{6p}^p + c\|\kappa\|_{2p}.$$

This implies that $F(x,u)$ is from $L^{6p}(D)$ into $L^{2p}(D)$. Moreover, from (3.23), we have

$$\begin{aligned} &\left\|F(x,u+h) - F(x,u) - F'(x,u)h\right\|_{2p}\\ &\quad = \left\{\int_D \left[F'(x,u+\theta_1 h) - F'(x,u)\right]^{2p}|h|^{2p} dx\right\}^{\frac{1}{2p}}\\ &\quad \leq \left\{\int_D \left|F''(x,u+\theta_2 h)\right|^{2p}|h|^{4p} dx\right\}^{\frac{1}{2p}}\\ &\quad \leq c\left\{\int_D \left(1+|u+\theta_2 h|^{p-2}\right)^{2p}|h|^{4p} dx\right\}^{\frac{1}{2p}}\\ &\quad \leq c\left(1+\|u\|_{6p(p-2)}^{p-2} + \|h\|_{6p(p-2)}^{p-2}\right)\|h\|_{6p}^2\\ &\quad \leq c_{17}\left(1+\|u\|_{6p}^{p-2} + \|h\|_{6p}^{p-2}\right)\|h\|_{6p}^2, \end{aligned}$$

where $0 < \theta_1, \theta_2 < 1$, this suggests that $F(x,u)$ is Fréchet differentiable and (1) holds.

Using (3.23) again, we obtain

$$\left\|F'(x,u)h\right\|_{2p} = \left\{\int_D \left|F'(x,u)\right|^{2p} h^{2p} dx\right\}^{\frac{1}{2p}}$$

$$
\begin{aligned}
&\leq c\left\{\int_D \left(1+|u|^{p-1}\right)^{2p} h^{2p}\,dx\right\}^{\frac{1}{2p}} \leq c\left\{\int_D \left(1+|u|^{2p(p-1)}\right) h^{2p}\,dx\right\}^{\frac{1}{2p}} \\
&\leq c\left(1+\|u\|_{3p(p-1)}^{p-1}\right)\|h\|_{6p} \leq c\left(1+\|u\|_{6p}^{p-1}\right)\|h\|_{6p},
\end{aligned} \tag{3.24}
$$

and

$$
\begin{aligned}
\left\|F'(x,u_1)h - F'(x,u_2)h\right\|_{2p} &= \left\{\int_D \left|F'(x,u_1)-F'(x,u_2)\right|^{2p} h^{2p}\,dx\right\}^{\frac{1}{2p}} \\
&= \left\{\int_D \left|F''\left(x,u_1+\theta_3(u_2-u_1)\right)\right|^{2p} |u_1-u_2|^{2p} h^{2p}\,dx\right\}^{\frac{1}{2p}} \\
&\leq c\left\{\int_D \left(1+\left|u_1+\theta_3(u_2-u_1)\right|^{p-2}\right)^{2p} |u_1-u_2|^{2p} h^{2p}\,dx\right\}^{\frac{1}{2p}} \\
&\leq c\left(1+\|u_1\|_{6p}^{p-2}+\|u_2\|_{6p}^{p-2}\right)\|u_1-u_2\|_{6p}\|h\|_{6p},
\end{aligned} \tag{3.25}
$$

where $0<\theta_3<1$. Thus, (3.24) and (3.25) imply (2) and (3), respectively. The proof is completed. □

Lemma 3.3 *Suppose that* (1.3)∼(1.5) *hold. Then* $v(t,\omega)$ *is Fréchet differentiable in* $\chi_1(\omega)$ *for every* $t\in\mathbb{R}$ *and a.s.* $\omega\in\Omega$.

Proof For any v_0 and $v_0+h\in\chi_1(\omega)$, we assume that $v_1=v_1(t)=v(t,\omega,v_0+h), v_2=v_2(t)=v(t,\omega,v_0)$ are two solutions of (3.3) starting from v_0+h and v_0, respectively, and set $w(t)=v_1(t)-v_2(t)$, then $w(t)$ satisfies

$$
\frac{dw}{dt} - \Delta w + \alpha(\theta_t\omega)\left[F\left(x,\alpha^{-1}(\theta_t\omega)v_1\right) - F\left(x,\alpha^{-1}(\theta_t\omega)v_2\right)\right] = bz(\theta_t\omega)w, \tag{3.26}
$$

The linearization is

$$
\frac{dU}{dt} - \Delta U + F'\left(x,\alpha^{-1}(\theta_t\omega)v_1\right)U = bz(\theta_t\omega)U. \tag{3.27}
$$

Setting $\varphi = w-U$, then from (3.26) and (3.27), we have

$$
\begin{aligned}
&\frac{d\varphi}{dt} - \Delta\varphi + F'\left(x,\alpha^{-1}(\theta_t\omega)v_1\right)\varphi \\
&\quad + \left[F'\left(x,\alpha^{-1}(\theta_t\omega)\left(v_1+\theta_4(v_2-v_1)\right)\right) - F'\left(x,\alpha^{-1}(\theta_t\omega)v_1\right)\right]w = bz(\theta_t)\varphi,
\end{aligned} \tag{3.28}
$$

where $0<\theta_4<1$ and

$$
\varphi(0)=0. \tag{3.29}
$$

Multiplying (3.26) by $|w|^{2p-2}w$ and using (3.8), (3.9), and (3.23), we get

$$
\frac{d}{dt}\left\|w(t)\right\|_{2p}^{2p} - \left(c_{18}+c_{19}z(\theta_s\omega)\right)\left\|w(t)\right\|_{2p}^{2p} + c\left\|w(t)\right\|_{6p}^{2p} \leq 0. \tag{3.30}
$$

Therefore, for all $t \geq 0$,

$$\|w(t)\|_{2p}^{2p} \leq e^{\int_0^t (c_{18}+c_{19}z(\theta_s\omega))\,ds}\|h\|_{2p}^{2p}. \tag{3.31}$$

Multiplying (3.30) by $e^{-\int_0^t (c_{18}+c_{19}z(\theta_l\omega))\,dl}$ then integrating in $[0,t]$ yields

$$c\int_0^t e^{-\int_0^s (c_{18}+c_{19}z(\theta_l\omega))\,dl}\|w(s)\|_{6p}^{2p}\,ds \leq \|w(0)\|_{2p}^{2p} = \|h\|_{2p}^{2p}.$$

Thus, there is a $t_2 = t_2(\omega) \in (0,t)$ such that

$$\|w(t_2)\|_{6p}^{2p} \leq ce^{\int_0^{t_2} (c_{18}+c_{19}z(\theta_s\omega))\,ds}\|h\|_{2p}^{2p}. \tag{3.32}$$

Next, taking the inner product (3.26) with $|w|^{6p-2}w$, using (3.23) and $(-\Delta w, |w|^{6p-2}w) \geq c\|w\|_{6p}^{6p}$, we obtain

$$\frac{d}{dt}\|w(t)\|_{6p}^{6p} - \big(c_{20} + c_{21}z(\theta_s\omega)\big)\|w(t)\|_{6p}^{6p} \leq 0.$$

Thus, by integrating the above inequality over (t_2,t) and using (3.32), we have

$$\|w(t)\|_{6p}^{6p} \leq ce^{\int_{t_2}^t (c_{20}+c_{21}z(\theta_s\omega))\,ds}e^{\int_0^{t_2} (c_{22}+c_{23}z(\theta_s\omega))\,ds}\|h\|_{2p}^{6p} \leq ce^{c_{24}\int_0^t (1+|z(\theta_s\omega)|)\,ds}\|h\|_{2p}^{6p}. \tag{3.33}$$

Taking the inner product of (3.28) with $|\varphi|^{2p-2}\varphi$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\varphi\|_{2p}^{2p} + 2p\big(-\Delta\varphi, |\varphi|^{2p-2}\varphi\big) + 2p\big(F'\big(x,\alpha^{-1}(\theta_t\omega)v_1\big)\varphi, |\varphi|^{2p-2}\varphi\big) \\
&\quad + 2p\big(\big[F'\big(x,\alpha^{-1}(\theta_t\omega)\big(v_1 + \theta_4(v_2 - v_1)\big)\big) - F'\big(x,\alpha^{-1}(\theta_t\omega)v_1\big)\big]w, |\varphi|^{2p-2}\varphi\big) \\
&\quad = 2pbz(\theta_t\omega)\|\varphi\|_{2p}^{2p}.
\end{aligned} \tag{3.34}$$

Using Lemma 3.1 and (3) in Lemma 3.2, we have

$$\begin{aligned}
&2p\big|\big(\big[F'\big(x,\alpha^{-1}(\theta_t\omega)\big(v_1 + \theta_4(v_2 - v_1)\big)\big) - F'\big(x,\alpha^{-1}(\theta_t\omega)v_1\big)\big]w, |\varphi|^{2p-2}\varphi\big)\big| \\
&\quad \leq 2p\big\|F'\big(x,\alpha^{-1}(\theta_t\omega)\big(v_1 + \theta_4(v_2 - v_1)\big)\big) \\
&\qquad - F'\big(x,\alpha^{-1}(\theta_t\omega)v_1\big)\big\|_{\mathcal{L}(L^{6p}(D),L^{2p}(D))}\|w\|_{6p}\|\varphi\|_{2p}^{2p-1} \\
&\quad \leq c\alpha^{-1}(\theta_t\omega)\big(1 + \alpha^{2-p}(\theta_t\omega)\|v_1\|_{6p}^{p-2} + \alpha^{2-p}(\theta_t\omega)\|v_2\|_{6p}^{p-2}\big)\|w\|_{6p}^2\|\varphi\|_{2p}^{2p-1} \\
&\quad \leq c\alpha^{-2p}(\theta_t\omega)\big(1 + \alpha^{2p(2-p)}(\theta_t\omega)\|v_1\|_{6p}^{2p(p-2)} + \alpha^{2p(2-p)}(\theta_t\omega)\|v_2\|_{6p}^{2p(p-2)}\big)\|w\|_{6p}^{4p} + c\|\varphi\|_{2p}^{2p} \\
&\quad \leq c_{25}\alpha^{-2p}(\theta_t\omega)\big(1 + 2\alpha^{2p(2-p)}(\theta_t\omega)M_0^{2p(p-2)}(\theta_t\omega)\big)\|w\|_{6p}^{4p} + c\|\varphi\|_{2p}^{2p} \\
&\quad = M_3(\theta_t\omega)\|w\|_{6p}^{4p} + c\|\varphi\|_{2p}^{2p},
\end{aligned} \tag{3.35}$$

where

$$M_3(\omega) = c_{25}\alpha^{-2p}(\omega)\big(1 + 2\alpha^{2p(2-p)}(\omega)M_0^{2p(p-2)}(\omega)\big). \tag{3.36}$$

Putting (3.35) into (3.34), we have

$$\frac{d}{dt}\|\varphi\|_{2p}^{2p} \le (c_{26} + c_{27}z(\theta_t\omega)\|\varphi\|_{2p}^{2p} + M_3(\theta_t\omega)\|w\|_{6p}^{4p},$$

Apply Gronwall's lemma to get

$$\|\varphi(t)\|_{2p}^{2p} \le \int_0^t e^{-\int_t^s (c_{26}+c_{27}z(\theta_l\omega))\,dl} M_3(\theta_s\omega)\|w(s)\|_{6p}^{4p}\,ds. \tag{3.37}$$

Then from (3.33) and (3.37), we obtain, for any $t > 0$,

$$\begin{aligned}\|\varphi(t)\|_{2p}^{2p} &\le c_{28}\int_0^t M_3(\theta_s\omega)e^{-\int_t^s (c_{26}+c_{27}z(\theta_l\omega))\,dl} e^{c_{29}\int_0^s (1+|z(\theta_s\omega)|)\,ds}\,ds\|h\|_{2p}^{4p} \\ &\le c_{28}e^{c_{29}\int_0^t (1+|z(\theta_s\omega)|)\,ds}\int_0^t M_3(\theta_s\omega)e^{-\int_t^s (c_{26}+c_{27}z(\theta_l\omega))\,dl}\,ds\|h\|_{2p}^{4p}.\end{aligned} \tag{3.38}$$

(3.38) implies that, for fixed t and ω, $v(t,\omega)$ is uniformly Fréchet differentiable for every point in $\chi_1(\omega)$ in the topology of $L^{2p}(D)$. The proof is completed. □

3.3 Decomposition of $v(t,\omega)$

In this subsection, we consider the linear version in $L^{2p}(D)$

$$D_{u_0}\big(v(t,\omega)\big) = \lim_{h\to 0}\frac{v(t,\omega,u_0+hv_0) - v(t,\omega,u_0)}{h}.$$

Let $\overline{w}(t) = v_1(t) - v_2(t) = v(t,\omega,u_0+hv_0) - v(t,\omega,u_0)$, then

$$\begin{cases} \frac{d\overline{w}}{dt} - \Delta\overline{w} + \alpha^{1-p}(\theta_t\omega)(|v_1|^{p-1}v_1 - |v_2|^{p-1}v_2) \\ \quad + \alpha(\theta_t\omega)[f(x,\alpha^{-1}(\theta_t\omega)v_1) - f(x,\alpha^{-1}(\theta_t\omega)v_2)] = bz(\theta_t\omega)\overline{w}, & x\in D, t>0, \\ \overline{w} = 0, & x\in\partial D, t>0, \\ \overline{w}(0) = hv_0, & x\in D. \end{cases} \tag{3.39}$$

Defining $\overline{W}(t) = \lim_{h\to 0}\frac{\overline{w}}{h}$, we get

$$\begin{cases} \frac{d\overline{W}}{dt} - \Delta\overline{W} + p\alpha^{1-p}(\theta_t\omega)|v_1|^{p-1}\overline{W} \\ \quad + f'(x,\alpha^{-1}(\theta_t\omega)v_1)W = bz(\theta_t\omega)\overline{W}, & x\in D, t>0, \\ \overline{W} = 0, & x\in\partial D, t>0, \\ \overline{W}(0) = v_0, & x\in D. \end{cases} \tag{3.40}$$

We split (3.40) into

$$\begin{cases} \frac{dW_1}{dt} - \Delta W_1 + p\alpha^{1-p}(\theta_t\omega)|v_1|^{p-1}W_1 = bz(\theta_t\omega)W_1, & x\in D, t>0, \\ W_1 = 0, & x\in\partial D, t>0, \\ W_1(0) = v_0, & x\in D, \end{cases} \tag{3.41}$$

and

$$\begin{cases} \frac{dW_2}{dt} - \Delta W_2 + p\alpha^{1-p}(\theta_t\omega)|v_1|^{p-1}W_2 \\ \quad +f'(x,\alpha^{-1}(\theta_t\omega)v_1)(W_1 + W_2) = bz(\theta_t\omega)W_2, & x \in D, t > 0, \\ W_2 = 0, & x \in \partial D, t > 0, \\ W_2(0) = 0, & x \in D. \end{cases} \tag{3.42}$$

Clearly, we have $\overline{W}(t) = W_1(t) + W_2(t)$.

In the following, we prove that $\mathrm{W}_2(T)$ is compact in $L^{2p}(D)$ and $\mathrm{W}_1(T)$ is contractive in the mean in $L^{2p}(D)$ for some $T > 1$. The proof of the following lemma is similar to some parts of Lemma 3.1, here we only give the sketch.

Lemma 3.4 *Suppose* (1.3)∼(1.5) *hold, then for a.s.* $\omega \in \Omega$,

(1) *for any* $t > 1$, $W_2(t,\omega)$ *is compact in* $L^{2p}(D)$;

(2) *there exists a tempered random variable* $\lambda_{t,\omega}$, *such that for any* $t > 0$ *it holds that* $\| W_1(t,\omega)\|_{\mathcal{L}(L^{2p}(D))} < \lambda_{t,\omega}$;

(3) *for any* $\varepsilon_0 \in (0, \frac{1}{2})$, $\varepsilon_1 > 0$, *there exists* $T > 1$, *which is independent of* ω *such that* $-\infty < \mathbb{E}[\ln \beta_\omega] < 0$, *where* $\beta_\omega = 4(1 + \varepsilon_1)\lambda_\omega + 2\varepsilon_0$, $\lambda_\omega = \lambda_{T,\omega}$.

Sketch of Proof (1) As the proof of (3.20), by taking the inner product of (3.40) and (3.41) with $|\overline{W}|^{6p-2}\overline{W}$ and $|W_1|^{6p-2}W_1$ respectively we obtain that $\overline{W}(t)$ and $W_1(t)$ are bounded from $L^{2p}(D)$ into $L^{6p}(D)$ for every $t \geq t^*$ and for some $t^* = t^*(\omega) \in (0,1)$. Thus $W_2(t) = \overline{W}(t) - W_1(t)$ is bounded from $L^{2p}(D)$ into $L^{6p}(D)$. similarly, it is also a standard procedure to get that $W_2(t)$ is bounded from $L^{2p}(D)$ into $H_0^1(D)$ for any $t > 1$. Therefore, by the compact embedding $H_0^1(D) \hookrightarrow L^2(D)$ and the interpolative inequality

$$\|u\|_{2p} \leq \|u\|_{6p}^{\theta}\|u\|^{1-\theta},$$

it is easy to prove that $W_2(t)$ is compact in $L^{2p}(D)$ for $t > 1$.

(2) Multiplying (3.41) by $|W_1|^{2p-2}W_1$, and using (3.8) and (3.9), we get

$$\frac{d}{dt}\| W_1\|_{2p}^{2p} + \big(c_{31} - c_{32}z(\theta_t\omega)\big)\| W_1\|_{2p}^{2p} \leq 0,$$

Thus,

$$\big\| W_1(t)\big\|_{2p}^{2p} \leq e^{-\int_0^t (c_{31}-c_{32}z(\theta_s\omega))\,ds}\|v_0\|_{2p}^{2p}.$$

Let

$$\lambda_{t,\omega} = 2e^{-\frac{1}{2p}\int_0^t (c_{31}-c_{32}z(\theta_s\omega))\,ds}, \tag{3.43}$$

then for any $t > 0$ we have $\| W_1(t,\omega)\|_{\mathcal{L}(L^{2p}(D))} < \lambda_{t,\omega}$.

(3) From (3.2), we see $\lim_{t\to+\infty}\lambda_{t,\omega} = 0$, thus $\lim_{t\to+\infty}\mathbb{E}[\lambda_{t,\omega}] = 0$. There exists a T, which is independent of ω such that, for any $t \geq T$, it holds that $\ln(2\varepsilon_0) + \frac{2(1+\varepsilon_1)}{\varepsilon_0}\mathrm{E}[\lambda_{t,\omega}] < 0$. Let $\lambda_\omega = \lambda_{T,\omega}$, then we have

$$\mathbb{E}[\ln \beta_\omega] = \mathbb{E}\big[\ln\big(4(1 + \varepsilon_1)\lambda_\omega + 2\varepsilon_0\big)\big]$$

$$= \ln(2\varepsilon_0) + \mathbb{E}\left[\ln\left(1 + \frac{2(1+\varepsilon_1)}{\varepsilon_0}\lambda_\omega\right)\right] \le \ln(2\varepsilon_0) + \frac{2(1+\varepsilon_1)}{\varepsilon_0}\mathbb{E}[\lambda_\omega] < 0.$$

The proof is completed. □

We choose the constant T obtained in Lemma 3.4, and set

$$M_4(\omega) = \left(c_{28}e^{c_{29}\int_0^T(1+|z(\theta_s\omega)|)\,ds}\int_0^T M_3(\theta_s\omega)e^{-\int_T^s(c_{26}+c_{27}z(\theta_l\omega))\,dl}\,ds\right)^{\frac{1}{2p}}. \tag{3.44}$$

Then $M_4(\omega)$ is positive and tempered, and we can rewrite (3.38) as

$$\left\|\varphi(T,\omega)\right\|_{2p} \le M_4(\omega)\|h\|_{2p}^2. \tag{3.45}$$

Let

$$M_4(\omega)\|h\|_{2p} \le \varepsilon_0, \tag{3.46}$$

and define

$$r_{0,\omega} = \varepsilon_0 M_4^{-1}(\omega), \tag{3.47}$$

then from Proposition 4.3.3 in [33] we know that $r_{0,\omega}$ is tempered. Moreover, $r_{0,\omega}$ satisfies (2.1). (3.45)–(3.47)implies that for any fixed $\varepsilon_0 > 0$, we have $\|\varphi(T,\omega)\|_{2p} \le \varepsilon_0\|h\|_{2p}$ for all $0 < \|h\|_{2p} < r_{0,\omega}$, thus (2.1) hold.

3.4 Construction of $\chi(\omega)$ and the main result

In this subsection, we construct N_ω and the positively invariant set $\chi(\omega)$ described in (**H1**) $\sim$ (**H5**) for the RDS $\phi(t,\omega)$ and prove that K_ω, N_ω and β_ω satisfy (**H5**). Since $A_{2p}(\omega)$ is the random attractor for the RDS $\phi(t,\omega)$ in $L^{2p}(D)$ (see Theorem 3.1), $A_{2p}(\omega)$ is compact in $L^{2p}(D)$. We assume the $\bigcup_{i=1}^{N_{r_{0,\omega}}(A_{2p}(\omega))} B(u_i; r_{0,\omega})$ is the covering of $A_{2p}(\omega)$ in $L^{2p}(D)$, where $N_{r_{0,\omega}}(A_{2p}(\omega))$ is the minimal number of balls with radius $r_{0,\omega}$ covering $A_{2p}(\omega)$ in $L^{2p}(D)$, then by Lemma 2.3 in [20] we get:

Lemma 3.5 *There exists a random variable* $\widehat{r}_\omega$ $(0 < \widehat{r}_\omega < r_{0,\omega})$ *such that*

$$\bigcup_{i=1}^{N_{r_{0,\omega}}(A_{2p}(\omega))} B(u_i; r_{0,\omega}) \supset \mathcal{N}_{\widehat{r}_\omega}\big(A_{2p}(\omega)\big),$$

where $\mathcal{N}_{\widehat{r}_\omega}(A_{2p}(\omega))$ *denotes the closed* $\widehat{r}_\omega$*-neighborhood of* $A_{2p}(\omega)$.

Define

$$\chi(\omega) = \left(\bigcup_{s>T_{\hat{B}_0}(\omega)+1}\phi(s,\theta_{-s}\omega)B_0(\theta_{-s}\omega)\right) \cap \mathcal{N}_{\widehat{r}_\omega}\big(A_{2p}(\omega)\big)\big(\subset \chi_1(\omega)\big). \tag{3.48}$$

Since the $\widehat{r}_\omega$-neighborhood of $A_{2p}(\omega)$ is absorbing, $\chi(\omega)$ is nonempty and satisfies

$$N_\omega = N_{r_{0,\omega}}\big(\chi(\omega)\big) \le N_{r_{0,\omega}}\big(A_{2p}(\omega)\big). \tag{3.49}$$

Lemma 3.6 *Suppose* (1.3)∼(1.5) *hold, then* $-\infty < \mathbb{E}[\ln\beta_\omega] < 0$, $0 \le \mathbb{E}[\ln K_\omega] < \infty$ *and* $0 \le \mathbb{E}[\ln N_\omega] < \infty$.

Proof The first result has been proved in Lemma 3.4.

From [18, 19], we have the following results:

$$\mathbb{E}\big[e^{\epsilon\int_\tau^{\tau+t}|z(\theta_s\omega)|\,ds}\big] \le e^{\frac{\epsilon}{\sqrt{\alpha}}t}, \quad \alpha^3 \ge \epsilon^2 \ge 1, \tau \in \mathbb{R}, t \ge 0, \tag{3.50}$$

$$\mathbb{E}\big[e^{\epsilon|z(\omega)|}\big] \le \left(1 + \frac{|\epsilon|}{\sqrt{\pi}}\right)e^{\epsilon^2}. \tag{3.51}$$

Taking the inner product of (3.27) by $|U|^{2p-2}U$, we can get

$$\frac{d}{dt}\|U\|_{2p}^{2p} - \big(c_{33} + c_{34}z(\theta_s\omega)\big)\|U\|_{2p}^{2p} \le 0,$$

Thus

$$\big\|U(t)\big\|_{2p}^{2p} \le e^{\int_0^t (c_{33}+c_{34}z(\theta_s\omega))\,ds}\|h\|_{2p}^{2p}.$$

This implies that

$$\sup_{v\in\chi(\omega)}\big\|D_v\phi(t,\omega)\big\|_{2p} \le e^{\int_0^t (c_{33}+c_{34}z(\theta_s\omega))\,ds}. \tag{3.52}$$

From Lemma 2.2 in [21] (see also Lemma 2.2 in [20]), we can get that $\nu(\omega) \le c\sup_{v\in\chi(\omega)}\|v\|_{2p}$. Since $\chi(\omega) \subset B_0(\omega)$, we have

$$\nu(\omega) \le c\sup_{v\in\chi(\omega)}\|v\|_{2p} \le c\sup_{v\in\chi(\omega)}\|v\|_{6p} \le cM_0(\omega). \tag{3.53}$$

Recall that (see (3.21) and (3.15))

$$M_0(\omega) = M_2^{\frac{1}{6}}(\omega),$$

$$M_2(\omega) = 1 + c_{13}e^{c_{15}\int_{-1}^0 |z(\theta_s\omega)|\,ds}M_1^3(\theta_{-1}\omega) + c_{14}e^{c_{16}\int_{-1}^0 |z(\theta_s\omega))|\,ds}\int_{-1}^0 \big(1 + \alpha^{6p}(\theta_s\omega)\big)\,ds,$$

and

$$M_1(\omega) = c_7\int_{-\infty}^0 e^{\int_0^s (\frac{c_4}{2} - c_6 z(\theta_l\omega))\,dl}\big(1 + \alpha^{2p}(\theta_s\omega)\big)\,ds.$$

Using Young's inequality and $\sqrt{x} \le e^x$, we get

$$M_1(\omega) = c\int_{-\infty}^0 e^{\frac{c_4}{2}s - 2c_6\int_0^s z(\theta_l\omega)\,dl}\,ds + c\int_{-\infty}^0 e^{\frac{c_4}{2}s}\big(1 + \alpha^{4p}(\theta_s\omega)\big)\,ds. \tag{3.54}$$

and

$$\ln M_1(\omega) = \ln\left(c_7\int_{-\infty}^0 e^{\int_0^s (\frac{c_4}{2} - c_6 z(\theta_l\omega))\,dl}\big(1 + \alpha^{2p}(\theta_s\omega)\big)\,ds\right)$$

$$\begin{aligned}&\leq \ln\left[c\left(\int_{-\infty}^{0} e^{\frac{c_4}{2}s-2c_6\int_0^s z(\theta_l\omega)\,dl}\,ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0} e^{\frac{c_4}{2}s}\big(1+\alpha^{4p}(\theta_s\omega)\big)\,ds\right)^{\frac{1}{2}}\right]\\&\leq c+\int_{-\infty}^{0} e^{\frac{c_4}{2}s-2c_6\int_0^s z(\theta_l\omega)\,dl}\,ds+\int_{-\infty}^{0} e^{\frac{c_4}{2}s}\big(1+\alpha^{4p}(\theta_s\omega)\big)\,ds. \qquad (3.55)\end{aligned}$$

From (3.54)–(3.55) and applying (3.50)–(3.51), we have $\mathbb{E}[M_1(\omega)]<+\infty, \mathbb{E}[\ln M_1(\omega)]<+\infty$. Similarly, we get $\mathbb{E}[M_2(\omega)]<+\infty, \mathbb{E}[\ln M_2(\omega)]<+\infty$. Thus, from (3.53), we see

$$\mathbb{E}\big[\nu(\omega)\big]<+\infty, \qquad \mathbb{E}\big[\ln\nu(\omega)\big]<+\infty. \qquad (3.56)$$

By using (2.3), (3.43), and (3.52) with $t=T$, we have the following estimate

$$\begin{aligned}\ln K_\omega &= \ln\nu(\omega)+\nu(\omega)\left(\ln 2+\ln\left(1+\frac{\sup_{v\in\chi(\omega)}\|D_v\phi(T,\omega)\|_{2p}+2\lambda_\omega}{2\lambda_\omega\varepsilon}\right)\right)\\&\leq \ln\nu(\omega)+\nu(\omega)\left(\ln 2+\frac{1}{\varepsilon}+\frac{\sup_{v\in\chi(\omega)}\|D_v\phi(T,\omega)\|_{2p}}{2\lambda_\omega\varepsilon}\right)\\&\leq c_{35}+\ln\nu(\omega)+c_{36}\nu^{2p}(\omega)+c_{37}\left(\frac{\sup_{v\in\chi(\omega)}\|D_v\phi(T,\omega)\|_{2p}}{2\lambda_\omega\varepsilon}\right)^{\frac{2p}{2p-1}}\\&\leq c_{35}+\ln\nu(\omega)+c_{36}\nu^{2p}(\omega)+c_{40}e^{\int_0^T(c_{38}+c_{39}|z(\theta_s\omega)|)\,ds}. \qquad (3.57)\end{aligned}$$

Therefore, from (3.50), (3.51), (3.56), and (3.57), we conclude that $0\leq\mathbb{E}[\ln K_\omega]<\infty$.

To prove $0\leq\mathbb{E}[\ln N_\omega]<\infty$, we assume that the sequence $\{\lambda_j\}_{j=1}^{\infty}$, $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_j\to\infty, j\to\infty$, and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $D(-\Delta)$, which forms an orthogonal basis in both $L^2(D)$ and $H_0^1(D)$ such that

$$-\Delta e_j=\lambda_j e_j, \quad \forall j\in N.$$

Given n, let $X_n=\operatorname{span}\{e_1,\dots,e_n\}$ and $P_n: L^2(D)\to X_n$ be the projection operator. For any $v\in H_0^1(D)$, we write $v=P_m v+(I-P_m)v:=v_1+v_2$.

Let $w_1=P_m w$, $w_2=(I-P_m)w$, where w is the solution of (3.26). Now we multiply (3.26) with $-\Delta w_1$ to get

$$\frac{d}{dt}\|\nabla w_1\|^2+\|\Delta w_1\|^2+\big(F'\big(x,\alpha^{-1}(\theta_t\omega)\big(v_1+\theta(v_2-v_1)\big)\big)w,-\Delta w_1\big)=bz(\theta_t\omega)\|\nabla w_1\|^2.$$

Since $v_1, v_2\in\chi(\omega)$, by applying (3.23), we can estimate the nonlinearity as

$$\begin{aligned}&\big|\big(F'\big(x,\alpha^{-1}(\theta_t\omega)\big(v_1+\theta(v_2-v_1)\big)\big)w,-\Delta w_1\big)\big|\\&\quad\leq c\int_D\big(1+\big|\alpha^{-1}(\theta_t\omega)v_1\big|^{2p-2}+\big|\alpha^{-1}(\theta_t\omega)v_2\big|^{2p-2}\big)|w|^2+\frac{1}{2}\|\Delta w_1\|^2\\&\quad\leq c\big(1+\alpha^{2-2p}(\theta_t\omega)\|v_1\|_{2p}^{2p-2}+\alpha^{2-2p}(\theta_t\omega)\|v_2\|_{2p}^{2p-2}\big)\|w\|_{2p}^2+\frac{1}{2}\|\Delta w_1\|^2\\&\quad\leq c_{41}\big(1+\alpha^{2-2p}(\theta_t\omega)\|v_1\|_{6p}^{2p-2}+\alpha^{2-2p}(\theta_t\omega)\|v_2\|_{6p}^{2p-2}\big)\|w\|_{2p}^2+\frac{1}{2}\|\Delta w_1\|^2\\&\quad\leq M_5(\theta_t\omega)\|w\|_{2p}^2+\frac{1}{2}\|\Delta w_1\|^2, \qquad (3.58)\end{aligned}$$

where

$$M_5(\omega) = c_{41}\left[1 + 2\alpha^{2-2p}(\omega)M_0^{2p-2}(\omega)\right]. \tag{3.59}$$

Therefore

$$\frac{d}{dt}\|\nabla w_1\|^2 - bz(\theta_t\omega)\|\nabla w_1\|^2 \le M_5(\theta_t\omega)\|w\|_{2p}^2.$$

Applying the Gronwall's inequality

$$\begin{aligned}\left\|\nabla w_1(t)\right\|^2 &\le e^{b\int_0^t z(\theta_s\omega)\,ds}\left\|\nabla w_1(0)\right\|^2 + \int_0^t e^{-b\int_t^s z(\theta_l\omega)\,dl} M_5(\theta_s\omega)\left\|w(s)\right\|_{2p}^2\,ds \\ &\le e^{b\int_0^t z(\theta_s\omega)\,ds}\|\nabla h\|^2 + \int_0^t e^{-b\int_t^s z(\theta_l\omega)\,dl} M_5(\theta_s\omega)\left\|w(s)\right\|_{2p}^2\,ds.\end{aligned} \tag{3.60}$$

Putting (3.31) into (3.60) and using the inequality $\sqrt{x} \le e^x$, we have

$$\begin{aligned}\left\|\nabla w_1(t)\right\|^2 &\le e^{b\int_0^t z(\theta_s\omega))\,ds}\|\nabla h\|^2 + \int_0^t e^{-b\int_t^s z(\theta_l\omega)\,dl} e^{\int_0^s (c_{42}+c_{43}z(\theta_l\omega))\,dl} M_5(\theta_s\omega)\,ds\|h\|_{2p}^2 \\ &\le c\left(e^{b\int_0^t z(\theta_s\omega)\,ds} + \int_0^t e^{-b\int_t^s z(\theta_l\omega)\,dl} e^{\int_0^s (c_{42}+c_{43}z(\theta_l\omega))\,dl} M_5(\theta_s\omega)\,ds\right)\|\nabla h\|^2 \\ &\le c\left(e^{b\int_0^t |z(\theta_s\omega)|\,ds} + e^{b\int_0^t |z(\theta_s\omega)|\,ds} e^{\int_0^t (c_{42}+c_{43}|z(\theta_l\omega)|)\,dl} \int_0^t M_5(\theta_s\omega)\,ds\right)\|\nabla h\|^2 \\ &\le c\left(e^{b\int_0^t |z(\theta_s\omega)|\,ds} + e^{b\int_0^t |z(\theta_s\omega)|\,ds} e^{\int_0^t (c_{42}+c_{43}|z(\theta_l\omega)|)\,dl} e^{2\int_0^t M_5(\theta_s\omega)\,ds}\right)\|\nabla h\|^2 \\ &\le c e^{\int_0^t (2b|z(\theta_s\omega)|+c_{42}+c_{43}|z(\theta_s\omega)|+2M_5(\theta_s\omega))\,ds}\|\nabla h\|^2 = c_{44} e^{\int_0^t C_1(\theta_s\omega)\,ds}\|\nabla h\|^2,\end{aligned} \tag{3.61}$$

with

$$C_1(\omega) = 2b\left|z(\omega)\right| + c_{42} + c_{43}\left|z(\omega)\right| + 2M_5(\omega). \tag{3.62}$$

Similarly, we take the inner product of (3.26) with $-\Delta w_2$ to get

$$\frac{d}{dt}\|\nabla w_2\|^2 + \frac{1}{2}\|\Delta w_2\|^2 \le M_5(\theta_t\omega)\|w\|_{2p}^2 + bz(\theta_t\omega)\|\nabla w_2\|^2.$$

By applying Poincaré inequality

$$\|\Delta v_2\|^2 \ge \lambda_{m+1}\|\nabla v_2\|^2, \quad \forall v \in D(-\Delta),$$

we get

$$\frac{d}{dt}\|\nabla w_2\|^2 + \frac{\lambda_{m+1}}{2}\|\nabla w_2\|^2 \le M_5(\theta_t\omega)\|w\|_{2p}^2 + bz(\theta_t\omega)\|\nabla w_2\|^2,$$

Thus

$$\left\|\nabla w_2(t)\right\|^2 \le e^{-\frac{\lambda_{m+1}}{2}t}\left\|\nabla w_2(0)\right\|^2 + e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} M_5(\theta_s\omega)\left\|w(s)\right\|_{2p}^2\,ds$$

$$
\begin{aligned}
&+ e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} bz(\theta_s\omega)\|\nabla w_2(s)\|^2\,ds\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} M_5(\theta_s\omega)\|w(s)\|_{2p}^2\,ds\\
&+ e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} bz(\theta_s\omega)\|\nabla w(s)\|^2\,ds.
\end{aligned}
\tag{3.63}
$$

To estimate $\|\nabla w(t)\|^2$, we multiply (3.26) by $-\Delta w$ and using a similar inequality as (3.58) to get

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla w\|^2 + \frac{1}{2}\|\Delta w\|^2\\
&\le M_5(\theta_t\omega)\|w\|_{2p}^2 + bz(\theta_t\omega)\|\nabla w\|^2 \le \big(bz(\theta_t\omega) + cM_5(\theta_t\omega)\big)\|\nabla w\|^2.
\end{aligned}
$$

Thus

$$
\|\nabla w\|^2 \le e^{\int_0^t (bz(\theta_s\omega)+cM_5(\theta_s\omega))\,ds}\|\nabla h\|^2.
\tag{3.64}
$$

From (3.31), (3.63), and (3.64), we obtain

$$
\begin{aligned}
&\|\nabla w_2(t)\|^2\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} M_5(\theta_s\omega) e^{\int_0^s (c_{42}+c_{43}z(\theta_l\omega))\,dl}\,ds\|h\|_{2p}^2\\
&+ e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} bz(\theta_s\omega) e^{\int_0^s (bz(\theta_l\omega)+cM_5(\theta_l\omega))\,dl}\,ds\|\nabla h\|^2\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + ce^{\int_0^t (c_{42}+c_{43}|z(\theta_s\omega)|)\,ds} e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} M_5(\theta_s\omega)\,ds\|\nabla h\|^2\\
&+ e^{\int_0^t (b|z(\theta_l\omega)|+cM_5(\theta_l\omega))\,dl} e^{-\frac{\lambda_{m+1}}{2}t}\int_0^t e^{\frac{\lambda_{m+1}}{2}s} bz(\theta_s\omega)\,ds\|\nabla h\|^2\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + ce^{\int_0^t (c_{42}+c_{43}|z(\theta_l\omega)|)\,ds}\\
&\times e^{-\frac{\lambda_{m+1}}{2}t}\left(\int_0^t e^{\lambda_{m+1}s}\,ds\right)^{\frac{1}{2}}\left(\int_0^t M_5^2(\theta_s\omega)\,ds\right)^{\frac{1}{2}}\|\nabla h\|^2\\
&+ e^{\int_0^t (b|z(\theta_l\omega)|+cM_5(\theta_l\omega))\,dl} e^{-\frac{\lambda_{m+1}}{2}t}\left(\int_0^t e^{\lambda_{m+1}s}\,ds\right)^{\frac{1}{2}}\left(\int_0^t b^2z^2(\theta_s\omega)\,ds\right)^{\frac{1}{2}}\|\nabla h\|^2\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + ce^{\int_0^t (c_{42}+c_{43}|z(\theta_s\omega)|)\,ds}\frac{1}{\sqrt{\lambda_{m+1}}}\left(\int_0^t M_5^2(\theta_s\omega)\,ds\right)^{\frac{1}{2}}\|\nabla h\|^2\\
&+ e^{\int_0^t (b|z(\theta_l\omega)|+cM_5(\theta_l\omega))\,dl}\frac{1}{\sqrt{\lambda_{m+1}}}\left(\int_0^t b^2z^2(\theta_s\omega)\,ds\right)^{\frac{1}{2}}\|\nabla h\|^2\\
&\le e^{-\frac{\lambda_{m+1}}{2}t}\|\nabla h\|^2 + \frac{c_{45}}{\sqrt{\lambda_{m+1}}} e^{c_{46}\int_0^t (1+|z(\theta_s\omega)|+z^2(\theta_s\omega)+M_5(\theta_s\omega)+M_5^2(\theta_s\omega))\,ds}\|\nabla h\|^2\\
&\le e^{-t}\|\nabla h\|^2 + \delta e^{\int_0^t C_2(\theta_s\omega)\,ds}\|\nabla h\|^2,
\end{aligned}
\tag{3.65}
$$

where

$$\delta = \frac{c_{45}}{\sqrt{\lambda_{m+1}}}, C_2(\omega) = c_{46}\big(1 + \big|z(\omega)\big| + z^2(\omega) + M_5(\omega) + M_5^2(\omega)\big). \tag{3.66}$$

Setting

$$C_0(\omega) = C_1(\omega) + C_2(\omega), \tag{3.67}$$

then from (3.50) and (3.51), one can check that $\mathbb{E}[C_0^2(\omega)] < \infty$. Since (3.61) and (3.65) hold for any $t > 0$, we first choose $t_0 \geq \ln 4$. Then fix m large enough such that $0 < \delta \leq \frac{1}{8}e^{-\frac{2}{\ln\frac{3}{2}}t_0^2\mathbb{E}[C_0^2(\omega)]}$, thus (3.61) and (3.65) implies that $\phi(t,\omega)$ satisfies (II) in Theorem 2.2 in [34]. Thanks to the case (II) in Theorem 2.2 in [34], we have the following covering:

$$A_{2p}(\omega) \subset \bigcup_{j=1}^{n_1 n_2 \cdots n_k} B_{H_0^1(D)}\big(u_{0k}^j; e^{-\frac{k}{2}\ln\frac{4}{3}} b_\omega\big), \tag{3.68}$$

where $B_{H_0^1(D)}(u_{0k}^j; e^{-\frac{k}{2}\ln\frac{4}{3}} b_\omega)$ denotes the ball in $H_0^1(D)$, $\mathbb{E}[\ln b_\omega] < \infty$ and

$$n_l \leq \left(\frac{\sqrt{m}}{\delta} + 1\right)^m, \quad l = 1, 2 \cdots k. \tag{3.69}$$

Since $\|u\|_{2p} \leq \widetilde{c}\|u\|_{H_0^1(D)}$, (3.68) implies

$$A_{2p}(\omega) \subset \bigcup_{j=1}^{n_1 n_2 \cdots n_k} B\big(u_{0k}^j; \widetilde{c}e^{-\frac{k}{2}\ln\frac{4}{3}} b_\omega\big), \tag{3.70}$$

where $B(u_{0k}^j; \widetilde{c}e^{-\frac{k}{2}\ln\frac{4}{3}} b_\omega)$ denotes the ball in $L^{2p}(D)$.

Since $e^{-\frac{k}{2}\ln\frac{4}{3}} b_\omega \to 0$, $k \to \infty$, there exists k_ω such that

$$\widetilde{c}e^{-\frac{k_\omega}{2}\ln\frac{4}{3}} b_\omega < r_{0,\omega} \leq \widetilde{c}e^{-\frac{k_\omega - 1}{2}\ln\frac{4}{3}} b_\omega, \tag{3.71}$$

This implies

$$k_\omega = \left[\frac{2(\ln\widetilde{c} + \ln b_\omega - \ln r_{0,\omega})}{\ln\frac{4}{3}}\right] + 1, \tag{3.72}$$

here $[\cdot]$ denotes the greatest integer function. Using (3.50), (3.51), and the inequality $\sqrt{x} \leq e^x$, one can show that

$$\mathbb{E}\left[\ln \int_0^T M_3(\theta_s\omega) e^{-\int_T^s (c_{26} + c_{27} z(\theta_l\omega))\, dl} ds\right] < +\infty.$$

By the above inequality and the definition of $r_{0,\omega}$ in (3.44) and (3.47), we can get $\mathbb{E}[\ln r_{0,\omega}] < \infty$. Therefore, (3.72) implies $\mathbb{E}[k_\omega] < \infty$. Combining (3.69)∼(3.71), we have

$$N_{r_{0,\omega}}\big(A_{2p}(\omega)\big) \leq n_1 \cdots n_{k_\omega} \leq \left(\frac{\sqrt{m}}{\delta} + 1\right)^{m k_\omega}.$$

Therefore, from (3.49) we have

$$\ln N_{\omega} \leq \ln N_{r_{0,\omega}}\big(A_{2p}(\omega)\big) \leq mk_{\omega} \ln\left(\frac{\sqrt{m}}{\delta} + 1\right), \tag{3.73}$$

this suggests that $\mathbb{E}[\ln N_{\omega}] < \infty$. The proof is completed. □

It is easy to check the assumption (**H3**), that is, the Lipschitz continuity for $\phi(t,\omega)$ in $L^{2p}(D)$ for any $t > 0$. From Theorem 3.1, Lemma 3.3, Lemma 3.4, Lemma 3.6 and the construction of $\chi(\omega)$ in (3.48), we see that $\phi(t,\omega)$ and $\chi(\omega)$ satisfy (**H0**)∼(**H5**). Therefore, as a consequence of Theorem 2.1, we have:

Theorem 3.2 *Suppose* (1.3)∼(1.5) *hold. Then the RDS* $\{\phi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ *defined in* (3.6) *possesses a random exponential attractor* $\{\mathcal{E}(\omega)\}_{\omega\in\Omega}$ *in* $L^{2p}(D)$.

Remark By the definition $\phi(t,\omega,v_0) = v(t,\omega,v_0)$, $\Phi(t,\omega,u_0) = u(t,\omega,u_0)$ and the relationship $u(t,\omega,u_0) = \alpha^{-1}(\theta_t\omega)v(t,\omega,\alpha(\omega)u_0)$, one can immediately get that the RDS $\{\Phi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$, which is generated by (1.1)∼(1.5), possesses a random exponential attractor in $L^{2p}(D)$.

4 Conclusion

In this paper, we have studied the asymptotic behavior of the RDS $\phi(t,\omega)$ generated by (3.3)–(3.5). First, an abstract result for the existence of a random exponential attractor is established in general Banach space. Second, a useful asymptotic a priori estimate in $L^{6p}(D)$ is given. Third, a positively invariant random set $\chi(\omega)$ in $L^{2p}(D)$ is constructed and the Fréchet differentiability of $\phi(T,\omega)$ in $\chi(\omega)$ is proved for a large time T. Then $\phi(T,\omega)$ is split into two parts, i. e. $W_1(T,\omega)$ and $W_2(T,\omega)$, and $W_1(T,\omega)$ is proved to be contractive in the mean in $L^{2p}(D)$ and $W_2(T,\omega)$ to be compact in $L^{2p}(D)$. Finally, by checking the assumptions (**H0**)∼(**H5**) presented in the abstract result for $\phi(t,\omega)$ and $\chi(\omega)$, the existence of a random exponential attractor is proved in $L^{2p}(D)$.

It is worth noticing that our case is different from that of [18]. In [18], the author proved the existence of a random exponential attractor for a stochastic non-autonomous reaction-diffusion equation with multiplicative white noise in the entire space $\mathbb{R}^3$. The author decomposed the solutions into two parts, of whose, one part is finite-dimensional which satisfies the flattening propety [11] and the "tail" part is "quickly decay" for suitable large $x \in \mathbb{R}^3$ and large time t. This implies the existence of a finite dimensional random exponential attractor in the Hilbert space $L^2(\mathbb{R}^3)$. However, the technique relies on the orthogonal basis $\{e_j\}_{j=1}^{\infty}$ in $L^2 \bigcap H_0^1$ and the orthogonal projections $P_n : L^2(D) \to X_n$, where X_n = span $\{e_1, \dots, e_n\}$, so that it cannot be applied directly to general Banach space.

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Declarations

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The authors declare no competing interests.

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