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Rich dynamics of a delayed SIRS epidemic model with two-age structure and logistic growth

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Abstract

This paper studies a two-age structured SIRS epidemic model with logistic growth of susceptible population and two-time delays. We simultaneously introduce two-time delays, i.e., the immunity and incubation periods, into this dynamic system and investigate their impact on different dynamic behaviors for the model. By means of the C_0-semigroup theory, the model is transformed into a non-densely defined abstract Cauchy problem, and the condition of the existence and uniqueness of the endemic equilibrium is obtained. Following the spectral analysis, the characteristic equation technique, and the Hopf bifurcation theorem, we show that different combinations of the two delays perform a vital role in the instability/stability as well as the Hopf bifurcation results of equilibrium solutions. We numerically provide some graphical representations to check the main theoretical results and show the rich dynamics by varying the two delay parameters.

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1 Introduction

Mathematical models are fundamental tools for understanding the spread and control of epidemic diseases. Since the ground-breaking work of Kermack and McKendrick [1], the susceptible-infected-recovered (SIR) model has served as the foundational mathematical theory for the spread of infectious diseases in populations and applied to specific diseases, such as measles, malaria, cholera, seasonal flu, COVID-19, and so on [2–7]. The classical SIR model put forth by Kermack and McKendrick [1] divided the population into three classes named susceptible population S, infected population I, and recovered population R. Assume that the immunity acquired following recovery is temporary, and the SIRS model can be written as below [8]:

$$\begin{cases} \frac{dS(t)}{dt} = \alpha - dS(t) - kI(t)S(t) + mR(t), \\ \frac{dI(t)}{dt} = kI(t)S(t) - (d+\gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - (d+m)R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ are the number of individuals in the corresponding classes at time t, and the model coefficients are all positive. In this model, α represents the constant input rate of new susceptible population; d is the common rate at which the individuals die naturally; γ stands for the recovery rate of infected population; the recovered population has a rate m of losing immunity and going back to the S class; the bilinear term kIS called the incidence rate, and the constant k means the transmission rate at which the disease spreads between the individuals in S and I classes.

However, there are two common features of the epidemic disease that the classical SIRS model (1.1) does not capture. First, in model (1.1), the input of the susceptible individuals is typically assumed to be constant. However, for a disease with a high mortality rate or a relatively long duration, it may be more reasonable to assume the logistic growth input of the susceptible individuals in some practical circumstances [9–11]. Li et al. [10] considered the logistic growth rate and saturated treatment, as well as bilinear incidence in an SIR disease model, and investigated the stability and bifurcation analysis of the model. In [11], Avila-Vales and Pérez studied an SIR vector-borne disease model with logistic growth, nonlinear incidence rate, Holling type II saturated treatment, and latency time delay. They revealed the existence of Bogdanov-Takens bifurcation and backward bifurcation. Indeed, many SIR/SIRS models and the approximation models have investigate the varying total population problem with a logistic equation recently, and as an example, we mention the studies [12–14] and the references therein.

In addition, we note that both the transmission and recovery coefficients in model (1.1) are constant, which means the infected person's infectivity is the same during their periodic infection. Over the last decades, many more advanced population models added age structure to the approximations of the classic SIR/SIRS model (1.1), and as an example, we mention the studies [12, 15–26] and the references therein. Numerous empirical studies have highlighted the necessity of considering age structure in epidemic prediction [27, 28]. Indeed, age is one of the essential factors in the spreading, controlling, preventing, and modeling of epidemic diseases since different age groups may experience different mortality rates and infection rates for the same disease [29–31]. Considering a nonconstant transmission rate where infected individuals transmit disease to susceptible ones differently depending on infection age (time passed since infection) and a constant input of susceptible population, Magal et al. [23] investigated the following model:

$$\begin{cases} \frac{dS(t)}{dt} = \alpha - dS(t) - \eta S(t)\int_0^{+\infty}\beta(a)i(t,a)\,da, \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\nu_I(a)i(t,a), \\ i(t,0) = \eta S(t)\int_0^{+\infty}\beta(a)i(t,a)\,da, \\ S(0) = S_0 \geq 0, \qquad i(0,\cdot) = i_0 \in L^1_+(0,+\infty), \end{cases}$$

where $i(t,a)$ is the density of the infected population at time t with infection age a; the infection age-dependent functions $\beta(a)$ and $\nu_I(a)$ are the transmission rate and mortality rate of the infected population. They pioneered the Lyapunov functional approach to show that the endemic equilibrium is globally asymptotically stable. Recent works have generalized this model of infection-age type, and we mention here, for example, [32–36].

In the case of SIRS models, the removed population in the R class can acquire a certain immunity period before losing the preservation to be susceptible. Regarding age as a continuous variable, Duan et al. [37] developed the following SIRS disease system with a recovery age structure to describe the immune protection process:

$$\begin{cases} \frac{dS(t)}{dt} = \alpha - dS(t) - kS(t)I(t) + \int_0^{\infty} m(a)r(t,a)\,da, \\ \frac{dI(t)}{dt} = kS(t)I(t) - (d+\mu+\gamma)I(t), \\ \frac{\partial r(t,a)}{\partial t} + \frac{\partial r(t,a)}{\partial a} = -(d+m(a))r(t,a), \\ r(t,0) = \gamma I(t), \\ S(0) = S_0, \qquad I(0) = I_0, \qquad r(0,a) = r_0(a) \in L^1_+(0,+\infty), \end{cases}$$

where $r(t,a)$ stands for the density of recovered population at time t with a recovery age a; the recovery age-dependent function $m(a)$ represents the progression rate of the removed population to the susceptible one; μ is the mortality rate caused by the disease. Using the Hopf bifurcation theorem, they showed that a local Hopf bifurcation exists under certain conditions. Several recent studies depend on such types of models, see, for example, [30, 38–41].

To extend the above age-structured models with constant input of S class where single infection age or recovery age is considered, this paper further investigates the dynamics of an age-structured SIRS model that integrates both two ages of infection and recovery (describing the latent and immunity periods) and also a logistic growth of susceptible population as below:

$$\begin{cases} \frac{dS(t)}{dt} = bS(t)(1-\frac{S(t)}{K}) - S(t)\int_0^{+\infty} \beta(a)i(t,a)\,da + \int_0^{+\infty} m(a)r(t,a)\,da, \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -(d_1+\mu+\gamma(a))i(t,a), \quad t\geq 0, a\geq 0, \\ \frac{\partial r(t,a)}{\partial t} + \frac{\partial r(t,a)}{\partial a} = -(d_2+m(a))r(t,a), \quad t\geq 0, a\geq 0, \\ i(t,0) = S(t)\int_0^{+\infty} \beta(a)i(t,a)\,da, \quad t\geq 0, \\ r(t,0) = \int_0^{+\infty} \gamma(a)i(t,a)\,da, \quad t\geq 0, \end{cases} \tag{1.2}$$

with the initial condition

$$S(0) = S_0 \geq 0, \qquad i(0,a) = i_0(a) \in L^1_+(0,+\infty), \qquad r(0,a) = r_0(a) \in L^1_+(0,+\infty).$$

In our model (1.2), the term $bS(t)(1-\frac{S(t)}{K})$ is the common logistic growth of susceptible individuals; $b = \alpha - d$ means the intrinsic growth rate; K means the carrying capacity of susceptible individuals; d_1 and d_2 are the natural mortality rates of infected and recovery individuals, respectively; $\gamma(a)$ means the recovery rate of the infected population with age a. We further assume the functions $\beta(a), \gamma(a), m(a) \in L^{\infty}_+(\mathbb{R}_+)$, with an essential upper bound β_*, γ_*, and m_*, respectively.

To the best of our knowledge, there are no prior age-structured SIRS models considering both infection and recovery age as well as the logistic growth of susceptible population. In this paper, we concentrate on investigating the impact of both the latent and immunity periods (described by infection and recovery ages) on the rich dynamics of system (1.2). We organize the rest as below. Section 2 gives the preliminaries and establishes the well-posedness result of system (1.2) transformed into a non-densely defined abstract Cauchy problem. In Sect. 3, the linearized system, the basic reproduction number, and the existence of the equilibria are obtained. We calculate the characteristic equation and show that the disease-free equilibrium is locally/globally asymptotically stable in Sect. 4. Section 5 investigates the stability and Hopf bifurcation results for the epidemic equilibrium under different combinations of latent and immunity periods described by two delays. Section 6 provides some graphical representations to check the obtained theoretical results. We conclude in Sect. 7.

2 Preliminaries and well-posedness

This part deals with rewriting model (1.2) into an abstract Cauchy problem on a suitable Banach lattice, and the well-posedness results will be established for (1.2). Before starting our discussion, we first gather some background information on linear operator, C_0-semigroup theory, and some notations.

Let $A : D(A) \subset Y \to Y$ be a linear operator on a Banach space Y. The resolvent set, spectrum, and point spectrum of operator A are denoted by $\rho(A)$, $\sigma(A) = \mathbb{C} \setminus \rho(A)$, and $\sigma_p(A) := \{\lambda \in \mathbb{C} : N(\lambda I - A) \neq \{0\}\}$, respectively.

Definition 2.1 (see [42]) Assume that $A : D(A) \subseteq Y \to Y$ is a linear operator, and there are real constants $H \geq 1$, and $\xi \in \mathbb{R}$, satisfying $(\xi, +\infty) \subseteq \rho(A)$ and

$$\left\|(\lambda - A)^{-n}\right\| \leq \frac{H}{(\lambda - \xi)^n}, \quad \text{for } n \in \mathbb{N}_+, \lambda > \xi.$$

The operator $(A, D(A))$ is then known as a Hille-Yosida operator.

The perturbation result for a Hille-Yosida operator is as follows.

Lemma 2.1 (see [42, 43]) *If $(A, D(A))$ is a Hille-Yosida operator on Y, and B is a bounded linear operator on Y, then the sum $C = A + B$ is also a Hille-Yosida operator.*

Let

$$\begin{aligned} &Y_0 := \left(\overline{D(A)}, \|\cdot\|\right), \\ &D(A_0) := \left\{x \in D(A) : Ax \in Y_0\right\}, \\ &A_0 x := Ax, \quad \text{for } x \in D(A_0). \end{aligned}$$

Then, $(A_0, D(A_0))$ is known as the part of A in Y_0, and the lemma follows.

Lemma 2.2 (see [42, 43]) *If $(A, D(A))$ is a Hille-Yosida operator, then its part $(A_0, D(A_0))$ generates a C_0-semigroup $(S_0(t))_{t\geq 0}$ on Y_0.*

Denote

$$Y = \mathbb{R} \times L^1\big((0, +\infty), \mathbb{R}\big) \times L^1\big((0, +\infty), \mathbb{R}\big) \times \mathbb{R} \times \mathbb{R}.$$

The linear operator $\mathcal{F}: D(\mathcal{F}) \subseteq Y \longrightarrow Y$ is defined as below:

$$\mathcal{F}\begin{pmatrix} x \\ y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -dx \\ -y_1'(a) - (d_1 + \mu + \gamma(a))y_1 \\ -y_2'(a) - (d_2 + m(a))y_2 \\ -y_1(0) \\ -y_2(0) \end{pmatrix},$$

where $D(\mathcal{F}) = \mathbb{R} \times W^{1,1}((0,+\infty),\mathbb{R}) \times W^{1,1}((0,+\infty),\mathbb{R}) \times \{0\} \times \{0\}$. So $\overline{D(\mathcal{F})} = \mathbb{R} \times L^1((0,+\infty),\mathbb{R}) \times L^1((0,+\infty),\mathbb{R}) \times \{0\} \times \{0\}$ is not dense on Y. Further, a nonlinear operator $\mathcal{L}: \overline{D(\mathcal{F})} \to Y$ is introduced as follows:

$$\mathcal{L}\begin{pmatrix} x \\ y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha x(1 - \frac{x}{K}) + \frac{d}{K}x^2 - x\int_0^{+\infty} \beta(a)y_1(a)\,da + \int_0^{+\infty} m(a)y_2(a)\,da \\ 0 \\ 0 \\ x\int_0^{+\infty} \beta(a)y_1(a)\,da \\ \int_0^{+\infty} \gamma(a)y_1(a)\,da \end{pmatrix},$$

here $b = \alpha - d$, recall that α and d represent the recruitment rate and natural mortality rate of the suspectable populations, respectively. Put $v(t) = (S(t), i(t,\cdot), r(t,\cdot), 0, 0)^\top$ so that model (1.2) can be reformulated as an abstract Cauchy problem (ACP):

$$\begin{cases} \frac{d}{dt}(v(t)) = \mathcal{F}v(t) + \mathcal{L}v(t), & t \geq 0, \\ v(0) = v_0, \end{cases} \tag{2.1}$$

where $v_0 = (S_0, i_0(a), r_0(a), 0, 0)^\top$.

Generally speaking, finding a strong solution to equation (2.1) is challenging. Therefore, we find a weak solution of (2.1) as integrated form

$$v(t) = v_0 + \mathcal{F}\int_0^t v(s)\,ds + \int_0^t \mathcal{L}\big(v(s)\big)\,ds. \tag{2.2}$$

Let

$$Y_0 = \overline{D(\mathcal{F})} = \mathbb{R} \times L^1\big((0,+\infty),\mathbb{R}\big) \times L^1\big((0,+\infty),\mathbb{R}\big) \times \{0\} \times \{0\},$$
$$Y_{0+} = \mathbb{R}_+ \times L^1_+\big((0,+\infty),\mathbb{R}\big) \times L^1_+\big((0,+\infty),\mathbb{R}\big) \times \{0\} \times \{0\},$$

and indicate that $\xi := \min\{d, d_1, d_2\} > 0$, $\Upsilon := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\xi\}$. Then the following result can be obtained.

Theorem 2.1 *The linear operator* $(\mathcal{F}, D(\mathcal{F}))$ *is a Hille-Yosida operator.*

Proof Suppose that $(\delta,\varphi_1,\varphi_2,\phi_1,\phi_2)\in Y$, $(\tilde\delta,\tilde\varphi_1,\tilde\varphi_2,0,0)\in D(\mathcal{F})$, $\lambda\in\Upsilon$, then it implies

$$(\lambda-\mathcal{F})^{-1}\begin{pmatrix}\delta\\ \varphi_1\\ \varphi_2\\ \phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}\tilde\delta\\ \tilde\varphi_1\\ \tilde\varphi_2\\ 0\\ 0\end{pmatrix}\quad\Leftrightarrow\quad\begin{cases}(\lambda+d)\tilde\delta=\delta,\\ \tilde\varphi_1'=-(\lambda+d_1+\mu+\gamma(a))\tilde\varphi_1+\varphi_1,\\ \tilde\varphi_2'=-(\lambda+d_2+m(a))\tilde\varphi_2+\varphi_2,\\ \tilde\varphi_1(0)=\phi_1,\\ \tilde\varphi_2(0)=\phi_2.\end{cases}$$

Hence, we have

$$\begin{cases}\tilde\delta=\frac{1}{(\lambda+d)}\delta,\\ \tilde\varphi_1=e^{-\int_0^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}\phi_1+\int_0^a e^{-\int_s^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}\varphi_1(s)\,ds,\\ \tilde\varphi_2=e^{-\int_0^a(\lambda+d_2+m(\theta))\,d\theta}\phi_2+\int_0^a e^{-\int_s^a(\lambda+d_2+m(\theta))\,d\theta}\varphi_2(s)\,ds.\end{cases}\tag{2.3}$$

Integrating $\tilde\varphi_1$ and $\tilde\varphi_2$ about the variable a and adding all equations of (2.3) yields

$$|\tilde\delta|+\|\tilde\varphi_1\|_{L^1}+\|\tilde\varphi_2\|_{L^1}\le\frac{1}{\lambda+\xi}\big(|\delta|+\|\varphi_1\|_{L^1}+\|\varphi_2\|_{L^1}+|\phi_1|+|\phi_2|\big).$$

Thus,

$$\big\|(\lambda-\mathcal{F})^{-1}\big\|\le\frac{1}{\lambda+\xi},\quad\text{for any }\lambda\in\Upsilon.$$

As a result, $(\mathcal{F},D(\mathcal{F}))$ is a Hille-Yosida operator. □

In terms of Lemma 2.2, $(\mathcal{F},D(\mathcal{F}))$ generates a C_0-semigroup on Y_0. Therefore, system (2.1) is well-posed according to the theorem below.

Theorem 2.2 *For any $v_0\in Y_{0+}$, model (1.2) has a unique continuous solution denoted by the integrated form (2.2) with values in Y_{0+}. Furthermore, the map $T:[0,+\infty)\times Y_{0+}\mapsto Y_{0+}$ defined by $T(t,v_0)=v(t,v_0)$ is a continuous semi-flow, i.e. the map T is continuous and satisfies the condition that $T(0,\cdot)$ is the identity map and $T(t,T(s,\cdot))=T(t+s,\cdot)$ on Y_{0+}.*

It is worthwhile to only consider nonnegative solutions due to the biological meaning of system (1.2). So, we turn to discuss the positivity and boundedness of the solutions for system (1.2).

Theorem 2.3 *Under nonnegative initial condition, all solutions of system (1.2) are nonnegative for all $t\ge0$ and are ultimately bounded.*

Proof First, the positivity of $i(t,a)$ and $r(t,a)$ are proved. Solving $i(t,a)$ and $r(t,a)$ through integrating the second and third equations of system (1.2) along the characteristic lines respectively, we get

$$i(t,a)=\begin{cases}i(t-a,0)e^{-\int_0^a(d_1+\mu+\gamma(\theta))\,d\theta},& a\le t,\\ i_0(a-t)e^{-\int_{a-t}^a(d_1+\mu+\gamma(\theta))\,d\theta},& a>t,\end{cases}\tag{2.4}$$

and

$$r(t,a)=\begin{cases} r(t-a,0)e^{-\int_0^a (d_2+m(\theta))\,d\theta}, & a\le t,\\ r_0(a-t)e^{-\int_{a-t}^a (d_2+m(\theta))\,d\theta}, & a>t.\end{cases}$$

Obviously, $i(t,a)$ and $r(t,a)$ remain nonnegative with nonnegative initial values. Now, we show the positivity of $S(t)$. Actually, if there is t_o, s.t. $S(t_o)=0$, and $S(t)>0$ for any $t\in(0,t_o)$, from the first equation of model (1.2), we get $S'(t_o)=\int_0^{+\infty} m(a)i(t_o,a)\,da\ge 0$, which is contradictory. Consequently, $S(t)\ge 0$, for any $t\ge 0$.

Next, we explore the ultimate boundness of the solutions for model (1.2). To this end, set $I(t)=\int_0^{+\infty} i(t,a)\,da$ and $R(t)=\int_0^{+\infty} r(t,a)\,da$, which represent the total number of infected and recovery individuals, respectively, at time t. Biologically, the maximum age should be finite, and it is rational to make the assumption $\lim_{a\to+\infty} i(t,a)=0$ and $\lim_{a\to+\infty} r(t,a)=0$. Further, denote $N(t)=: S(t)+I(t)+R(t)$, then based on model (1.2), we get

$$\begin{aligned} N'(t) &= bS(t)\left(1-\frac{S(t)}{K}\right)-i(t,0)+\int_0^{+\infty} m(a)r(t,a)\,da\\ &\quad+\int_0^{+\infty}\left(-\frac{\partial i(t,a)}{\partial a}-\big(d_1+\mu+\gamma(a)\big)i(t,a)\right)da\\ &\quad+\int_0^{+\infty}\left(-\frac{\partial r(t,a)}{\partial a}-\big(d_2+m(a)\big)r(t,a)\right)da\\ &= bS(t)\left(1-\frac{S(t)}{K}\right)-(d_1+\mu)\int_0^{+\infty} i(t,a)\,da-d_2\int_0^{+\infty} r(t,a)\,da\\ &\le(\alpha-d)S(t)-d_1I(t)-d_2R(t)\\ &\le \alpha K-\min\{d,d_1,d_2\}N(t). \end{aligned}$$

Therefore,

$$\limsup_{t\to+\infty} N(t)\le\frac{\alpha K}{\min\{d,d_1,d_2\}},$$

which implies that $S(t)$, $i(t,a)$, and $r(t,a)$ are ultimately bounded. Furthermore, we obtain the bounded feasible region of model (1.2):

$$\Gamma=\left\{(S,i,r): S\ge 0, i,r\in L_+^1(0,+\infty),\right.\\ \left. S+\int_0^{+\infty} i(t,a)\,da+\int_0^{+\infty} r(t,a)\,da\le\frac{\alpha K}{\min\{d,d_1,d_2\}}\right\},$$

which contains the omega limit set of (1.2) and is obviously positively invariant according to model (1.2). □

3 Equilibriums and linearized system

This part first investigates the existence of steady states for system (1.2) to obtain the linearized system of (1.2) around these steady states.

Obviously, the trivial steady state (disease-free equilibrium) $E_0 = (\overline{S}_0, 0, 0) = (K, 0, 0)$ of the system (1.2) is always existed. Now, we turn to explore the positive steady state $E_* = (S_*, i_*(a), r_*(a))$ of (1.2). Set

$$\begin{cases} bS_*(1 - \frac{S_*}{K}) - S_* \int_0^{+\infty} \beta(a) i_*(a)\, da + \int_0^{+\infty} m(a) r_*(a)\, da = 0, \\ \frac{di_*(a)}{da} = -(d_1 + \mu + \gamma(a)) i_*(a), \\ \frac{dr_*(a)}{da} = -(d_2 + m(a)) r_*(a), \\ i_*(0) = S_* \int_0^{+\infty} \beta(a) i_*(a)\, da, \\ r_*(0) = \int_0^{+\infty} \gamma(a) i_*(a)\, da. \end{cases} \tag{3.1}$$

We solve the equations about $i_*(a)$ and $r_*(a)$ in (3.1), which implies that

$$i_*(a) = i_*(0) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}, \qquad r_*(a) = r_*(0) e^{-\int_0^a (d_2 + m(\theta))\, d\theta}.$$

Substituting $i_*(a)$ into the fourth equation of the system (3.1), we obtain

$$S_* = \frac{1}{\int_0^{+\infty} \beta(a) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}\, da}. \tag{3.2}$$

Combining the above three formulations about $i_*(a)$, $r_*(a)$, S_* and the first equation in (3.1), we get

$$i_*(0) = \frac{bS_*(1 - \frac{S_*}{K})}{P},$$

where

$$P = 1 - \int_0^{+\infty} m(a) e^{-\int_0^a (d_2 + m(\theta))\, d\theta}\, da \cdot \int_0^{+\infty} \gamma(a) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}\, da.$$

With the assumptions about $m(a)$ and $\gamma(a)$ in the Introduction section, we can see that $0 \leq \int_0^{+\infty} m(a) e^{-\int_0^a (d_2 + m(\theta))\, d\theta}\, da < \int_0^{+\infty} m(a) e^{-\int_0^a m(\theta)\, d\theta}\, da = 1 - e^{-\int_0^{+\infty} m(\theta)\, d\theta} < 1$, and similarly, $0 \leq \int_0^{+\infty} \gamma(a) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}\, da < 1$. Therefore, we have $P > 0$.

Consequently, we define the basic reproduction number R_0 as

$$R_0 := \frac{K}{S_*} = K \int_0^{+\infty} \beta(a) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}\, da.$$

Thus,

$$i_*(0) = \frac{bS_*(1 - \frac{1}{R_0})}{P}.$$

Hence, when $R_0 > 1$, we have $i_*(0) > 0$, which leads to

$$r_*(0) = i_*(0) \int_0^{+\infty} \gamma(a) e^{-\int_0^a (d_1 + \mu + \gamma(\theta))\, d\theta}\, da > 0.$$

Therefore, there is only one positive equilibrium for system (1.2). In particular, we arrive at

Theorem 3.1 *The disease-free equilibrium $E_0 = (\overline{S}_0, 0, 0)$ always exists for system (1.2). When $R_0 > 1$, system (1.2) has a unique positive equilibrium $E_* = (S_*, i_*(a), r_*(a))$.*

Then, set $S(t) = x(t) + \overline{S}$, $i(t,a) = y_1(t,a) + \overline{i}(a)$, $r(t,a) = y_2(t,a) + \overline{r}(a)$, where $\overline{E} = (\overline{S}, \overline{i}(a), \overline{r}(a))$ is any equilibrium of model (1.2), and let $\tilde{u}(t) = (x(t), y_1(t,a), y_2(t,a), 0, 0)$, $\bar{u} = (\overline{S}, \overline{i}(a), \overline{r}(a), 0, 0)$. Consequently, system (2.1) corresponds to the below system

$$\begin{cases} \frac{d}{dt}\tilde{u}(t) = \mathcal{F}\tilde{u}(t) + \mathcal{L}(\tilde{u}(t) + \bar{u}) - \mathcal{L}(\bar{u}(t)), & t \geq 0, \\ \tilde{u}(0) = u(0) - \bar{u}, \end{cases}$$

Direct calculations yield the following linearized system of (2.1) around the equilibrium $\bar{u}$

$$\begin{cases} \frac{d}{dt}\tilde{u}(t) = \mathcal{F}\tilde{u}(t) + D\mathcal{L}(\bar{u})(\tilde{u}(t)), & t \geq 0, \\ \tilde{u}(0) = u(0) - \bar{u}, \end{cases} \tag{3.3}$$

with

$$D\mathcal{L}(\bar{u}) \begin{pmatrix} x(t) \\ y_1(t,a) \\ y_2(t,a) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} [b(1 - \frac{2\overline{S}}{K}) - \int_0^{+\infty} \beta(a)\overline{i}(a)\,da]x(t) - \overline{S}\int_0^{+\infty} \beta(a)y_1(t,a)\,da + \int_0^{+\infty} m(a)y_2(t,a)\,da \\ 0 \\ 0 \\ x(t)\int_0^{+\infty} \beta(a)\overline{i}(a)\,da + \overline{S}\int_0^{+\infty} \beta(a)y_1(t,a)\,da \\ \int_0^{+\infty} \gamma(a)y_1(t,a)\,da \end{pmatrix}.$$

As the range of $D\mathcal{L}(\bar{u})$ is finite dimension, $D\mathcal{L}(\bar{u})$ is compact on Y.

Then Lemma 2.1 and Theorem 2.1 imply that

Lemma 3.1 *The operator $\mathcal{F} + D\mathcal{L}(\bar{u})$ is a Hille-Yosida operator.*

Combing Lemma 2.2 yields the following theorem.

Theorem 3.2 *The part of $(\mathcal{F}, D(\mathcal{F}))$ and $(\mathcal{F} + D\mathcal{L}(\bar{u}), D(\mathcal{F} + D\mathcal{L}(\bar{u})))$ generates C_0-semigroups $(\mathscr{T}(t))_{t\geq 0}$ and $(\mathscr{C}(t))_{t\geq 0}$, respectively, on space Y_0.*

To discuss the stability/instability of steady states and whether or not the Hopf bifurcation exists for model (1.2), it is necessary to first examine the compactness of the C_0-semigroup related to above system (3.3). Exactly, we have to demonstrate the quasi-compactness of the associated C_0-semigroup so that the spectrum of the generators just contains spectrum points. We start by defining quasi-compactness for a C_0-semigroup in the following.

Definition 3.1 (see [44]) Call a C_0-semigroup $(\mathcal{S}(t))_{t\geq 0}$ quasi-compact if $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ with the operator families $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ satisfying

(i) $\mathcal{S}_1(t) \to 0$, as $t \to +\infty$,
(ii) $\mathcal{S}_2(t)$ is eventually compact, i.e. there exists $t_0 > 0$, s.t. $\mathcal{S}_2(t)$ is compact for any $t \geq t_0$.

The quasi-compact C_0-semigroup has the following property.

Lemma 3.2 (see [44]) *Let $(\mathcal{S}(t))_{t\geq 0}$ be a quasi-compact C_0-semigroup and $(A, D(A))$ its infinitesimal generator. Then, for $\delta > 0$, $e^{\delta t}\|\mathcal{S}(t)\| \to 0$ as $t \to +\infty$ if and only if all eigenvalues of A have strictly negative real part.*

From the proof of Theorem 2.1 about the Hille-Yosida estimate, we obtain that $\|\mathscr{T}(t)\| \leq e^{-\xi t}$. Moreover, $D\mathcal{L}(\bar{u})\mathscr{T}(t) : Y_0 \to Y$ is compact, for any $t \geq 0$. As

$$\mathscr{C}(t) = e^{D\mathcal{L}(\bar{u})t}\mathscr{T}(t) = \mathscr{T}(t) + \sum_{k=1}^{+\infty} \frac{(D\mathcal{L}(\bar{u})t)^k}{k!}\mathscr{T}(t),$$

which yields that $(\mathscr{C}(t))_{t\geq 0}$ is quasi-compact. Thus, based on Lemma 3.2, there exits $\delta > 0$, s.t. $e^{\delta t}\|\mathscr{C}(t)\| \to 0$ as $t \to +\infty$ if and only if all the eigenvalues of the operator $(\mathcal{F} + D\mathcal{L}(\bar{u}))$ have negative real parts.

We can derive the following conclusions from the discussion above.

Theorem 3.3 *The solution of model (1.2), semi-flow $T(t, v_0)$, defined as in Theorem 2.2, has the following properties.*
(i) *If all the eigenvalues of $(\mathcal{F} + D\mathcal{L}(\bar{u}))$ have strictly negative real part, then the steady state $\bar{u}$ is locally asymptotically stable.*
(ii) *If, however, at least one eigenvalue of $(\mathcal{F} + D\mathcal{L}(\bar{u}))$ has strictly positive part, then the steady state $\bar{u}$ is unstable.*

4 Global stability of disease-free equilibrium

By the above results, now we can analyze the dynamical behaviors of the equilibria for system (1.2). For this part, we focus on the locally and globally asymptotic stability results of the disease-free equilibrium E_0. The local stability of $E_0 = (\overline{S}_0, 0, 0)$ will be first investigated.

Theorem 4.1 *When $R_0 < 1$, the disease-free equilibrium $E_0 = (\overline{S}_0, 0, 0)$ of system (1.2) is locally asymptotically stable. While $R_0 > 1$, E_0 is not stable.*

Proof To linearize system (3.3) at E_0, we assume $x(t) = S(t) - \overline{S}_0$, $y_1(t,a) = i(t,a)$, $y_2(t,a) = r(t,a)$, then the linearized system of (3.3) is derived

$$\begin{cases} x'(t) = -bx(t) - K\int_0^{+\infty} \beta(a)y_1(t,a)\,da + \int_0^{+\infty} m(a)y_2(t,a)\,da, \\ \frac{\partial y_1(t,a)}{\partial t} + \frac{\partial y_1(t,a)}{\partial a} = -(d_1 + \mu + \gamma(a))y_1(t,a), \\ \frac{\partial y_2(t,a)}{\partial t} + \frac{\partial y_2(t,a)}{\partial a} = -(d_2 + m(a))y_2(t,a), \\ y_1(t,0) = K\int_0^{+\infty} \beta(a)y_1(t,a)\,da, \\ y_2(t,0) = \int_0^{+\infty} \gamma(a)y_1(t,a)\,da. \end{cases} \tag{4.1}$$

Substituting $x(t) = x_0 e^{\lambda t}$, $y_1(t,a) = y_{10}(a)e^{\lambda t}$, $y_2(t,a) = y_{20}(a)e^{\lambda t}$ into equations (4.1), we obtain

$$\begin{cases} \lambda x_0 = \int_0^{+\infty} m(a)y_{20}(a)\,da - K\int_0^{+\infty} \beta(a)y_{10}(a)\,da - bx_0, \\ \frac{dy_{10}(a)}{da} = -(\lambda + d_1 + \mu + \gamma(a))y_{10}(a), \\ \frac{dy_{20}(a)}{da} = -(\lambda + d_2 + m(a))y_{20}(a), \\ y_{10}(0) = K\int_0^{+\infty} \beta(a)y_{10}(a)\,da, \\ y_{20}(0) = \int_0^{+\infty} \gamma(a)y_{10}(a)\,da. \end{cases} \tag{4.2}$$

Solving (4.2) yields

$$y_{10}(a) = y_{10}(0)e^{-\int_0^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}, \qquad y_{20}(a) = y_{20}(0)e^{-\int_0^a(\lambda+d_2+m(\theta))\,d\theta}.$$

By substituting $y_{10}(0)$, $y_{20}(0)$ into the above two equations and some computations, we then obtain the characteristic equation of (4.1)

$$\Delta_0(\lambda) := (\lambda + b)f(\lambda) = 0, \tag{4.3}$$

where $f(\lambda) := 1 - K\int_0^{+\infty} \beta(a)e^{-\int_0^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}\,da$. Obviously, $\Delta_0(\lambda) = 0$ has a negative solution: $\lambda = -b$. So, the distribution of the solutions for $f(\lambda) = 0$ just needs to be discussed. We notice that when $\lambda \in \mathbb{R}$, $f(\lambda)$ is strictly increasing, continuous, real and has the following property

$$\lim_{\lambda\to+\infty} f(\lambda) = 1, \qquad f(0) = 1 - R_0.$$

Thus, when $R_0 > 1$, there exists at least one positive real solution for $f(\lambda) = 0$, as well as the characteristic equation $\Delta_0(\lambda) = 0$; therefore, E_0 is not stable. While $R_0 < 1$, there are no complex roots with nonnegative real parts for $f(\lambda) = 0$. Actually, let $\lambda = \sigma + \varpi i$ be any complex solution with $\sigma \geq 0$, then

$$\begin{aligned} 1 = \left|1 - f(\lambda)\right| &= \left|K\int_0^{+\infty} \beta(a)e^{-\int_0^a(\sigma+\varpi i+d_1+\mu+\gamma(\theta))\,d\theta}\,da\right| \\ &\leq K\int_0^{+\infty} \beta(a)e^{-\int_0^a(\sigma+d_1+\mu+\gamma(\theta))\,d\theta}\,da \\ &= 1 - f(\sigma) \leq 1 - f(0) = R_0, \end{aligned}$$

which contradicts $R_0 < 1$. Thus, $f(\lambda) = 0$ does not have roots with nonnegative real parts, as well as the characteristic equation $\Delta_0(\lambda) = 0$. Therefore, the steady state E_0 is locally stable when $R_0 < 1$. □

Moreover, when $R_0 < 1$, the globally asymptotic stability of E_0 can be proved as follows.

Theorem 4.2 *When $R_0 < 1$, the disease-free steady state E_0 of system (1.2) is globally asymptotically stable.*

Proof Based on Theorem 4.1, we just need to show that E_0 has the property of global attractivity, when $R_0 < 1$, that is to say, it merely remains to prove $\lim_{t\to+\infty}(S(t), i(t,a), r(t,a)) = (\overline{S}_0, 0, 0)$, where $(T(t), i(t,a), r(t,a))$ is any nonnegative solutions of model (1.2).

We assume that $\limsup_{t\to+\infty}(S(t), i(t,0)) = (S^0, i^0)$, thus there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with the feature

$$\lim_{n\to\infty} t_n = +\infty, \qquad \lim_{n\to\infty} i(t_n, 0) = i^0.$$

Substituting $t = t_n$ into $i(t,0)$ in the system (1.2) and combing the expression about $i(t,a)$ given by (2.4), we have

$$\begin{aligned} i(t_n, 0) = S(t_n)\biggl(&\int_0^{t_n} \beta(a) i(t_n - a, 0) e^{-\int_0^a (d_1+\mu+\gamma(\theta))\,d\theta}\,da \\ &+ \int_{t_n}^{\infty} \beta(a) i_0(a - t_n) e^{-\int_{a-t}^a (d_1+\mu+\gamma(\theta))\,d\theta}\,da\biggr), \end{aligned}$$

then, when n goes to infinity, we obtain

$$i^0 \le i^0 S^0 \int_0^{\infty} \beta(a) e^{-\int_0^a (d_1+\mu+\gamma(\theta))\,d\theta}\,da.$$

By the fact that K is the carrying capacity of susceptible individuals, so $S(t) \le K$ for any t. Hence, $S^0 \le K$. We then derive

$$i^0 \le i^0 K \int_0^{\infty} \beta(a) e^{-\int_0^a (d_1+\mu+\gamma(\theta))\,d\theta}\,da = i^0 R_0.$$

Therefore, the case $R_0 < 1$ implies $i^0 = 0$, and hence $\lim_{t\to+\infty} i(t,0) = 0$. Furthermore, by formulation (2.4) $i(t,a) = i(t-a,0)e^{-\int_0^a (d_1+\mu+\gamma(\theta))\,d\theta}$, for $t > a$, and $\lim_{t\to+\infty} i(t,0) = 0$, we obtain $\lim_{t\to+\infty} i(t,a) = 0$. Similarly, $\lim_{t\to+\infty} r(t,a) = 0$ can also be proved.

Due to the above discussion, the first equation of (1.2) is asymptotic to the equation below

$$\frac{d\check{S}(t)}{dt} = b\check{S}(t)\left(1 - \frac{\check{S}(t)}{K}\right)$$

from which we see

$$\lim_{t\to+\infty} \check{S}(t) = K = \overline{S}_0.$$

Finally, in terms of Corollary 4.3 in [45], the asymptotic autonomous semi-flow theory deduces that

$$\lim_{t\to+\infty} S(t) = \overline{S}_0.$$

Consequently, when $R_0 < 1$, $\lim_{t\to+\infty}(S(t), i(t,a), r(t,a)) = (\overline{S}_0, 0, 0)$. Thus, E_0 is globally stable. □

Remark 1 It appears to be challenging to construct Liapunov functions as in Refs. [46–49] to get the global attractivity. Therefore, here, to explore this property, we use asymptotic methods, which are very different from those in [46–49].

5 Hopf bifurcation analysis around the epidemic equilibrium E_*

For this part, the stability and the existence of Hopf bifurcations around the epidemic equilibrium E_* will be explored. Note that due to the quasi-compactness of the C_0-semigroup $(\mathscr{C}(t))_{t\geq 0}$, as demonstrated in Sect. 3, the linearized system will be reduced to an ODE system in finite dimension by applying the center manifold Theorem 4.21 and Proposition 4.22 in [50]. Hence, based on Hassard Hopf's bifurcation theorem proven in [51], we can establish the Hopf bifurcation results for system (1.2) as described in this section (see Theorems 5.4, 5.5, and 5.6 in the following).

The linearized system of (1.2) around the positive steady state E_* will be first discussed to investigate its local stability. Specifically, introducing the perturbation variables $\widetilde{x}(t) = S(t) - S_*$, $\widetilde{y}_1(t,a) = i(t,a) - i_*(a)$, $\widetilde{y}_2(t,a) = r(t,a) - r_*(a)$ leads to

$$\begin{cases} \widetilde{x}'(t) = b(1-\frac{2S_*}{K})\widetilde{x}(t) - \widetilde{x}(t)\int_0^{+\infty}\beta(a)i_*(a)\,da - S_*\int_0^{+\infty}\beta(a)\widetilde{y}_1(t,a)\,da \\ \qquad + \int_0^{+\infty} m(a)\widetilde{y}_2(t,a)\,da, \\ \frac{\partial\widetilde{y}_1(t,a)}{\partial t} + \frac{\partial\widetilde{y}_1(t,a)}{\partial a} = -(d_1+\mu+\gamma(a))\widetilde{y}_1(t,a), \\ \frac{\partial\widetilde{y}_2(t,a)}{\partial t} + \frac{\partial\widetilde{y}_2(t,a)}{\partial a} = -(d_2+m(a))\widetilde{y}_2(t,a), \\ \widetilde{y}_1(t,0) = \widetilde{x}(t)\int_0^{+\infty}\beta(a)i_*(a)\,da + S_*\int_0^{+\infty}\beta(a)\widetilde{y}_1(t,a)\,da, \\ \widetilde{y}_2(t,0) = \int_0^{+\infty}\gamma(a)\widetilde{y}_1(t,a)\,da. \end{cases}$$

As in Sect. 4, we search for the following solutions in form of $\widetilde{x}(t) = \widetilde{x}_0 e^{\lambda t}$, $\widetilde{y}_1(t,a) = \widetilde{y}_{10}(a)e^{\lambda t}$, $\widetilde{y}_2(t,a) = \widetilde{y}_{20}(a)e^{\lambda t}$. Then, we can get

$$\begin{cases} [\lambda - b(1-\frac{2S_*}{K})]\widetilde{x}_0 = -\widetilde{x}_0\int_0^{+\infty}\beta(a)i_*(a)\,da - S_*\int_0^{+\infty}\beta(a)\widetilde{y}_{10}(a)\,da \\ \qquad + \int_0^{+\infty} m(a)\widetilde{y}_{20}(a)\,da, \\ \frac{d\widetilde{y}_{10}(a)}{da} = -(\lambda+d_1+\mu+\gamma(a))\widetilde{y}_{10}(a), \\ \frac{d\widetilde{y}_{20}(a)}{da} = -(\lambda+d_2+m(a))\widetilde{y}_{20}(a), \\ \widetilde{y}_{10}(0) = \widetilde{x}_0\int_0^{+\infty}\beta(a)i_*(a)\,da + S_*\int_0^{+\infty}\beta(a)\widetilde{y}_{10}(a)\,da, \\ \widetilde{y}_{20}(0) = \int_0^{+\infty}\gamma(a)\widetilde{y}_{10}(a)\,da. \end{cases} \tag{5.1}$$

We solve the equations about $\widetilde{y}_{10}(a)$ and $\widetilde{y}_{20}(a)$, then we obtain

$$\widetilde{y}_{10}(a) = \widetilde{y}_{10}(0)e^{-\int_0^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}, \qquad \widetilde{y}_{20}(a) = \widetilde{y}_{20}(0)e^{-\int_0^a(\lambda+d_2+m(\theta))\,d\theta}. \tag{5.2}$$

Substituting $\widetilde{y}_{10}(a)$, $\widetilde{y}_{20}(0)$ and $\widetilde{y}_{20}(a)$ into the first equation of (5.1) implies that

$$\begin{aligned} \left(\lambda - b\left(1-\frac{2S_*}{K}\right)\right)\widetilde{x}_0 = -\widetilde{y}_{10}(0)\bigg(1 &- \int_0^{+\infty} m(a)e^{-\int_0^a(\lambda+d_2+m(\theta))\,d\theta}\,da \\ &\cdot \int_0^{+\infty}\gamma(a)e^{-\int_0^a(\lambda+d_1+\mu+\gamma(\theta))\,d\theta}\,da\bigg). \end{aligned} \tag{5.3}$$

Multiplying $(\lambda - b(1 - \frac{2S_*}{K}))$ at the both sides of the fourth equation about $\widetilde{y}_{10}(0)$ in the above system (5.1), further (5.2) and (5.3) leads to

$$\begin{aligned}\left(\lambda - b\left(1 - \frac{2S_*}{K}\right)\right) = -\int_0^{+\infty} \beta(a) i_*(a)\, da\left(1 - \int_0^{+\infty} m(a) e^{-\int_0^a (\lambda + d_2 + m(\theta))\, d\theta}\, da\right.\\ \left.\cdot \int_0^{+\infty} \gamma(a) e^{-\int_0^a (\lambda + d_1 + \mu + \gamma(\theta))\, d\theta}\, da\right)\\ + \left(\lambda - b\left(1 - \frac{2S_*}{K}\right)\right) S_* \int_0^{+\infty} \beta(a) e^{-\int_0^a (\lambda + d_1 + \mu + \gamma(\theta))\, d\theta}\, da.\end{aligned} \tag{5.4}$$

Note that the complexity of the characteristic equation is a major problem for the bifurcation analysis. Hence, to simplify our reasoning, we make the following assumptions.

Assumption 5.1

(i) The function $\gamma(a)$ is constant, i.e. $\gamma(a) \equiv \gamma$.

(ii) The age-dependent functions $\beta(a)$ and $m(a)$ take the following forms:

$$\beta(a) = \begin{cases} \beta_0, & a \geq \tau_1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.5}$$

and

$$m(a) = \begin{cases} m_0, & a \geq \tau_2, \\ 0, & \text{otherwise}, \end{cases} \tag{5.6}$$

where $\beta_0, m_0 > 0$ and $\tau_1, \tau_2 \geq 0$ are all constants.

Assumption 5.1(ii) indicates that the latent and immunity periods of the infectious disease are described by the infection and recovery ages, which is in line with the age-structured models [12, 15, 17, 19, 20, 22–25, 29–41]. Concretely, we introduce two constant delays $\tau_1 > 0$ and $\tau_2 > 0$ to represent the average latent period (the average time passed since infection) and immune period (the average time passed for recovered individuals to become susceptible again). $\tau_1 = 0$ or $\tau_2 = 0$ means that there is no latent or immune period. In addition, the latent and immune periods increase as τ_1 and τ_2 increase. As mentioned in the Introduction section, the change in the latent and immune periods plays a vital role in the spread and control of infectious diseases. Thus, we next concentrate on discussing the rich dynamics of the endemic equilibrium E_* by varying the two delay parameters.

If we take new forms (5.5) and (5.6) in (5.4) and carry out complex calculation, the characteristic equation will be obtained as follows

$$\begin{aligned}\Pi(\lambda, \tau_1, \tau_2) &:= \frac{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau_1} + c_0 e^{-\lambda\tau_2}}{(\lambda + d_1 + \mu + \gamma)(\lambda + d_2 + m_0)}\\ &=: \frac{g(\lambda, \tau_1, \tau_2)}{s(\lambda)} = 0,\end{aligned}$$

where $\lambda \in \Upsilon$ (and thus $s(\lambda) \neq 0$),

$$\begin{aligned}
a_2 &= d_1 + \mu + \gamma + d_2 + m_0 - b\left(1 - \frac{2S_*}{K}\right) + \int_0^{+\infty} \beta(a) i_*(a)\,da, \\
a_1 &= (d_1 + \mu + \gamma)(d_2 + m_0) \\
&\quad + \left(\int_0^{+\infty} \beta(a) i_*(a)\,da - b\left(1 - \frac{2S_*}{K}\right)\right)(d_1 + \mu + \gamma + d_2 + m_0), \\
a_0 &= (d_1 + \mu + \gamma)(d_2 + m_0)\left(\int_0^{+\infty} \beta(a) i_*(a)\,da - b\left(1 - \frac{2S_*}{K}\right)\right), \\
b_2 &= -\beta_0 S_* e^{-(d_1+\mu+\gamma)\tau_1}, \\
b_1 &= -\beta_0 S_*\left(d_2 + m_0 - b\left(1 - \frac{2S_*}{K}\right)\right)e^{-(d_1+\mu+\gamma)\tau_1}, \\
b_0 &= \beta_0 S_*(d_2 + m_0) b\left(1 - \frac{2S_*}{K}\right)e^{-(d_1+\mu+\gamma)\tau_1}, \\
c_0 &= -\gamma m_0 \int_0^{+\infty} \beta(a) i_*(a)\,da e^{-d_2\tau_2}.
\end{aligned}$$

Note that (3.2) implies

$$\beta_0 S_* e^{-(d_1+\mu+\gamma)\tau_1} = d_1 + \mu + \gamma,$$

and,

$$Q := \int_0^{+\infty} \beta(a) i_*(a)\,da = b\left(1 - \frac{1}{R_0}\right)\frac{(d_1 + \mu + \gamma)(d_2 + m_0)}{(d_1 + \mu + \gamma)(d_2 + m_0) - m_0\gamma e^{-d_2\tau_2}} > 0,$$

hence,

$$\begin{aligned}
Q - b\left(1 - \frac{2S_*}{K}\right) &= b\left(1 - \frac{1}{R_0}\right)\left(\frac{(d_1 + \mu + \gamma)(d_2 + m_0)}{(d_1 + \mu + \gamma)(d_2 + m_0) - m_0\gamma e^{-d_2\tau_2}} - 1\right) + \frac{bS_*}{K} \\
&> \frac{bS_*}{K} > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_2 &= d_1 + \mu + \gamma + d_2 + m_0 + Q - b\left(1 - \frac{2S_*}{K}\right) > 0, \\
a_1 &= (d_1 + \mu + \gamma)(d_2 + m_0) + \left(Q - b\left(1 - \frac{2S_*}{K}\right)\right)(d_1 + \mu + \gamma + d_2 + m_0) > 0, \\
a_0 &= (d_1 + \mu + \gamma)(d_2 + m_0)\left(Q - b\left(1 - \frac{2S_*}{K}\right)\right) > 0, \\
b_2 &= -(d_1 + \mu + \gamma) < 0, \\
b_1 &= -(d_1 + \mu + \gamma)\left(d_2 + m_0 - b\left(1 - \frac{2S_*}{K}\right)\right), \\
b_0 &= (d_1 + \mu + \gamma)(d_2 + m_0) b\left(1 - \frac{2S_*}{K}\right),
\end{aligned}$$

$$c_0 = -\gamma m_0 Q e^{-d_2\tau_2} < 0.$$

It is easy to notice that

$$\{\lambda \in \Upsilon : \Pi(\lambda, \tau_1, \tau_2) = 0\} = \{\lambda \in \Upsilon : g(\lambda, \tau_1, \tau_2) = 0\}.$$

Next, we analyze the asymptotic stability and the existence of the Hopf bifurcations about E_* under different combinations of two delays and $R_0 > 1$, aiming to show the complex dynamic behaviors of E_*.

Case (I): $\tau_1 = \tau_2 = 0$.

In this case, we can get

$$g(\lambda, 0, 0) = \lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + a_0 + b_0 + c_0 = 0,$$

then with some algebra computations, it is easy to get

$$\begin{aligned}
&a_2 + b_2 = d_2 + m_0 + Q - b\left(1 - \frac{2S_*}{K}\right) > 0,\\
&a_1 + b_1 = \left(Q - b\left(1 - \frac{2S_*}{K}\right)\right)(d_2 + m_0) + Q(d_1 + \mu + \gamma) > 0,\\
&a_0 + b_0 + c_0 = Q\big[d_2(d_1 + \mu + \gamma) + m_0(d_1 + \mu)\big] > 0,
\end{aligned}$$

then, we derive that

$$\begin{aligned}
&(a_2 + b_2)(a_1 + b_1) - (a_0 + b_0 + c_0)\\
&\quad = (d_2 + m_0)\left(Q - b\left(1 - \frac{2S_*}{K}\right)\right)\left(Q - b\left(1 - \frac{2S_*}{K}\right) + d_2 + m_0\right)\\
&\qquad + \left[\left(Q - b\left(1 - \frac{2S_*}{K}\right)\right)(d_1 + \mu + \gamma) + m_0\gamma\right]Q > 0.
\end{aligned}$$

Therefore, $(a_2 + b_2)(a_1 + b_1) > (a_0 + b_0 + c_0)$, then according to the Routh-Hurwitz criterion, we obtain that all the real parts of the solutions for $g(\lambda, 0, 0) = 0$ are negative. Thus, when $\tau_1 = \tau_2 = 0$, we deduce that E_* is locally asymptotically stable, i.e.

Theorem 5.1 *If $R_0 > 1$ and $\tau_i = 0$ ($i = 1, 2$), the endemic equilibrium E_* of* (1.2) *is locally asymptotically stable.*

Case (II): $\tau_1 > 0$, $\tau_2 = 0$.

In this situation, the characteristic equation will be written

$$g(\lambda, \tau_1, 0) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + c_0 + \left(b_2\lambda^2 + b_1\lambda + b_0\right)e^{-\lambda\tau_1} = 0.$$

Suppose $\lambda = i\omega_1$ with $\omega_1 > 0$ is a pure imaginary solution of $g(\lambda, \tau_1, 0) = 0$ if and only if λ satisfies

$$-i\omega_1^3 - a_2\omega_1^2 + ia_1\omega_1 + a_0 + c_0 + \left(-b_2\omega_1^2 + ib_1\omega_1 + b_0\right)(\cos\omega_1\tau_1 - i\sin\omega_1\tau_1) = 0,$$

which leads to

$$\begin{cases} a_2\omega_1^2 - (a_0 + c_0) = (b_0 - b_2\omega_1^2)\cos\omega_1\tau_1 + b_1\omega_1\sin\omega_1\tau_1, \\ \omega_1^3 - a_1\omega_1 = b_1\omega_1\cos\omega_1\tau_1 - (b_0 - b_2\omega_1^2)\sin\omega_1\tau_1. \end{cases}$$

Using above equations, we have

$$\begin{cases} \cos\omega_1\tau_1 = \frac{(b_1-a_2b_2)\omega_1^4+(a_2b_0+a_0b_2+c_0b_2-a_1b_1)\omega_1^2-(a_0+c_0)b_0}{b_1^2\omega_1^2+(b_0-b_2\omega_1^2)^2}, \\ \sin\omega_1\tau_1 = \frac{b_2\omega_1^5+(a_2b_1-b_0-a_1b_2)\omega_1^3+(a_1b_0-a_0b_1-c_0b_1)\omega_1}{b_1^2\omega_1^2+(b_0-b_2\omega_1^2)^2}. \end{cases} \tag{5.7}$$

and

$$\omega_1^6 + \left(a_2^2 - 2a_1 - b_2^2\right)\omega_1^4 + \left(a_1^2 - 2a_2(a_0 + c_0) + 2b_0b_2 - b_1^2\right)\omega_1^2 + (a_0 + c_0)^2 - b_0^2 = 0. \tag{5.8}$$

Let $z_1 = \omega_1^2$, and

$$p_2 = a_2^2 - 2a_1 - b_2^2, \qquad p_1 = a_1^2 - 2a_2(a_0 + c_0) + 2b_0b_2 - b_1^2, \qquad p_0 = (a_0 + c_0)^2 - b_0^2,$$

then (5.8) becomes

$$z_1^3 + p_2z_1^2 + p_1z_1 + p_0 = 0. \tag{5.9}$$

Put

$$h_1(z_1) = z_1^3 + p_2z_1^2 + p_1z_1 + p_0.$$

From the formulations of a_0, b_0, and c_0, it is easy to verify $p_0 > 0$. Further, combining $\lim_{z_1\to+\infty} h_1(z_1) = +\infty$, we conclude that equation (5.9) may have no positive roots or have at least one positive root. Differentiating $h_1(z_1)$ leads to

$$\frac{h_1(z_1)}{dz_1} = 3z_1^2 + 2p_2z_1 + p_1.$$

Let $\Delta = p_2^2 - 3p_1$, if $\Delta \le 0$, then $h_1(z_1)$ $(z_1 \ge 0)$ is increasing, and (5.9) has no positive roots. When $\Delta > 0$, there exist two real roots for the equation $\frac{h_1(z_1)}{dz_1} = 3z_1^2 + 2p_2z_1 + p_1 = 0$, denoted by

$$z_{11}^* = \frac{-p_2 + \sqrt{\Delta}}{3}, \qquad z_{12}^* = \frac{-p_2 - \sqrt{\Delta}}{3}.$$

It is obvious that $h_1''(z_{11}^*) = 2\sqrt{\Delta} > 0$ and $h_1''(z_{12}^*) = -2\sqrt{\Delta} < 0$, which implies that z_{11}^* and z_{12}^* are the minimum and maximum points of $h_1(z_1)$, respectively. Based on the discussions above, we have

Lemma 5.1

(i) *When* $\Delta = p_2^2 - 3p_1 \le 0$, *there is no positive root for equation* (5.9);

(ii) *When* $\Delta = p_2^2 - 3p_1 > 0$, *there are positive roots for equation* (5.9) *if and only if* $z_{11}^* = \frac{-p_2+\sqrt{\Delta}}{3} > 0$ *and* $h_1(z_{11}^*) \leq 0$.

Assume that there exist positive roots for (5.9), without loss of generality, we suppose that there exist three positive roots for (5.9), defined by z_{1k}, $k = 1, 2, 3$. Hence, equation (5.8) has three positive solutions $\omega_{1k} = \sqrt{z_{1k}}$, $k = 1, 2, 3$, $g(\lambda, 0, \tau_2) = 0$ has three pairs of purely imaginary roots $\pm i\omega_{1k}$, $k = 1, 2, 3$. Moreover, from (5.7), we find

$$\tau_{1,k}^j = \begin{cases} \frac{1}{\omega_{1k}}(\arccos \frac{(b_1-a_2b_2)\omega_{1k}^4+(a_2b_0+a_0b_2+c_0b_2-a_1b_1)\omega_{1k}^2-(a_0+c_0)b_0}{b_1^2\omega_{1k}^2+(b_0-b_2\omega_{1k}^2)^2} + 2\pi j), & \eta \geq 0, \\ \frac{1}{\omega_{1k}}(-\arccos \frac{(b_1-a_2b_2)\omega_{1k}^4+(a_2b_0+a_0b_2+c_0b_2-a_1b_1)\omega_{1k}^2-(a_0+c_0)b_0}{b_1^2\omega_{1k}^2+(b_0-b_2\omega_{1k}^2)^2} + (2\pi+1)j), & \eta < 0, \end{cases}$$

with $k = 1, 2, 3$, $j = 0, 1, 2, \ldots$, and

$$\eta = \frac{b_2\omega_{1k}^5 + (a_2b_1 - b_0 - a_1b_2)\omega_{1k}^3 + (a_1b_0 - a_0b_1 - c_0b_1)\omega_{1k}}{b_1^2\omega_{1k}^2 + (b_0 - b_2\omega_{1k}^2)^2}.$$

Now, we define

$$\tau_{10} := \tau_{1,k_0}^0 = \min\{\tau_{1,1}^0, \tau_{1,2}^0, \tau_{1,3}^0\}, \qquad \omega_{1*} = \omega_{1k_0}, \qquad z_{1*} = z_{1k_0}.$$

To explore the transversality condition, we first establish the necessary theorem as below.

Theorem 5.2 *If conditions* (*ii*) *in Lemma* 5.1 *hold and* $R_0 > 1$, $\tau_1 > 0$, $\tau_2 = 0$, *then*

$$\left.\frac{\partial g(\lambda, \tau_1, 0)}{\partial \lambda}\right|_{\lambda = i\omega_{1*}} \neq 0.$$

Proof Calculating the derivative of the equation $g(\lambda, \tau_1, 0) = 0$ about λ, we obtain

$$\frac{\partial g(\lambda, \tau_1, 0)}{\partial \lambda} = 3\lambda^2 + 2a_2\lambda + a_1 + (2b_2\lambda + b_1)e^{-\lambda\tau_1} - \tau_1(b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau_1}.$$

Then, differentiating λ according to τ_1 in the equation $g(\lambda, \tau_1, 0) = 0$ yields that

$$\begin{aligned} &\left(3\lambda^2 + 2a_2\lambda + a_1 + (2b_2\lambda + b_1)e^{-\lambda\tau_1} - \tau_1(b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau_1}\right)\frac{d\lambda(\tau_1)}{d\tau_1} \\ &\quad = \lambda(b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau_1}. \end{aligned} \tag{5.10}$$

Thus, if $\frac{\partial g(\lambda,\tau_1,0)}{\partial\lambda}|_{\lambda=i\omega_{1*}} = 0$, then

$$i\omega_*(-b_2\omega_{1*}^2 + ib_1\omega_{1*} + b_0)e^{-i\omega_{1*}\tau_1} = 0,$$

which implies

$$b_2\omega_{1*}^2 - ib_1\omega_{1*} - b_0 = 0,$$

hence $b_1 = 0$ and $b_0 = b_2\omega_{1*}^2$. However, from $b_1 = 0$, we can get $b_0 > 0$, which contradicts $b_0 = b_2\omega_{1*}^2 < 0$. Therefore,

$$\left.\frac{\partial g(\lambda,\tau_1,0)}{\partial\lambda}\right|_{\lambda=i\omega_{1*}} \neq 0. \qquad \square$$

Based on the above theorem, we can show that the transversality condition is well satisfied under $R_0 > 1$, $\tau_1 > 0$, $\tau_2 = 0$.

Theorem 5.3 *Suppose that conditions* (*ii*) *in Lemma* 5.1 *are fulfilled and* $h_1'(z_{1*}) \neq 0$, *set* $\lambda(\tau_1) = \vartheta(\tau_1) + i\zeta(\tau_1)$ *to be the solution of* $g(\lambda,\tau_1,0) = 0$, *satisfying* $\vartheta(\tau_{1,k_0}^j) = 0$, $\zeta(\tau_{1,k_0}^j) = \omega_{1*}$, *then*

$$\operatorname{Re}\left[\left.\left(\frac{d\lambda}{d\tau_1}\right)\right|_{\tau_1=\tau_{1,k_0}^j}\right] \neq 0.$$

Proof From (5.10), we obtain

$$\left(\frac{d\lambda(\tau_1)}{d\tau_1}\right)^{-1} = -\frac{3\lambda^2 + 2a_2\lambda + a_1}{\lambda(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + c_0)} + \frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau_1}{\lambda}. \tag{5.11}$$

Substituting $\lambda = i\omega_{1*}$ into (5.11), we get

$$\begin{aligned}
&\operatorname{Re}\left\{\left[\left.\left(\frac{d\lambda(\tau_1)}{d\tau_1}\right)\right|_{\tau_1=\tau_{1,k_0}^j}\right]^{-1}\right\} \\
&\quad= \frac{3\omega_{1*}^4 + 2(a_2^2 - 2a_1 - b_2^2)\omega_{1*}^2 + a_1^2 - 2a_2(a_0 + c_0) + 2b_0b_2 - b_1^2}{b_1^2\omega_{1*}^2 + (b_0 - b_2\omega_{1*}^2)^2} \\
&\quad= \frac{3z_{1*}^2 + 2p_2z_{1*} + p_1}{b_1^2\omega_{1*}^2 + (b_0 - b_2\omega_{1*}^2)^2} = \frac{h_1'(z_{1*})}{b_1^2\omega_{1*}^2 + (b_0 - b_2\omega_{1*}^2)^2} \neq 0.
\end{aligned}$$

It is well known that

$$\operatorname{sign}\left\{\operatorname{Re}\left[\left.\left(\frac{d\lambda(\tau_1)}{d\tau_1}\right)\right|_{\tau_1=\tau_{1,k_0}^j}\right]\right\} = \operatorname{sign}\left\{\operatorname{Re}\left\{\left[\left.\left(\frac{d\lambda(\tau_1)}{d\tau_1}\right)\right|_{\tau_1=\tau_{1,k_0}^j}\right]^{-1}\right\}\right\},$$

hence

$$\operatorname{Re}\left[\left.\left(\frac{d\lambda(\tau_1)}{d\tau_1}\right)\right|_{\tau_1=\tau_{1,k_0}^j}\right] \neq 0. \qquad \square$$

Finally, according to Theorem 5.2 and Theorem 5.3, the Hopf bifurcation results are derived.

Theorem 5.4 *Suppose that* $R_0 > 1$, $\tau_1 > 0$ *and* $\tau_2 = 0$,

(i) *if condition* (*i*) *in Lemma* 5.1 *holds, the positive steady state* E_* *of system* (1.2) *is locally asymptotically stable*;

(ii) *if conditions* (*ii*) *in Lemma* 5.1 *hold*, E_* *is locally asymptotically stable for* $\tau_1 \in [0,\tau_{10})$. *Moreover, if* $h_1(z_{1*}) \neq 0$, *system* (1.2) *occurs as a Hopf bifurcation around* E_* *when* τ_1 *crosses through* τ_{10}.

Remark 2 Using the similar method in Case (II), when $R_0 > 1$ and $\tau_1 = 0$, $\tau_2 > 0$, the parallel Hopf bifurcation theorem for E_* can also be established. That is, we can find τ_{20}, s.t., E_* is locally asymptotically stable, when $\tau_2 \in [0, \tau_{20})$; however, system (1.2) undergoes a Hopf bifurcation at τ_{20}.

In light of Theorem 5.4, we notice that when $\tau_2 = 0$ and $\tau_1 \in [0, \tau_{10})$, E_* has the property of locally asymptotic stability under certain conditions. Actually, the stability of E_* may switch or preserve as τ_2 increases. Now, we fix $\tau_1 \in [0, \tau_{10})$ and increase the bifurcation parameter τ_2 from zero to find a potential Hopf bifurcation. To make sure of this, we discuss the following case.

Case (III): $\overline{\tau}_1 \in [0, \tau_{10})$, $\tau_2 > 0$.

Then equation $g(\lambda, \overline{\tau}_1, \tau_2) = 0$ becomes

$$\left(b_2\lambda^2 + b_1\lambda + b_0\right)e^{-\lambda\overline{\tau}_1} + c_0 e^{-\lambda\tau_2} + \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

Let $\lambda = i\omega_2$, $\omega_2 > 0$, be a pure imaginary root of $g(\lambda, \overline{\tau}_1, \tau_2) = 0$, thus one has

$$\begin{aligned} &-i\omega_2^3 - a_2\omega_2^2 + ia_1\omega_2 + a_0 + \left(-b_2\omega_2^2 + ib_1\omega_2 + b_0\right)e^{-\omega_2\overline{\tau}_1 i} \\ &\quad + c_0(\cos\omega_2\tau_2 - i\sin\omega_2\tau_2) = 0. \end{aligned} \tag{5.12}$$

By the real and imaginary parts of (5.12), it is easy to see that

$$\begin{cases} P_1(\omega_2, \overline{\tau}_1) = c_0\cos\omega_2\tau_2, \\ P_2(\omega_2, \overline{\tau}_1) = c_0\sin\omega_2\tau_2, \end{cases} \tag{5.13}$$

with

$$\begin{aligned} P_1(\omega_2, \overline{\tau}_1) &= a_2\omega_2^2 - a_0 + \left(b_2\omega_2^2 - b_0\right)\cos\omega_2\overline{\tau}_1 - b_1\omega_2\sin\omega_2\overline{\tau}_1, \\ P_2(\omega_2, \overline{\tau}_1) &= a_1\omega_2 - \omega_2^3 + b_1\omega_2\cos\omega_2\overline{\tau}_1 + \left(b_2\omega_2^2 - b_0\right)\sin\omega_2\overline{\tau}_1. \end{aligned}$$

Clearly, (5.13) yields

$$P_1^2(\omega_2, \overline{\tau}_1) + P_2^2(\omega_2, \overline{\tau}_1) = c_0^2,$$

i.e.,

$$\begin{aligned} h_2(\omega_2) := {}& \omega_2^6 + \left(a_2^2 - 2a_1 + b_2^2\right)\omega_2^4 + \left(a_1^2 - 2a_0a_2 - 2b_0b_2 + b_1^2\right)\omega_2^2 + a_0^2 + b_0^2 - c_0^2 \\ & + 2\left[(a_2b_2 - b_1)\omega_2^4 + (a_1b_1 - a_0b_2 - a_2b_0)\omega_2^2 + a_0b_0\right]\cos\omega_2\overline{\tau}_1 \\ & + 2\left[-b_2\omega_2^5 + (b_0 + a_1b_2 - a_2b_1)\omega_2^3 + (a_0b_1 - a_1b_0)\omega_2\right]\sin\omega_2\overline{\tau}_1 = 0. \end{aligned} \tag{5.14}$$

As we know, there typically do not exist exact solutions for transcendental equations, and usually, it is impossible to determine the equations' solutions straightforwardly. Here, we suppose that there exists a positive real root ω_{2*} for Eq. (5.14), from (5.13), then there exists a sequence of τ_2^j $(j = 0, 1, 2, 3, \ldots)$ such that ω_{2*} is a pair of purely imaginary roots of $g(\lambda, \overline{\tau}_1, \tau_2) = 0$ when $\tau_2 = \tau_2^j$. Here

$$\tau_2^j = \begin{cases} \frac{1}{\omega_{2*}}(\arccos \frac{-P_1(\omega_{2*},\overline{\tau}_1)}{c_0} + 2\pi j), & \frac{P_2(\omega_{2*},\overline{\tau}_1)}{c_0} \geq 0, \\ \frac{1}{\omega_{2*}}(-\arccos \frac{-P_1(\omega_{2*},\overline{\tau}_1)}{c_0} + (2\pi + 1)j), & \frac{P_2(\omega_{2*},\overline{\tau}_1)}{c_0} < 0, \end{cases} \tag{5.15}$$

with $j = 0, 1, 2, \ldots$.

Next, we check the transversality conditions. In fact, let $\lambda(\tau_2) = \widetilde{\vartheta}(\tau_2) + \widetilde{\zeta}(\tau_2)i$ be the root of $g(\lambda, \overline{\tau}_1, \tau_2) = 0$ with $\widetilde{\vartheta}(\tau_2^j) = 0$, $\widetilde{\zeta}(\tau_2^j) = \omega_{2*}$. Differentiating the equation $g(\lambda, \overline{\tau}_1, \tau_2) = 0$ about τ_2 implies

$$\begin{aligned} \left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)^{-1} &= \frac{3\lambda^2 + 2a_2\lambda + a_1 + [2b_2\lambda + b_1 - \overline{\tau}_1(b_2\lambda^2 + b_1\lambda + b_0)]e^{-\lambda\overline{\tau}_1}}{c_0\lambda e^{-\lambda\tau_2}} - \frac{\tau_2}{\lambda} \\ &= -\frac{3\lambda^2 + 2a_2\lambda + a_1 + [2b_2\lambda + b_1 - \overline{\tau}_1(b_2\lambda^2 + b_1\lambda + b_0)]e^{-\lambda\overline{\tau}_1}}{\lambda[\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\overline{\tau}_1}]} - \frac{\tau_2}{\lambda} \end{aligned} \tag{5.16}$$

Substituting $\lambda = i\omega_{2*}$ into (5.16) and carrying out some direct and tedious calculations, we get

$$\mathrm{Re}\left\{\left[\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)\bigg|_{\tau_2=\tau_2^j}\right]^{-1}\right\} = \mathrm{Re}\left(-\frac{M_1 + N_1 i}{M_2 + N_2 i}\right) = -\frac{M_1M_2 + N_1N_2}{M_2^2 + N_2^2},$$

where

$$\begin{aligned} M_1 &= -3\omega_{2*}^2 + a_1 + (\overline{\tau}_1 b_2\omega_{2*}^2 + b_1 - \overline{\tau}_1 b_0)\cos\omega_{2*}\overline{\tau}_1 + (2b_2\omega_{2*} - \overline{\tau}_1 b_1\omega_{2*})\sin\omega_{2*}\overline{\tau}_1, \\ M_2 &= \omega_{2*}^4 - a_1\omega_{2*}^2 - b_1\omega_{2*}^2\cos\omega_{2*}\overline{\tau}_1 + (b_0\omega_{2*} - b_2\omega_{2*}^3)\sin\omega_{2*}\overline{\tau}_1, \\ N_1 &= 2a_2\omega_{2*} - (\overline{\tau}_1 b_2\omega_{2*}^2 + b_1 - \overline{\tau}_1 b_0)\sin\omega_{2*}\overline{\tau}_1 + (2b_2\omega_{2*} - \overline{\tau}_1 b_1\omega_{2*})\cos\omega_{2*}\overline{\tau}_1, \\ N_2 &= -a_2\omega_{2*}^3 + a_0\omega_{2*} + b_1\omega_{2*}^2\sin\omega_{2*}\overline{\tau}_1 + (b_0\omega_{2*} - b_2\omega_{2*}^3)\cos\omega_{2*}\overline{\tau}_1. \end{aligned}$$

Thus, we further introduce the assumption

$$M_1M_2 + N_1N_2 \neq 0. \tag{5.17}$$

It is well known that

$$\mathrm{sign}\left\{\mathrm{Re}\left[\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)\bigg|_{\tau_2=\tau_2^j}\right]\right\} = \mathrm{sign}\left\{\mathrm{Re}\left\{\left[\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)\bigg|_{\tau_2=\tau_2^j}\right]^{-1}\right\}\right\},$$

hence, if (5.17) holds, it follows that

$$\mathrm{Re}\left[\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)\bigg|_{\tau_2=\tau_2^j}\right] \neq 0.$$

In light of the above discussions, we deduce the Hopf bifurcation results.

Theorem 5.5 *Suppose that* $R_0 > 1$, $\overline{\tau}_1 \in [0, \tau_{10})$, *then*

(i) *If there are no positive real roots for Eq.* (5.14), *the positive steady state* E_* *of the system* (1.2) *is locally asymptotically stable, when* $\tau_2 \geq 0$.
(ii) *If there is a positive real root* ω_{2*} *for Eq.* (5.14), *and* (5.17) *holds, system* (1.2) *undergoes a Hopf bifurcation at* E_* *when* $\tau_2 = \tau_2^j$ *and here* τ_2^j *is given by* (5.15).

Remark 3 When $R_0 > 1$ with fixed $\tau_2 = \overline{\tau}_2 < \tau_2^0$ and as $\tau = \tau_1$ increases from zero, similar bifurcation result for E_* can be investigated by the same discussions as above.

Case (IV): $\tau_1 = \tau_2 > 0$.

Finally, we consider the special case $\tau_1 = \tau_2 = \tau$, and the corresponding characteristic equation can be written

$$g(\lambda,\tau) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \left(b_2\lambda^2 + b_1\lambda + b_0 + c_0\right)e^{-\lambda\tau} = 0.$$

Here, we omit the detailed discussion on the distribution of roots for $g(\lambda,\tau) = 0$ since it is very similar to that of Case (II), we just present the conditions and the conclusions below. Set

$$h(z) = z^3 + q_2 z^2 + q_1 z + q_0, \tag{5.18}$$

with $q_2 = a_2^2 - 2a_1 - b_2^2$, $q_1 = a_1^2 - 2a_2a_0 + 2(b_0 + c_0)b_2 - b_1^2$, $q_0 = a_0^2 - (b_0 + c_0)^2$, and then define $\tilde{\Delta} = q_2^2 - 3q_1$, $z_1^* = \frac{-q_2+\sqrt{\tilde{\Delta}}}{3}$, $z_2^* = \frac{-q_2-\sqrt{\tilde{\Delta}}}{3}$.

Lemma 5.2 *For equation* (5.18),
(i) *If* $\tilde{\Delta} = q_2^2 - 3q_1 \leq 0$, *for Eq.* (5.18), *positive roots do not exist*;
(ii) *If* $\tilde{\Delta} = q_2^2 - 3q_1 > 0$, *for Eq.* (5.18), *positive roots exist if and only if* $z_1^* = \frac{-q_2+\sqrt{\tilde{\Delta}}}{3} > 0$ *and* $h(z_1^*) \leq 0$.

If conditions (ii) in Lemma 5.2 are satisfied, Eq. (5.18) has at most three positive roots written as z_{k*}, $k = 1,2,3$. Hence, $g(\lambda,\tau) = 0$ has three pairs of purely imaginary roots $\pm i\omega_{k*}$ ($\omega_{k*} = \sqrt{z_{k*}}$), $k = 1,2,3$, when $\tau = \tau_k^j$. Here

$$\tau_k^j = \begin{cases} \frac{1}{\omega_{k*}}(\arccos \frac{(b_1-a_2b_2)\omega_{k*}^4+[a_2(b_0+c_0)+a_0b_2-a_1b_1]\omega_{k*}^2-a_0(b_0+c_0)}{b_1^2\omega_{k*}^2+(b_0+c_0-b_2\omega_{k*}^2)^2} + 2\pi j), & \tilde{\eta} \geq 0, \\ \frac{1}{\omega_{k*}}(-\arccos \frac{(b_1-a_2b_2)\omega_{k*}^4+[a_2(b_0+c_0)+a_0b_2-a_1b_1]\omega_{k*}^2-a_0(b_0+c_0)}{b_1^2\omega_{k*}^2+(b_0+c_0-b_2\omega_{k*}^2)^2} + (2\pi+1)j), & \tilde{\eta} < 0, \end{cases}$$

with $k = 1,2,3$, $j = 0,1,2,\dots$, and

$$\tilde{\eta} = \frac{b_2\omega_{k*}^5 + (a_2b_1 - b_0 - c_0 - a_1b_2)\omega_{k*}^3 + (a_1b_0 + a_1c_0 - a_0b_1)\omega_{k*}}{b_1^2\omega_{k*}^2 + (b_0 + c_0 - b_2\omega_{k*}^2)^2}.$$

Now, we define

$$\tau_0 := \tau_{k_0}^0 = \min\left\{\tau_1^0, \tau_2^0, \tau_3^0\right\}, \qquad \omega_* = \omega_{k_0*}, \qquad z_* = z_{k_0*}.$$

Theorem 5.6 *Assume that* $R_0 > 1$, $\tau_1 = \tau_2 = \tau$,

(i) *if condition (i) in Lemma 5.2 holds, the positive equilibrium* E_* *of system* (1.2) *is locally asymptotically stable for all* $\tau \geq 0$;
(ii) *if conditions (ii) in Lemma 5.2 are satisfied,* E_* *is locally asymptotically stable for* $\tau \in [0, \tau_0)$. *Moreover, if* $h(z_1^*) \neq 0$, *system* (1.2) *undergoes a Hopf bifurcation around* E_* *when* τ *crosses through* τ_0.

6 Numerical simulation

This section aims to conduct some numerical results with graphs to demonstrate the obtained theoretical results for system (1.2), such as stability/instability and Hopf bifurcations of the equilibria in different cases.

Example 1 Consider the parameters as follows

$$b = 0.8, \quad K = 20, \quad d_1 = 0.5, \quad d_2 = 0.3, \quad \mu = 0.5,$$
$$\gamma = 0.5, \quad \tau_1 = 10, \quad \tau_2 = 5, \quad \beta_0 = 0.5, \quad m_0 = 0.02.$$

In this case, $R_0 = 2.039 * 10^{-6} < 1$ and according to Theorem 4.2, $E_0 = (20, 0, 0)$ is globally asymptotically stable, as shown in Fig. 1.

Now, we examine the asymptotic behavior around the positive equilibrium E_* in different cases as follows.

Example 2 The following parameters are chosen

$$b = 2, \quad K = 30, \quad d_1 = 0.3, \quad d_2 = 0.3, \quad \mu = 0.5,$$
$$\gamma = 0.6, \quad \tau_1 = 0, \quad \tau_2 = 0, \quad \beta_0 = 0.3, \quad m_0 = 0.5,$$

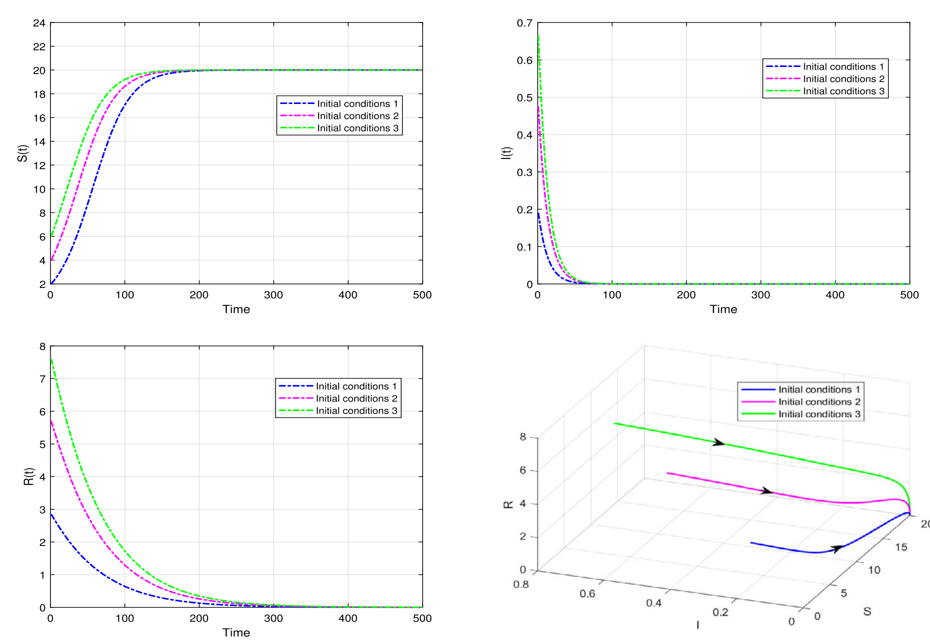


Figure 1 Global asymptotical stability of E_0

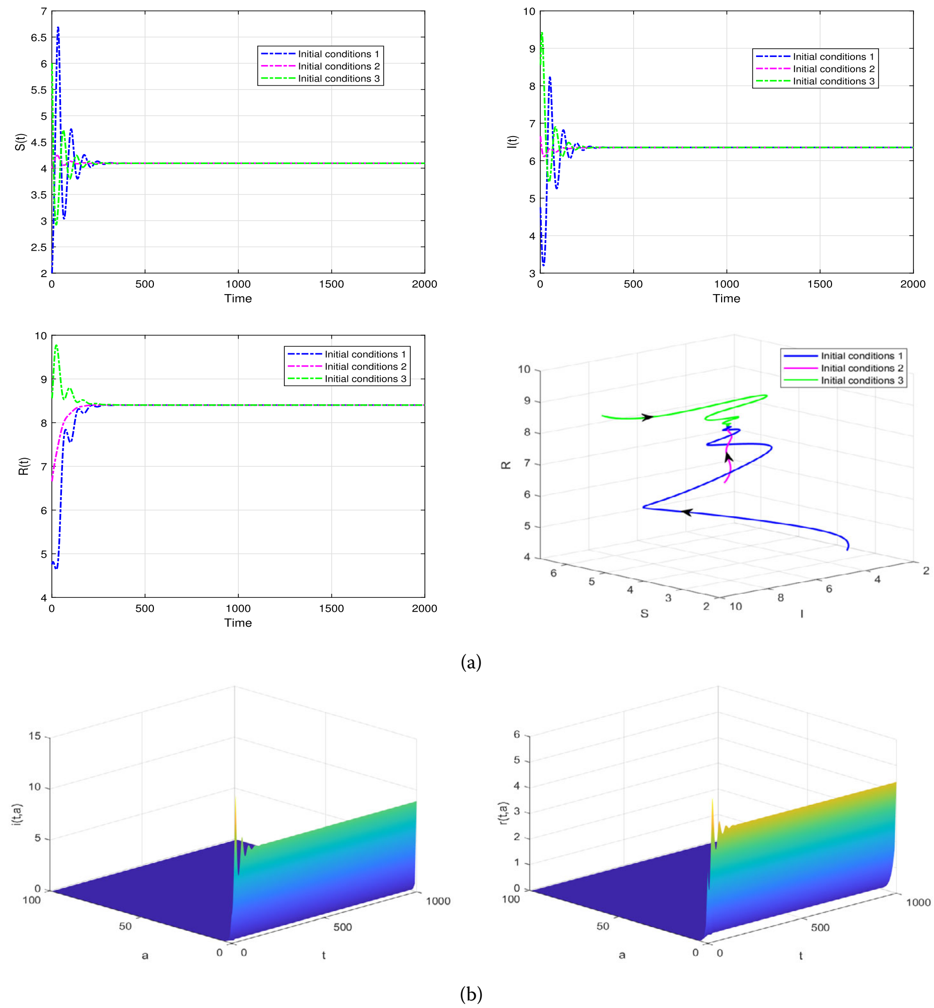


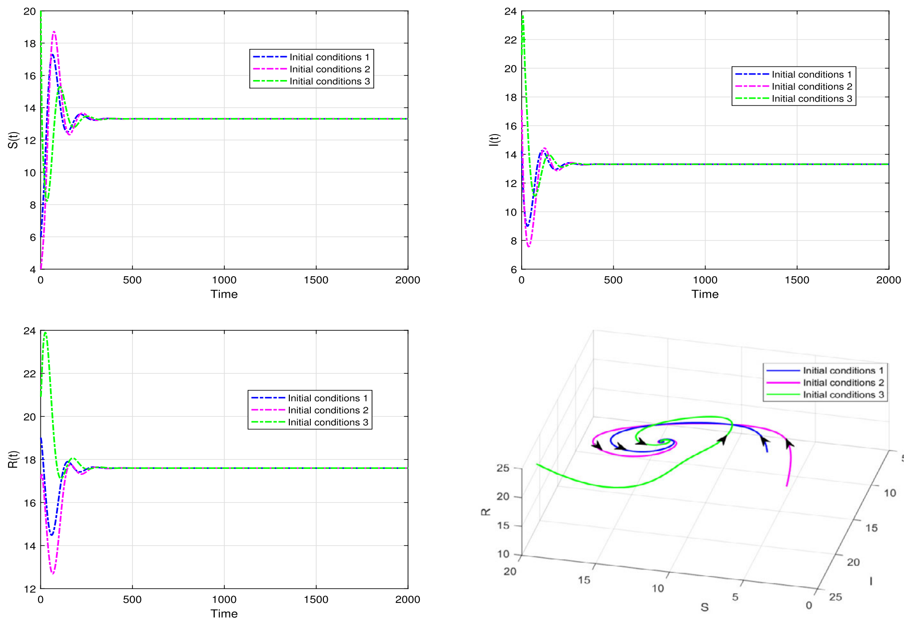
Figure 2 (a): The stability result for E_* in Case (I). (b): The distributions of $i(t,a)$ and $r(t,a)$ about time and infection age at E_* in Case (I)

then $R_0 = 6.428 > 1$, we find that the endemic steady state E_* is locally asymptotically stable, as shown in Fig. 2(a), which verifies the result of Theorem 5.1 in Case (I). Moreover, we represent the distributions of the infected class (I-class) and recovered class (R-class) about time t and infection age a in Fig. 2(b).

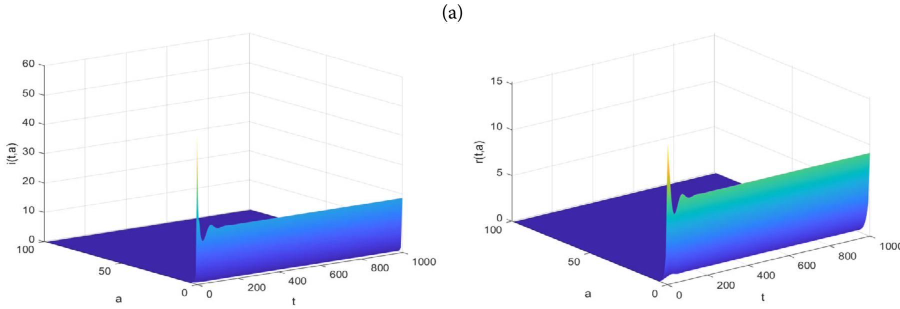
Example 3 Let the vital parameters in (1.2) be chosen as

$$b = 2, \quad K = 30, \quad d_1 = 0.3, \quad d_2 = 0.3, \quad \mu = 0.5,$$
$$\gamma = 0.6, \quad \tau_1 = 1, \quad \tau_2 = 0, \quad \beta_0 = 0.3, \quad m_0 = 0.2.$$

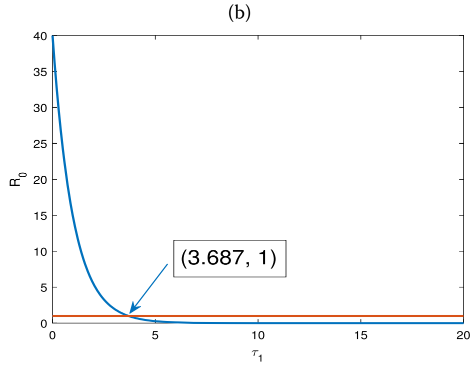
Then, we have $\Delta = -8.4721 < 0$ (defined in Lemma 5.1) satisfying the condition (i) in Theorem 5.4. Therefore, the epidemic steady state E_* is asymptotically stable, as shown in Fig. 3(a). Accordingly, the distributions of $i(t,a)$ and $r(t,a)$ with both time and infection age at E_* are shown in Fig. 3(b).



(a)



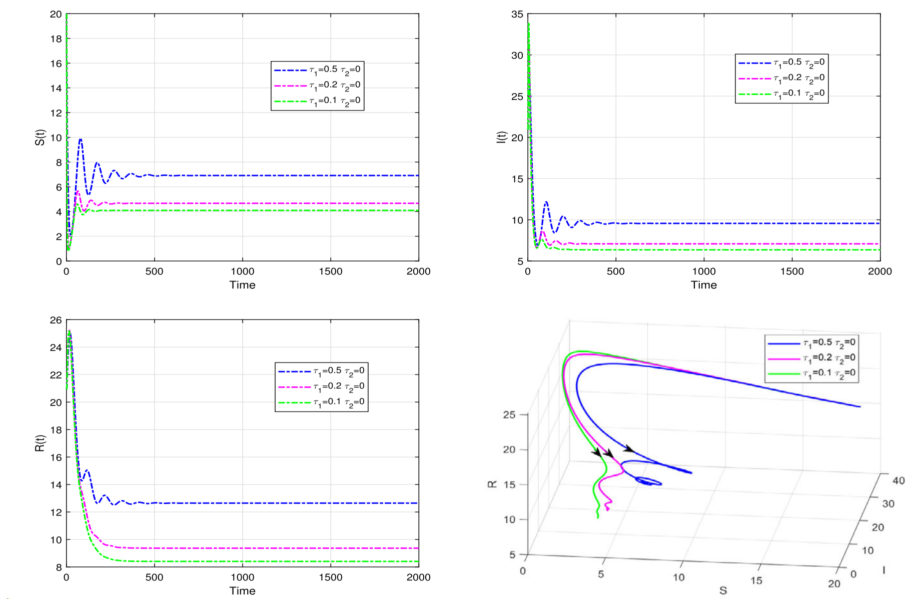
(b)



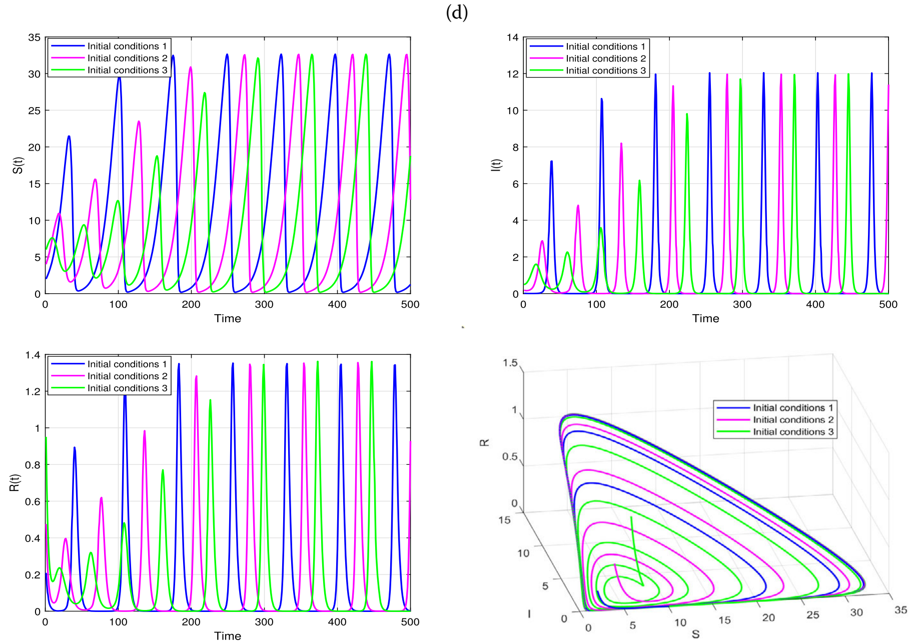
(c)

Figure 3 (a): The asymptotical stability of E_* in (i) of Theorem 5.4 for Case (II). (b): The stable distributions of $i(t,a)$ and $r(t,a)$ in Case (II). (c): The relationship between R_0 and τ_1. (d): The asymptotical stability of E_* in (ii) of Theorem 5.4 for Case (II). (e): The Hopf bifurcation result in Case (II). (f): The periodic distributions for $i(t,a)$ and $r(t,a)$ at E_* in Case (II)

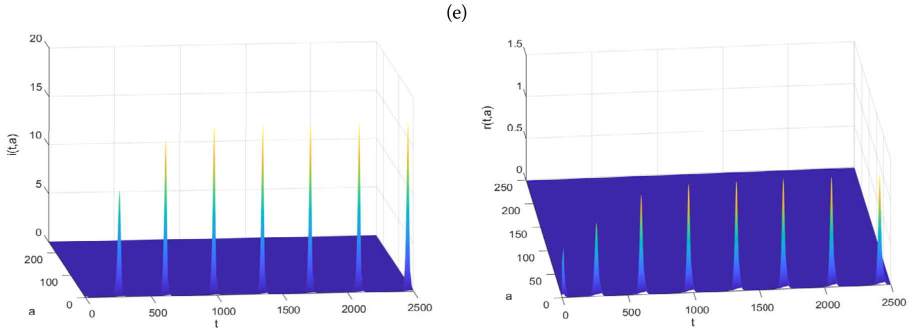
Afterward, we discuss the changes in dynamics of system (1.2) for various τ_1. Set $b = 0.2$, $K = 50$, $d_1 = 0.6$, $d_2 = 0.5$, $\mu = 0.3$, $\gamma = 0.1$, $\tau_2 = 0$, $\beta_0 = 0.8$, $m_0 = 0.2$. As shown in Fig. 3(c), the basic reproduction number R_0 decreases monotonically as τ_1 increases gradually, and R_0 will be less than 1 when $\tau_1 > 3.678$.



(d)



(e)



(f)

Figure 3 Continued

Take now $\tau_1 = 0.1; 0.2; 0.5 < \tau_{10}$, then $R_0 = 36.193; 32.749; 24.261 > 1$; respectively. The local stability of E_* with different τ_1 is shown in Fig. 3(d), which is consistent with the theoretical result (ii) in Theorem 5.4. Comparing the curves with different τ_1 in Fig. 3(d), we find that as the infectiousness delay $\tau_1 \in [0, \tau_{10})$ increases, the number of the susceptible, infected, and recovered classes increases gradually.

However, if τ_1 exceeds the threshold value τ_{10}, E_* will become unstable and will lead to the Hopf bifurcation around E_*, which can been seen in Fig. 3(e) and (f) with $\tau_1 = 1.5$.

Example 4 Let

$$b = 2, \quad K = 50, \quad d_1 = 0.2, \quad d_2 = 0.3,$$
$$\mu = 0.1, \quad \gamma = 0.3, \quad \beta_0 = 0.9, \quad m_0 = 0.2.$$

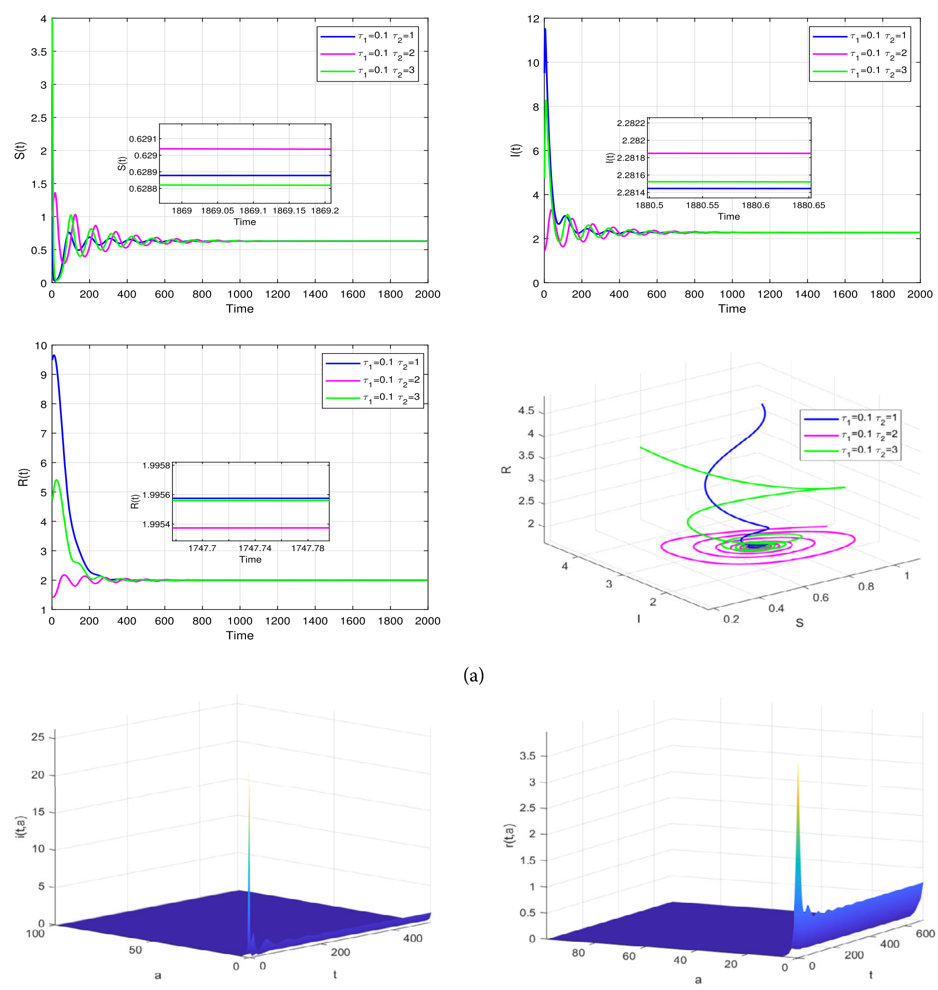


Figure 4 (a): The asymptotical stability of E_* in Case (III). (b): The stable distributions of $i(t,a)$ and $r(t,a)$ in Case (III). (c): The case of Hopf bifurcation in Case (III). (d): The periodic distributions of $i(t,a)$ and $r(t,a)$ under Case (III)

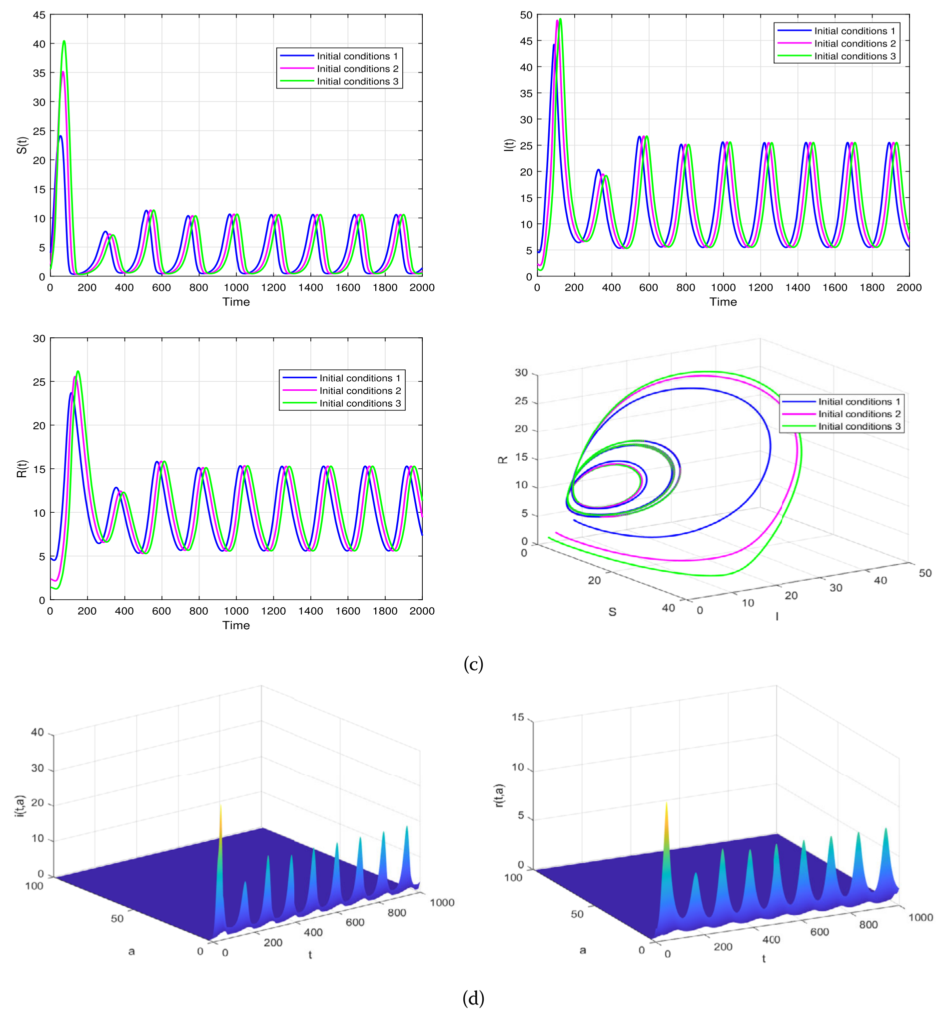


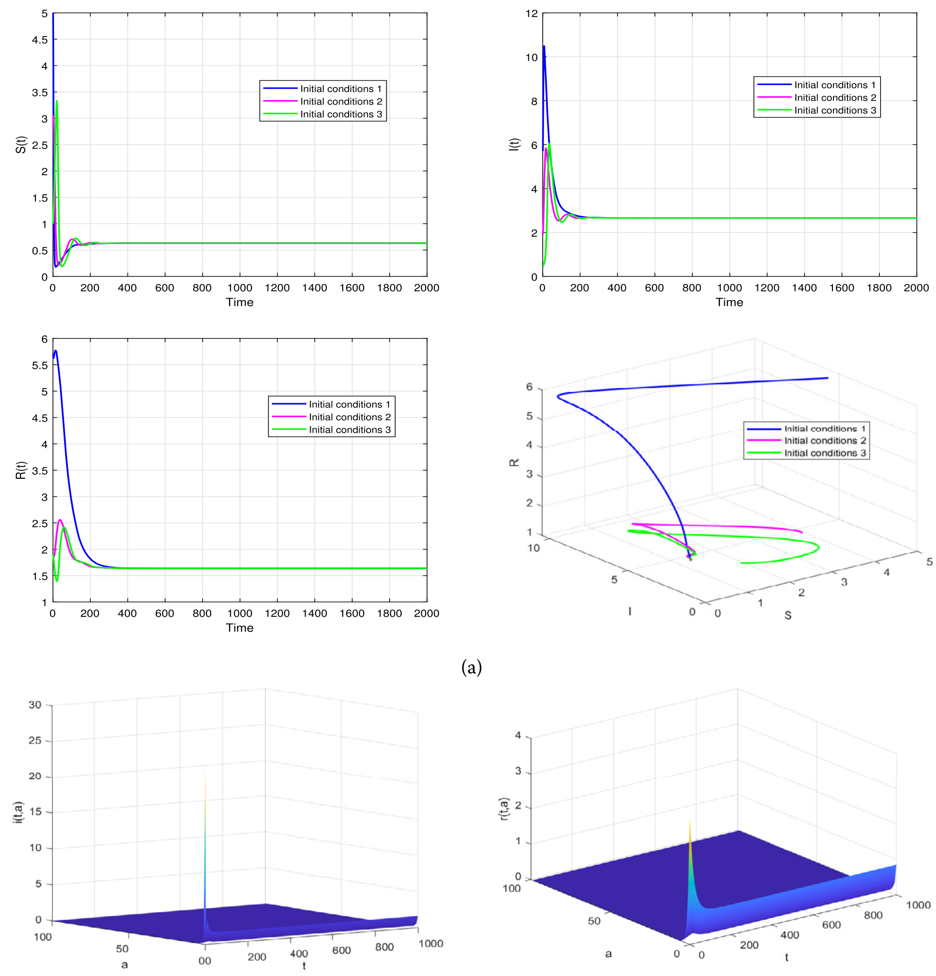
Figure 4 Continued

Thus, $\tau_1 = 0.1 < \tau_{10}$, $\tau_2 = 1, 2, 3$, satisfies the condition in Theorem 5.5(i) in Case (III). By applying Theorem 5.5(i), the locally asymptotic stability of E_* is verified, when $\tau_2 \geq 0$. In Fig. 4(a), E_* indeed remains stable when τ_2 takes different values, which agrees well with the result in Theorem 5.5(i). Besides, Fig. 4(b) describes the changes in the infected and recovered populations about time and infected age.

Now, we increase τ_2, but remain $\tau_1 \in [0, \tau_{10})$ and other parameters the same, as in Fig. 4(a). At this time, we have verified that the conditions in (ii) of Theorem 5.5 are satisfied. Therefore, the Hopf bifurcation occurs at the equilibrium E_*, which is displayed in Fig. 4(c) and (d) with $\tau_2 = 3.5$ and three different initial conditions.

Example 5 Let $\tau_1 = \tau_2 = 0.1$, and other parameters are the same as in Fig. 4(a). In this situation, in the light of Theorem 5.6(i), E_* is asymptotically stable, see, Fig. 5(a). Accordingly, the distributions of $i(t, a)$ and $r(t, a)$ about time and infection age are shown in Fig. 5(b).

However, if we choose $\tau_1 = \tau_2 = 1, 2.6, 3$, other parameters do not change, the epidemic equilibrium E_* loses its stability, and periodic oscillation occurs, see Fig. 5(c) and (d). From



(a)

(b)

Figure 5 (a): The asymptotical stability of E_* in Case (IV). (b): The distributions of $i(t,a)$ and $r(t,a)$ about time and infection age in Case (IV). (c): The case of the Hopf bifurcation in Case (IV). (d): The periodic distributions of $i(t,a)$ and $r(t,a)$ about time and infection age in Case (IV)

the three different curve lines in Fig. 5(c), we observe that the amplitude of periodic solution increases as delay $\tau = \tau_1 = \tau_2$ increases.

7 Conclusion

The transmission between infected and susceptible persons is influenced by the degree of the infectivity of the contagion person, measured by the age of infection. Infected individuals with differentiated infection age may result in different levels of infection among susceptible individuals. The time spent by a recovered person in the R-class before becoming susceptible is measured by the age of recovery, as it can highlight that immunity in the R-class is not permanent. In fact, the changes of virus in each year will lead to changes of individual immunity. Therefore, the impact of recovery age is also crucial to determine the outbreak of the contagious disease. This paper investigates an age-structured SIRS epidemic model with both infection age and recovery age, as well as the logistic growth of susceptible individuals. To better understand the dynamic behaviors of model (1.2),

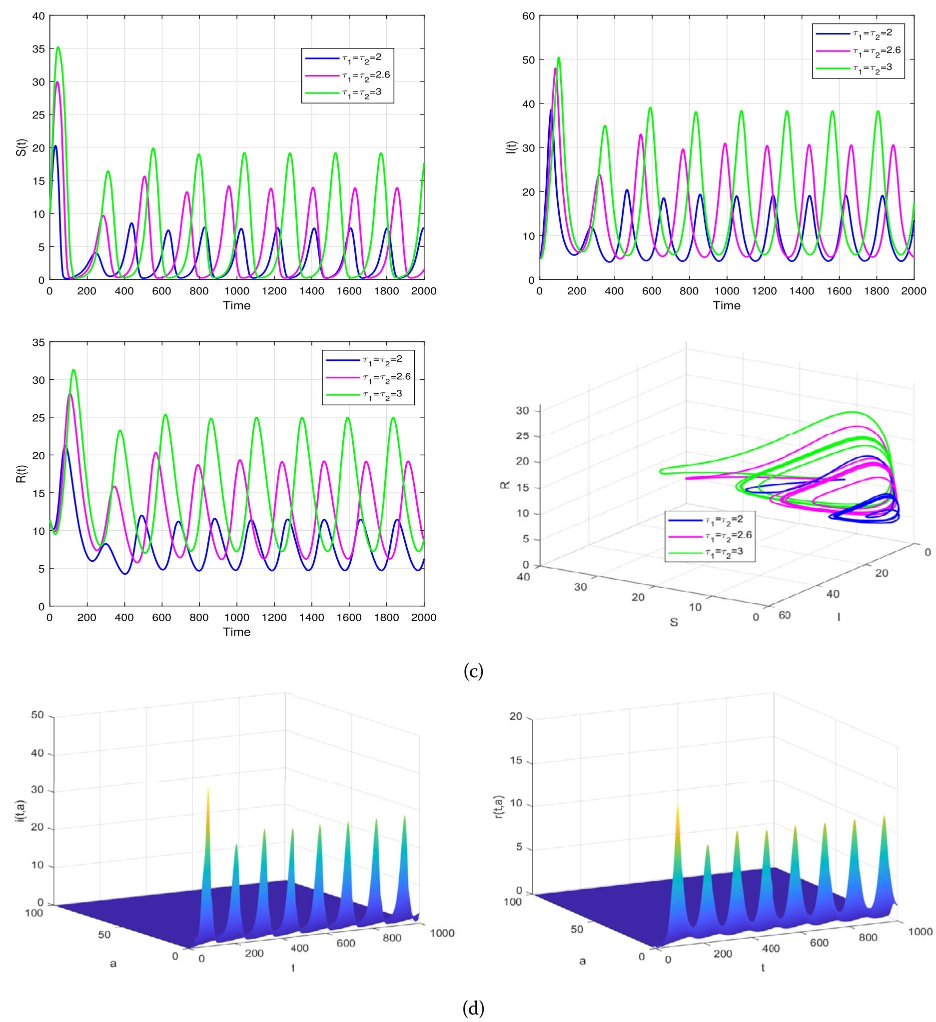


Figure 5 Continued

we first rewrite the model as a non-densely defined abstract Cauchy problem, and then establish the existence, positivity and boundedness of equilibria, and calculate the basic reproduction number R_0, linearized system, and characteristic equation at the equilibria. Following the characteristic equation and spectral analysis methods, we show that the disease-free equilibrium E_0 is globally asymptotically stable when $R_0 < 1$ (Theorem 4.2). Under Assumption 5.1, where the latent and immune periods of infection and recovery ages are introduced as two delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$, we show the local stability and the existence of Hopf bifurcation for the endemic equilibrium E_* when $R_0 > 1$ by applying the Hopf bifurcation theorem (Theorems 5.1, 5.4, 5.5, and 5.6). It is shown that the stability and Hopf bifurcation switches will appear as the two delays increase from zero. Finally, we conducted several numerical results with graphs to demonstrate the theoretical results.

The Hopf bifurcation for age-structured systems with only one delay and one age is very prevalent. Compared to only one-age system, the Hopf bifurcation analysis in the case of two ages and two delays is more complicated. The system in this paper with both infection age and recovery age results in incubation period τ_1 and immunity period τ_2. Hence, the analysis process of the characteristic equation in this paper is more complicated than one-age system. Moreover, as shown in the Numerical simulation section, the changes in the latent and immune periods play a vital role in the dynamic behaviors of infectious diseases and lead to stability and bifurcation switches at the positive equilibrium E_*, which means that the infection and recovery ages result in more richer dynamic behaviors, compared to the epidemic model with only one age. In Fig. 3(c), we remark that the delay τ_1 has a negative influence on the reproduction number R_0, where we observe the possibility of reducing the value of R_0 from a high value $R_0 = 40$ for $\tau_1 = 0$ to less than 1 for values of delay τ_1 larger than 3.687. The vaccines affecting the reproduction of the virus in the host cell allow the human immune system to react early to the new intruder virus and thus leads to the increase in the incubation period τ_1. Then, it is easier to control R_0 to be less than one. As a result, it is possible to eliminate the infection from the population if vaccines are adopted to reduce the speed of the reproduction of the virus. Besides, we find that $\beta(a)$ has a positive influence on the reproduction number R_0. Therefore, adopting some quarantine measures to reduce the infection rate $\beta(a)$ is an effective and practical policy to control disease transmission.

Two interesting questions remain for future research: what would be the results of stability and Hopf bifurcation if other function types of the infection age $\beta(a)$ and the recovery age $m(a)$ or the spatial diffusions are considered in our epidemic model.

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Data availability
This declaration is not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
DY: Conceptualization, Methodology, Supervision, Writing-original draft, Review & editing. YC: Conceptualization, Visualization, Software, Writing C review & editing. All authors read and approved the final manuscript.

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