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A central limit theorem for a classical gas

Hans Zessin[1*] and Suren Poghosyan[2]

Dedicated to Jean Ginibre whose work has been our constant source of inspiration

*Correspondence:
zessin@math.uni-bielefeld.de
[1]Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501, Bielefeld, Germany
Full list of author information is available at the end of the article

Abstract
For a class of translation-invariant pair potentials ϕ in $(\mathbb{R}^d, z\lambda)$ satisfying a stability and regularity condition, we choose z so small that the associated collection $\mathcal{G}(\phi, z\lambda)$ of Gibbs processes contains at least the stationary process G, which is a Gibbs process in the sense of DLR and is given by the limiting Gibbs process with empty boundary conditions. Using an abstract version of the method of cluster expansions and Dobrushin's approach to the central limit theorem, we present a central limit theorem for the particle numbers of G.

Keywords: Cluster expansions; Gibbs point processes; Classical continuous systems

1 Introduction

In 1970 Minlos and Halfina [12] proved for a limiting lattice gas with empty boundary conditions, i.e., for some phase, that the fluctuations of the particle number have asymptotically for large volumes a normal distribution if the activity z is small enough. They assumed Penrose stability in the sense of [18] and some regularity condition for the underlying pair potential. Martin-Löf [9] obtained in 1973 similar results for the Ising model. In 1975 Malyshev [7] proved a general result of this type under the assumption of exponential decay of correlations.

To our knowledge, such results have not yet been presented for a classical continuous gas without recurring to the lattice case. That the latter is possible has been noted already by Minlos and Halfina [12] and was realized in a number of works (cf. [3–6]).

Our aim is to show a central limit theorem for the particle numbers of a classical gas in Euclidean space by using point process methods, thereby avoiding the artificial recours to lattice gas technics. Our assumptions on the potential ϕ are similar to the one of Minlos and Halfina; particularly, we consider weakly Penrose stable pair potentials and a regularity condition, which implies that the activity z is small enough such that the limiting Gibbs process with empty boundary conditions, denoted by G, exists and is a Gibbs process in the sense of DLR. For this pure phase of a Gibbs process G, using an equivalent form of DLR equations given by Nguyen and Zessin [15], we show that the number of particles in a bounded large region satisfies the central limit theorem.

For a proof we use the method of cluster expansions in an abstract form, developed by Nehring [13] (see also [17]). Another important ingredient is Dobrushin's approach to the central limit theorem (cf. [1]), which is based on an application of elementary facts of the theory of analytic functions.

2 Synopsis of notions and methods

Notions not explained here can be found in [18].

2.1 Point processes

The underlying phase space $(X, \mathcal{B}(X), \mathcal{B}_0(X))$ is an lcscH, i.e., locally compact, second countable Hausdorff topological space.[1] $\mathcal{B}(X)$ denotes its Borel sets and $\mathcal{B}_0(X)$ is bounded, this means relatively compact, Borel sets. $\mathcal{M}(X)$ is the space of Radon measures on X, which is Polish with respect to vague convergence, and $\mathcal{M}^{\cdot}(X)$ is the Borel subspace of simple point measures, i.e., locally finite subsets of X. $\mathfrak{X} = \mathcal{M}^{\cdot}_f(X)$ denotes the measurable subset of finite elements ξ in $\mathcal{M}^{\cdot}(X)$ and $\mathfrak{X}'$ all $\xi \in \mathfrak{X}$ with $\xi \neq o$, where o denotes the zero-measure or vacuum. For notational convenience, we do not indicate always the underlying space in the sequel. Δ_o is the Dirac measure on $\mathcal{M}^{\cdot}$ in o. Dirac measures on X are denoted by ε_x.

Laws P on $(\mathcal{M}^{\cdot}, \mathcal{B}(\mathcal{M}^{\cdot}))$ are called (simple point) processes in X.[2] We use different function spaces: U is the collection of all bounded, nonnegative, and measurable functions f on X with bounded support. F denotes the set of all nonnegative measurable functions on the underlying space, which is not indicated. For $f \in U$, the functional $\zeta_f : \mu \mapsto \mu(f)$ is a well-defined measurable function on $\mathcal{M}^{\cdot}$, which is vaguely continuous for continuous $f \in U$. If $B \in \mathcal{B}_0(X)$, then we write ζ_B instead of $\zeta_{\mathbf{1}_B}$. ζ_B counts the number of particles in B.

Functionals of point processes $\mathscr{L}_P(f) = P(\mathrm{e}^{-\zeta_f}), f \in U$, denotes the Laplace transform of process P. Another important functional is the Campbell measure of P, defined by

$$C_P(h) = \int_{\mathcal{M}^{\cdot}} \int_X h(x, \mu)\mu(\mathrm{d}x)P(\mathrm{d}\mu), \quad h \in F.$$

P is uniquely determined by both functionals. The Campbell measure of a process P determines all moment measures of P. Indeed, the kth moment measure is given by

$$\nu_P^k(f_1 \otimes \cdots \otimes f_k) = C_P\big(f_1 \otimes (\zeta_{f_2} \cdots \zeta_{f_k})\big), \quad f_1, \ldots, f_k \in F.$$

If ν_P^k is a Radon measure, i.e., $\nu_P^k(f) < \infty$ for all $f \in U$, then P is called of order k. If P has moments of all orders, then P is called of infinite order. The factorial measure of P of order k is defined as the restriction of ν_P^k to $\widetilde{X}^k = \{(x_1, \ldots, x_k) : (i \neq j \Rightarrow x_i \neq x_j)\}$ and is denoted by $\widetilde{\nu}_P^k$. Thus for $f \in F(X^k)$

$$\widetilde{\nu}_P^k(f) = \int_{\mathcal{M}^{\cdot}} P(\mathrm{d}\mu) \int_{X^k} \widetilde{\mu}^k(\mathrm{d}x_1 \ldots \mathrm{d}x_k) f(x_1, \ldots, x_k).$$

[1] For a classical gas, X will be the Euclidean space $E = \mathbb{R}^d$.

[2] In the sequel we call them processes.

Here

$$\widetilde{\mu}^k(\mathrm{d}x_1\dots\mathrm{d}x_k) = \mu(\mathrm{d}x_1)(\mu-\varepsilon_{x_1})(\mathrm{d}x_2)\dots\left(\mu-\sum_{\ell=1}^{k-1}\varepsilon_{x_\ell}\right)(\mathrm{d}x_k).$$

If P is of order k, then obviously also $\widetilde{\nu}_P^k$.

2.2 The method of cluster expansions

We use the abstract form of this method as developed in [13, 14, 17] and explain it shortly in the special case of a classical gas in Euclidean space. Let E denote the space $\mathbb{R}^d$ with Lebesgue measure $\lambda(\mathrm{d}x) = \mathrm{d}x$. For a given activity $z > 0$, we set $\varrho = z\lambda$. Let Φ be a pair potential on E, i.e., a measurable function $\Phi : E \times E \to (-\infty, +\infty]$. We assume stability and some regularity condition, which we will make explicit in what follows.

Consider next the associated Ursell function, defined by $\varkappa(o) = 0$, $\varkappa(x) \equiv 1$, and

$$\varkappa(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n}) = \sum_{\gamma\in\mathcal{C}_n}\prod_{\{x,y\}\in\gamma}\left(\mathrm{e}^{-\Phi(x,y)}-1\right), \quad n\geq 2.$$

This is a sum over all unoriented graphs γ with n vertices, and the product is taken over all edges of γ.

The method of cluster expansions enables the construction of the limiting Gibbs process with empty boundary conditions. And strengthening the assumptions on Φ allows to show that the limiting process is even a Gibbs process.

Consider the cluster functional

$$\mathrm{L} = \varkappa.\Lambda_\varrho,$$

i.e., the measure Λ_ϱ with density $\varkappa$. Here Λ_ϱ is the measure ϱ lifted to the collection of finite configurations $\mathfrak{X}$, i.e.,

$$\Lambda_\varrho(\varphi) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{E^n}\varphi(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n})\varrho(\mathrm{d}x_1)\dots\varrho(\mathrm{d}x_n), \quad \varphi\in F.$$

In general, L is not a well-defined mathematical object. To give to it a precise meaning, we shall make assumptions ensuring that L is a signed Radon measure. This means that L is locally a finite signed measure.

To be more precise, we consider the so-called variation of L

$$|\mathrm{L}| = |\varkappa|.\Lambda_\varrho.$$

This is a positive measure on $\mathfrak{X}' := \mathfrak{X}\setminus\{o\}$, but may be infinite. Our assumptions below on the potential Φ will assure that $|\mathrm{L}|$ is of first order. This means

$$\nu^1_{|\mathrm{L}|}f := \int_{\mathfrak{X}'}\xi(f)|\varkappa|(\xi)\Lambda_\varrho(\mathrm{d}\xi) < \infty, \quad f\in U. \qquad (\mathcal{P})$$

By partial integration this is equivalent to saying that

$$\int_E f(x)\int_{\mathfrak{X}'}|\varkappa|(\xi+\varepsilon_x)\Lambda_\varrho(\mathrm{d}\xi)\varrho(\mathrm{d}x) < \infty, \quad f\in U. \qquad (\mathcal{P}')$$

Indeed, by Mecke's formula [10],

$$\int_{\mathfrak{X}'}\int_E f(x)|\varkappa|(\xi)\xi(\mathrm{d}x)\Lambda_\varrho(\mathrm{d}\xi)=\int_{\mathfrak{X}'}\int_E f(x)|\varkappa|(\xi+\varepsilon_x)\varrho(\mathrm{d}x)\Lambda_\varrho(\mathrm{d}\xi).$$

We now localize L in the following way: Let $B\in\mathcal{B}_0(E)$ and consider

$$\mathrm{L}_B:=\mathbf{1}_{\{\zeta_{B^c}=0\}}.\mathrm{L},$$

i.e., we restrict L to the collection $\mathfrak{X}'(B)$ of configurations in B. This is a finite signed measure on $\mathfrak{X}'$. Indeed, assumption $(\mathcal{P})$ implies

$$|\mathrm{L}_B|(\mathfrak{X}')\le\nu^1_{|\mathrm{L}|}(B)<\infty.$$

Moreover, condition $(\mathcal{P})$ implies that the so-called modified Laplace transform of $|\mathrm{L}|$, defined by

$$\mathscr{K}_{|\mathrm{L}|}f:=|\mathrm{L}|\big(1-\mathrm{e}^{-\zeta_f}\big),\quad f\in U,$$

is finite-valued, and thereby also $\mathscr{K}_{\mathrm{L}}(f)$. Note here that $1-\mathrm{e}^{-\zeta_f}\le\zeta_f, f\in U$.

The local construction Let $\mathcal{E}_{fs}$ denote the space of all finite signed measures on $\mathfrak{X}$: It is well known that the space $\mathcal{E}_{fs}$ is a real Banach algebra with respect to convolution $*$ and the total variation norm. Its unit is $\mathbf{1}:=\Delta_o$, the Dirac measure in the vacuum o.

Thus the finite signed measure L_B generates within $\mathcal{E}_{fs}$ the signed measure

$$\Im_{\mathrm{L}_B}=\frac{1}{\Xi_B}\cdot\exp\mathrm{L}_B,\tag{1}$$

where

$$\exp\mathrm{L}_B=\sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{L}_B)^{*n},$$

$$\Xi_B=\exp\mathrm{L}_B\big(\mathfrak{X}'\big)=\exp\bigg(\int_{\mathfrak{X}'(B)}\varkappa(\xi)\Lambda_\varrho(\mathrm{d}\xi)\bigg)=\exp\bigg(\int_{\mathfrak{X}'}\varkappa(\xi)\Lambda_{\varrho_B}(\mathrm{d}\xi)\bigg),$$

$$\varrho_B=\mathbf{1}_B.\varrho.$$

Note that the integral $\int_{\mathfrak{X}'}\varkappa(\xi)\Lambda_{\varrho_B}(\mathrm{d}\xi)$ is absolutely convergent because of condition $(\mathcal{P})$. The Laplace transform of L_B is

$$\mathscr{L}_{\mathrm{L}_B}=\exp(-\mathscr{K}_{\mathrm{L}_B}),\tag{2}$$

where

$$\mathscr{K}_{\mathrm{L}_B}f=\mathrm{L}_B\big(1-\mathrm{e}^{-\zeta_f}\big),\quad f\in U.$$

It is now decisive that $\Im_{\mathrm{L}_B}$ can be represented in the following form:

$$\Im_{\mathrm{L}_B} = \frac{1}{\Xi_B} \cdot \Gamma(\varkappa).\Lambda_{\varrho B}, \tag{3}$$

where $\Gamma(\varkappa)$ is the algebraic exponential of the Ursell function, explicitly written as

$$\Gamma(\varkappa)(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}) = \sum_{\mathcal{J} \in \wp[k]} \prod_{J \in \mathcal{J}} \varkappa\big((x_j)_{j\in J}\big).$$

Here $\wp[k]$ denotes the collection of partitions of the set $[k]$. On the other hand (cf. Ruelle [19]), this coincides with the Boltzmann factor, i.e.,

$$\Gamma(\varkappa) = \exp(-E_\Phi),$$

where E_Φ denotes the energy based on Φ, i.e.,

$$E_\Phi(\varepsilon_{x_1} + \cdots + \varepsilon_{x_k}) = \sum_{1\le i<j\le k} \phi(x_j - x_i), \quad x_1, \ldots, x_k \in E.$$

We thus see that the above method of abstract cluster expansions leads for finite signed L *to the local Gibbs process for* (ϱ, Φ) *with empty boundary conditions.*

The infinitely extended construction (*For the details, we refer to [17].*)

To construct the limiting process $\Im_{\mathrm{L}}$, we localize L as above and then go to the limit. To be more precise: Let $(B_n)_n$ be an increasing sequence of bounded Borel sets in E exhausting E, i.e., $B_n \uparrow E$ if $n \uparrow \infty$. We assume that $(B_n)_n$ is regular in the following sense:

$$\lim_{n\to\infty} \frac{\lambda(B_n \cap (B_n - y))}{\lambda(B_n)} = 1, \quad y \in E.$$

Consider the sequence of finite signed measures

$$\mathrm{L}_n = \mathbf{1}_{\mathfrak{X}'(B_n)}.\mathrm{L}$$

and the associated sequence $\Im_{\mathrm{L}_n}$ of local Gibbs processes. By means of Mecke's continuity theorem (cf. [11]), one then obtains the following.

Lemma 1 *Under assumption* $(\mathcal{P})$ *or* $(\mathcal{P}')$ *there exists a limiting process* $\Im_{\mathrm{L}}$ *in E such that* $\Im_{\mathrm{L}_n}$ *converges weakly to* $\Im_{\mathrm{L}}$ *if* $n \to \infty$. *The Laplace transform of* $\Im_{\mathrm{L}}$ *is given by* $\exp(-\mathcal{K}_{\mathrm{L}})$.

$\Im_{\mathrm{L}}$ is the limiting Gibbs process with empty boundary conditions. In the sequel we write G for $\Im_{\mathrm{L}}$.

2.3 The cluster equation (cf. Mecke [11] and Nehring [13])

If L is of first order, then $\Im_{\mathrm{L}}$ is of first order and a solution P of the cluster equation

$$C_P = C_{\mathrm{L}} \star P. \tag{$\mathscr{C}\ell$}$$

Here the operation $\star$ is defined by[3]

$$C_{\mathrm{L}} \star P(h) = \int_{\mathcal{M}^{\cdot}} \int_E \int_{\mathfrak{X}} h(x, \xi + \mu) C_{\mathrm{L}}(\mathrm{d}x\,\mathrm{d}\xi) P(\mathrm{d}\mu), \quad h \in F.$$

Conversely, if L is of first order and P is a process that solves $(\mathscr{C}\ell)$, then $P = \Im_{\mathrm{L}}$.

We will see below that the cluster equation is very useful to calculate moment measures of G as well as other properties.

2.4 Stationary processes

The group operation in E is written additively: $T_x z = z - x$. It induces on $\mathcal{M}^{\cdot}(E) = \mathcal{M}^{\cdot}$ the transformation

$$(T_x\mu)(B) = \mu(B + x), \quad B \in \mathcal{B}.$$

Write also $T_x\mu = \mu - x$. This in turn induces on the level of processes P the transformation T_xP, i.e., the image of P under T_x. P is called stationary if

$$T_xP = P, \quad x \in E.$$

More generally, we shall consider also the stationarity property for L.

For stationary P of first order, the first moment measure ν_P^1 of P is stationary and thereby of the form $\imath_P \cdot \lambda$ for some $\imath_P > 0$. $\imath_P$ thus is the expected number of particles in a region B with $\lambda(B) = 1$. [4]

3 Classical continuous systems

We consider from now on a classical gas in the Euclidean space $E = \mathbb{R}^d$ of the following type.

Conditions on the potential We consider pair potentials Φ in E of the form

$$\Phi(x, y) = \phi(x - y), \quad x, y \in E,$$

where ϕ is an even, measurable function on E, and assume stability and regularity in the following sense:

(A1) *(weak or w$\wp$-stability)* There exists a constant $\mathrm{c} \geq 0$ such that for every $\xi \in \mathfrak{X}'$ there exists $x \in \xi$ with

$$W_\phi(x, \xi_x) \geq -\mathrm{c} \cdot |\xi|, \quad \xi \in \mathfrak{X}.$$

Here $|\xi| = \xi(E)$ denotes the number of particles in ξ; $\xi_x = \xi - \varepsilon_x$ is the configuration without particle x and

$$W_\phi(x, \xi_x) = \sum_{y \in \xi_x} \phi(y - x)$$

is the conditional energy of x given ξ. If c is given, the *regularity condition* is

[3] Note that the Campbell measure, which was defined above for processes only, makes also sense for L.

[4] Throughout processes are assumed to be of first order without further mentioning.

(A2) *(modified* c*-regularity)*

$$z\,\mathrm{e}^{\mathrm{c}+1}\,C''_\phi \le 1.$$

Here $C''_\phi = \int_E |\overline{\phi}|(y)\,\mathrm{e}^{\phi^-(y)}\,\mathrm{d}y$, where $\overline{\phi} \equiv 1$ if $\phi \equiv +\infty$ and otherwise coincides with ϕ.

A remark is in order here: Ruelle [19] uses the constant $C_\phi = \int_E |1-\mathrm{e}^{-\phi(y)}|\,\mathrm{d}y$; whereas in [18] we used $C'_\phi = \int_E (1-\mathrm{e}^{-|\phi|(y)})\,\mathrm{d}y$. It is obvious that

$$C'_\phi \le C_\phi \le C''_\phi. \tag{4}$$

Furthermore, $w\wp$-stability of ϕ implies the usual stability for the same constant, i.e.,

$$E_\Phi(\xi) \ge -\mathrm{c}\cdot|\xi|, \quad \xi \in \mathfrak{X}.$$

Under conditions (A1) and (A2), the measure $|\mathrm{L}|$ is of first order by Theorem 2.1 from [16]. Lemma 1 therefore implies the existence of the limiting Gibbs process $\mathrm{G} = \Im_{\mathrm{L}}$ with empty boundary conditions. Moreover, G is a Gibbs process for (ϕ,ϱ) of infinite order by Theorem 3 in [18]. This means that G satisfies the DLR-equations in the following equivalent form (cf. Nguyen X.X. and Zessin [15]): G is a solution P of the following equation:

$$C_P(h) = \int_E \int_{\mathcal{M}^\cdot} h(x,\mu+\varepsilon_x)\,\mathrm{e}^{-W_\phi(x,\mu)}\,P(\mathrm{d}\mu)\varrho(\mathrm{d}x), \quad h \in F. \tag{Σ_ϱ}$$

Its correlation functions, i.e., the densities r^n_{G} of the factorial moment measures $\widetilde{\nu}^n_{\mathrm{G}}$ with respect to the product measures ϱ^n, have the representation

$$r^n_{\mathrm{G}}(x_1,\ldots,x_n) = \int_{\mathcal{M}^\cdot} \mathrm{e}^{-W_\phi(\varepsilon_{x_1}+\cdots\varepsilon_{x_n},\mu)}\,\mathrm{G}(\mathrm{d}\mu), \quad x_1,\ldots,x_n \in E, \tag{5}$$

and satisfy the Ruelle estimate

$$r^n_{\mathrm{G}}(x_1,\ldots,x_n) \le \mathrm{e}^{\mathrm{c}\,n}. \tag{6}$$

(For details, we refer to [18].) Here

$$W_\phi(\varepsilon_{x_1}+\cdots\varepsilon_{x_n},\mu) = W_\phi(x_1,\mu) + W_\phi(x_2,\mu+\varepsilon_{x_1}) + \cdots + W_\Phi(x_n,\mu+\varepsilon_{x_1}+\cdots+\varepsilon_{x_{n-1}}).$$

We remark also that L is stationary since Φ has this property. This implies that the limiting Gibbs process G is stationary too. Indeed, since G is the unique solution of the cluster equation for L, it follows that its translation $T_x\mathrm{G}$ equals $\Im_{T_x\mathrm{L}}$, which coincides with $\Im_{\mathrm{L}} = \mathrm{G}$.

Another implication of the cluster equation is

$$\nu^1_{\mathrm{G}} = \nu^1_{\mathrm{L}} = z\int_{\mathfrak{X}} \varkappa(\xi+\varepsilon_0)\Lambda_\varrho(\mathrm{d}\xi)\cdot\lambda.$$

Thus the intensity of G is given by

$$\imath_{\mathrm{G}} = z\int_{\mathfrak{X}} \varkappa(\xi+\varepsilon_0)\Lambda_\varrho(\mathrm{d}\xi).$$

Finally, we evaluate for later use the variance $V_{\mathrm{G}}(\zeta_K)$: Set

$$r^1_{\mathrm{L}}(0) = \int_{\mathfrak{X}} \varkappa(\xi + \varepsilon_0)\Lambda_\varrho(\mathrm{d}\xi), \tag{7}$$

$$r^2_{\mathrm{L}}(x,y) = \int_{\mathfrak{X}} \varkappa(\xi + \varepsilon_x + \varepsilon_y)\Lambda_\varrho(\mathrm{d}\xi). \tag{8}$$

The variance $V_{\mathrm{G}}(\zeta_K) = \nu^2_{\mathrm{G}}(K \times K) - \nu^1_{\mathrm{G}}(K)^2$ of the particle number ζ_K in the region $K \in \mathcal{B}_0(E)$ with respect to the limiting Gibbs process G is given by Lemma 2.

Lemma 2

$$V_{\mathrm{G}}(\zeta_K) = \int_{K^2} r^2_{\mathrm{L}}(x,y)\varrho(\mathrm{d}x)\varrho(\mathrm{d}y) + r^1_{\mathrm{L}}(0) \cdot \varrho(K). \tag{9}$$

Proof Again using the cluster equation for L, we obtain

$$\begin{aligned} \nu^2_{\mathrm{G}}(K \times K) &= C_{\mathrm{G}}(\mathbf{1}_K \otimes \zeta_K) \\ &= C_{\mathrm{L}} \star \mathrm{G}(\mathbf{1}_K \otimes \zeta_K) \\ &= \int \mathbf{1}_K(x)(\mu + \xi)(K)\varkappa(\xi)\xi(\mathrm{d}x)\Lambda_\varrho(\mathrm{d}\xi)\,\mathrm{G}(\mathrm{d}\mu). \end{aligned}$$

The right-hand side is a sum of two terms. The first is given by $\nu^1_{\mathrm{G}}(K)\nu^1_{\mathrm{L}}(K) = \nu^1_{\mathrm{G}}(K)^2$, whereas the second, which will give the right-hand side of (8), equals

$$\int_{\mathfrak{X}} \int_K \xi(K)\varkappa(\xi)\xi(\mathrm{d}x)\Lambda_\varrho(\mathrm{d}\xi) = \int_K \int_{\mathfrak{X}} \varkappa(\xi + \varepsilon_x)\big(\xi(K) + 1\big)\Lambda_\varrho(\mathrm{d}\xi)\varrho(\mathrm{d}x).$$

Partial integration then yields the assertion. □

4 A lemma of Dobrushin

Let $(B_n)_n$ be some regular sequence of bounded Borel sets in E; take for example an increasing sequence of balls of radius N centered in 0 where $N \uparrow \infty$. Consider the sequence $S_n = \zeta_{B_n}$ of counting variables defined on the probability space given by some simple point process P in E of order two. Thus expectations and variances of the variables, i.e.,

$$P(\zeta_{B_n}) = \nu^1_P(B_n), \qquad V_P(\zeta_{B_n}) = P\big(\zeta^2_{B_n}\big) - \nu^1_P(B_n)^2,$$

are well defined and finite. Assume that all limiting variances are strictly positive. The central limit theorem is valid for the sequence $(\zeta_{B_n})_n$ if for any real λ,

$$\lim_{n\to\infty} P\left\{\frac{\zeta_{B_n} - \nu^1_P(B_n)}{\sqrt{V_P(\zeta_{B_n})}} < \lambda\right\} = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\lambda} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t.$$

The main lemma for a proof will be the following. Denote for $R > 0$ the centered closed ball in the complex plane $\mathbb{C}$

$$\mathcal{O}_R = \big\{a \in \mathbb{C} : |a| \le R\big\}.$$

Lemma 3 *(Dobrushin [1]) Assume that for some $R > 0$ the Laplace transform of P, defined by*

$$\Psi_n(a) = P\big(\mathrm{e}^{a\zeta_{B_n}}\big), \quad a \in \mathcal{O}_R, \tag{$\wp$1}$$

is finite-valued and does not vanish for all $a \in \mathcal{O}_R$. Assume also that Ψ_n is continuous in $\mathcal{O}_R$ and analytic in its interior, and furthermore that there exists some constant $K > 0$ such that

$$|\ln \|\Psi_n\||(a) \leq K \cdot |B_n|, \quad a \in \mathcal{O}_R. \tag{$\wp$2}$$

Here $\|\Psi_n\|(a) = \mathrm{G}(|\,\mathrm{e}^{a\zeta_{B_n}}\,|)$. Assume finally that there exists a constant $\mathrm{d} > 0$ such that the variances satisfy the condition

$$V_P(\zeta_{B_n}) \geq \mathrm{d} \cdot |B_n|. \tag{$\wp$3}$$

Both ($\wp$2) and ($\wp$3) are valid for sufficiently large n. Then the central limit theorem is valid for the random variables ζ_{B_n}.

A proof can be found in [1] and is not repeated here.

5 Proof of the central limit theorem

We verify the assumptions of Dobrushin's lemma for the counting variables of the limiting Gibbs process G.

ad ($\wp$1)

Lemma 4 *For every $\mu \in \mathcal{M}^{\cdot}$ and $B \in \mathcal{B}_0(E)$,*

$$\mathrm{e}^{a\cdot\mu(B)} = 1 + \sum_{n=1}^{\infty} \frac{\widetilde{\mu}^n(B^n)}{n!}\big(\mathrm{e}^a - 1\big)^n, \quad a \in \mathbb{C}. \tag{10}$$

(*Note that the series on the right-hand side is a finite sum because $\mu(B)$ is finite.*)

Proof Indeed, given $\mu \in \mathcal{M}^{\cdot}$ with $\mu = \sum_j \varepsilon_{x_j}$ and $B \in \mathcal{B}_0(E)$ with $\ell = \mu(B)$,

$$\begin{aligned}
\mathrm{e}^{a\cdot\mu(B)} &= \sum_{n=0}^{\ell} \sum_{I\subseteq[\ell],|I|=n} \prod_{j\in I}\big(\mathrm{e}^{a\cdot\mathbf{1}_B(x_j)} - 1\big) \\
&= \sum_{n=0}^{\ell}\big(\mathrm{e}^a - 1\big)^n \sum_{I\subseteq[\ell],|I|=n} \prod_{j\in I} \mathbf{1}_B(x_j) \\
&= \sum_{n=0}^{\infty} \frac{(\mathrm{e}^a - 1)^n}{n!}\widetilde{\mu}^n\big(B^n\big).
\end{aligned}$$

□

The next result uses the Gibbsian character of G for (ϕ, ϱ).

Proposition 1 *Under assumptions* (*A*1) *and* (*A*2),

$$\Psi_B(a) = \mathrm{G}\big(\mathrm{e}^{a\zeta_B}\big) = 1 + \sum_{n=1}^{\infty} \frac{\widetilde{\nu}^n_{\mathrm{G}}(B^n)}{n!}\big(\mathrm{e}^a - 1\big)^n, \quad a \in \mathcal{O}_{\ln 2}.$$

Proof We show that the right-hand side of (10) converges absolutely almost surely with respect to G: Let $a \in \mathcal{O}_{\ln 2}$ and thereby $|\mathrm{e}^a - 1| \leq 1$. Iterating equation (Σ_ϱ) implies

$$\begin{aligned}\int_{\mathcal{M}^\cdot} \left| \sum_{n=1}^{\infty} \frac{\widetilde{\mu}^n(B^n)}{n!}\big(\mathrm{e}^a - 1\big)^n \right| \mathrm{G}(\mathrm{d}\mu) &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathcal{M}^\cdot} \widetilde{\mu}^n\big(B^n\big)\,\mathrm{G}(\mathrm{d}\mu) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \int_{B^n} \int_{\mathcal{M}^\cdot} \mathrm{e}^{-W_\phi(\varepsilon_{x_1}+\cdots+\varepsilon_{x_n},\mu)}\,\mathrm{G}(\mathrm{d}\mu)\varrho^n(\mathrm{d}x_1 \ldots \mathrm{d}x_n) \\ &\leq \exp\big(\mathrm{e}^{\mathrm{c}}\,\varrho(B)\big).\end{aligned}$$

Here we also used Ruelle's estimate (5). □

This shows that the function Ψ_B is finite-valued in $\mathcal{O}_{\ln 2}$. Moreover, it does not vanish for all $a \in \mathcal{O}_{\ln 2}$ since $\Psi_B(-\alpha) > 0$ for all $\alpha \geq 0$. Furthermore, Ψ_B is continuous in the closed ball $\mathcal{O}_{\ln 2}$ and analytic in its interior. Finally,

$$\|\Psi_B\|(a) \leq \mathrm{G}\big(\mathrm{e}^{|a|\zeta_B}\big), \quad a \in \mathcal{O}_{\ln 2}. \tag{11}$$

ad ($\wp$2) We know from Lemma 1 that

$$\mathrm{G}\big(\mathrm{e}^{-\alpha\zeta_B}\big) = \exp\big(\mathrm{L}\big(\mathrm{e}^{-\alpha\zeta_B} - 1\big)\big), \quad \alpha \geq 0.$$

Assume now that

$$\sum_{\ell=1}^{\infty} \frac{\widetilde{\nu}^\ell_{|\mathrm{L}|}(B^\ell)}{\ell!} < +\infty, \quad B \in \mathcal{B}_0(E). \tag{12}$$

This condition is due to Malyshev and Minlos in an equivalent form (cf. [8], Chap. 3, Theorem 2).

Lemma 5 *Under assumption* (12),

$$\mathrm{L}\big(\mathrm{e}^{a\zeta_B} - 1\big) = \sum_{n=1}^{\infty} \frac{\widetilde{\nu}^n_{\mathrm{L}}(B^n)}{n!}\big(\mathrm{e}^a - 1\big)^n, \quad a \in \mathcal{O}_{\ln 2}. \tag{13}$$

It follows that the left-hand side of (13), considered as a function of a, is continuous in $\mathcal{O}_{\ln 2}$ and analytic in its interior. The proof of the lemma is the same as for Proposition 1 if G is replaced with L.

Particularly, formula (13) holds for all $a \in [-\ln 2, \ln 2]$. Thus, using (11), one obtains for all $a \in \mathcal{O}_{\ln 2}$

$$|\ln\|\Psi_B\||(a) \leq |\mathrm{L}|\big(\mathrm{e}^{|a|\zeta_B} - 1\big)$$

$$\leq \sum_{n=1}^{\infty} \frac{\widetilde{\nu}^n_{|L|}(B^n)}{n!}. \tag{14}$$

To obtain ($\wp$2), we estimate the right-hand side by means of the following basic result due to Ruelle (Cf. [19], estimate (4.37) of Theorem 4.4.8). Let $r^n_{|L|}$ denote the Radon–Nikodym derivative of $\widetilde{\nu}^n_{|L|}$ with respect to ϱ^n, which is given by partial integration by

$$r^n_{|L|}(x_1,\dots,x_n) = \int_{\mathfrak{X}} |\varkappa|(\xi + \varepsilon_{x_1} + \cdots + \varepsilon_{x_n})\Lambda_\varrho(\mathrm{d}\xi). \tag{15}$$

It is worth to note that the result of Ruelle was given for the Radon–Nikodym derivative of $\widetilde{\nu}^n_{|L|}$ with respect to λ^n and for this reason he had an additional factor z^n in the front of integral (15).

Lemma 6 *Under* (*A*1) *and* (*A*2) (*which imply Ruelle's assumptions on the underlying potential*), *the following estimate holds true uniformly in x*:

$$\begin{aligned}&\int_{E^{n-1}} r^n_{|L|}(x,x_2,\dots,x_n)\varrho(\mathrm{d}x_2)\dots\varrho(\mathrm{d}x_n)\\ &\quad\leq (n-1)!\,\mathrm{e}^{-c}\,\frac{(z\,\mathrm{e}^{c+1}\,C_\phi)^{n-1}}{(1-z\,\mathrm{e}^{c+1}\,C_\phi)^n}.\end{aligned} \tag{16}$$

The lemma implies

$$\widetilde{\nu}^n_{|L|}\big(B^n\big) \leq \varrho(B)(n-1)!\,\mathrm{e}^{-c}\,\frac{\chi^{n-1}}{(1-\chi)^n},$$

where $\chi = z\,\mathrm{e}^{c+1}\,C_\phi$. It follows for all z satisfying $\chi < \frac{1}{2}$ that

$$\sum_{n=1}^{\infty} \frac{\widetilde{\nu}^n_{|L|}(B^n)}{n!} \leq \varrho(B)\cdot \frac{\mathrm{e}^{-c}}{\chi}\log\frac{1-\chi}{1-2\chi},$$

which shows assumption (12) and thereby assertion ($\wp$2).

ad ($\wp$3) Note first that the limiting variance is given by

$$\sigma^2 = z\cdot \int_{\mathfrak{X}} \varkappa(\xi + \varepsilon_0)\big(|\xi| + 1\big)\Lambda_\varrho(\mathrm{d}\xi). \tag{17}$$

Indeed, we know from Lemma 2 that

$$V_G(\zeta_B) = z\cdot r^1_L(0)\lambda(B) + \int_{B^2} r^2_L(x,y)\varrho(\mathrm{d}x)\varrho(\mathrm{d}y).$$

Translation-invariance of the potential implies for the second term on the right-hand side

$$\int_{B^2} r^2_L(x,y)\varrho(\mathrm{d}x)\varrho(\mathrm{d}y) = \int_E r^2_G(0,y)\varrho\big(B\cap(B-y)\big)\varrho(\mathrm{d}y).$$

And since the B_r are chosen in a regular way, we obtain

$$\sigma^2 = \lim_{r\to\infty} \frac{V_{\mathrm{G}}(\zeta_{B_r})}{\lambda(B_r)} = z \cdot \left(r_{\mathrm{L}}^1(0) + \int_E r_{\mathrm{L}}^2(0,y)\varrho(\mathrm{d}y) \right).$$

This implies assertion (17) in view of definitions (7) and (8).

It remains to show that σ^2 is strictly positive. This follows from representation (17) by means of Ueltschi's tree graph bound (cf. [20]), which is valid under c-stability:

$$\left|\varkappa(\xi)\right| \le \mathrm{e}^{\mathrm{c}|\xi|} \cdot \sum_{\tau\in\mathscr{T}(\xi)} \prod_{\{x,y\}\in\tau} \left(1 - \mathrm{e}^{-|\phi|(x,y)}\right), \quad \xi \in \mathfrak{X}. \tag{18}$$

Indeed, estimating the second term on the right-hand side of

$$\sigma^2 = z\left(1 + \int_{\mathfrak{X}'} \varkappa(\xi + \varepsilon_0)\left(|\xi| + 1\right)\Lambda_\varrho(\mathrm{d}\xi)\right), \tag{19}$$

we obtain from (18)

$$\begin{aligned}\left|\int_{\mathfrak{X}'} \varkappa(\xi + \varepsilon_0)\left(|\xi| + 1\right)\Lambda_\varrho(\mathrm{d}\xi)\right| &\le \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell!} \mathrm{e}^{\mathrm{c}(\ell+1)}(\ell+1)^\ell \left(C'_\phi\right)^\ell \\ &\le \mathrm{e}^{\mathrm{c}+1} \sum_{\ell=1}^{\infty} \left(z\,\mathrm{e}^{\mathrm{c}+1}\,C'_\phi\right)^\ell \\ &= \frac{z\,\mathrm{e}^{2(\mathrm{c}+1)}\,C'_\phi}{1 - z\,\mathrm{e}^{\mathrm{c}+1}\,C'_\phi}.\end{aligned}$$

In view of equation (19) it follows that the specific variance σ^2 is strictly positive if z satisfies

$$z < \frac{1}{\mathrm{e}^{\mathrm{c}+1}\,C'_\phi(1 + \mathrm{e}^{\mathrm{c}+1})}. \tag{20}$$

To summarize, we obtained the following central limit theorem.

Theorem 1 *Let* Φ *be a pair potential in* E*, which is weakly* $\wp$*-stable for some constant* $\mathrm{c} \ge 0$ *and satisfies the strengthened modified regularity condition*

$$z\,\mathrm{e}^{\mathrm{c}+1}\,C''_\phi < \frac{1}{1 + \mathrm{e}^{\mathrm{c}+1}}. \tag{21}$$

Then, along some regular sequence $(B_r)_r$ *in* E*, the sequence of particle numbers* $(\zeta_{B_r})_r$ *satisfies the central limit theorem for the underlying Gibbs process given by the limiting Gibbs process* G *with empty boundary conditions.*

Note that in view of inequalities (4) condition (21) implies condition (20) as well as condition $\chi < \frac{1}{2}$, which is needed to obtain ($\wp$2).

5.1 Historical remark

The method of cluster expansions originated in the 1930s in statistical mechanics with the aim to study classical gases. It was made rigorous by mathematical physicists in the 1960s, in particular by Jean Ginibre in his pioneering work [2], where he develops the method for quantum gases.

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Author details
[1]Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501, Bielefeld, Germany. [2]Institute of Mathematics, Armenian National Academy of Sciences, Bagramian Av. 24-5, 0019, Yerevan, Armenia.

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