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Conservative Fourier spectral method for a class of modified Zakharov system with high-order space fractional quantum correction

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Abstract

In this paper, we consider the Fourier spectral method and numerical investigation for a class of modified Zakharov system with high-order space fractional quantum correction. First, the numerical scheme of the system is developed with periodic boundary condition based on the Crank–Nicolson/leap-frog methods in time and the Fourier spectral method in space. Moreover, it is shown that the scheme preserves simultaneously mass and energy conservation laws. Second, we analyze stability and convergence of the numerical scheme. Last, the numerical experiments are given, and the results show the correctness of theoretical results and the efficiency of the conservative scheme.

Keywords: Modified Zakharov system; Fractional quantum correction; Fourier spectral method; Stability; Convergence; Conservativeness

1 Introduction

The classical Zakharov system is one of the best models in describing the coupling of high-frequency Langmuir waves and low-frequency ion-acoustic waves. Moreover, it has been widely applied to shallow water wave, nonlinear optics, etc. During the past decades, some attentions have been paid to study the properties of the classical Zakharov system, for example, solitary wave solution [1], well-posedness [2], chaotic behavior [3], etc. In particular, many numerical methods such as conservative difference scheme [4], energy-preserving scheme [5], multi-symplectic scheme [6], time-splitting schemes [7, 8] have been developed to solve the classical Zakharov system with homogeneous boundary condition or periodic boundary condition.

Quantum effect plays a very important role in the field of micro-equipment and laser plasma. In 2005, Garcia et al. [9] considered the Landau damping of Langmuir wave in the study of the plasma and obtained the quantum Zakharov system

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial^2 x}-H^2\frac{\partial^4 E}{\partial^4 x}-NE=0, \tag{1}$$

$$\frac{\partial^2 N}{\partial^2 t} - \frac{\partial^2 N}{\partial^2 x} + H^2 \frac{\partial^4 N}{\partial^4 x} - \frac{\partial^2}{\partial^2 x}\left(|E|^2\right) = 0, \tag{2}$$

where H is the dimensionless quantum parameter. In equations (1)–(2), E, N are complex and real-valued unknown functions, and E, N are the Langmuir envelope electric field and the density fluctuation respectively. Afterwards, some attentions have been paid to study theory and numerical methods of the quantum Zakharov system [10–13]. In [11], Misra et al. studied the pattern dynamics and spatiotemporal chaos of the quantum modified Zakharov system. In [14], Fang et al. obtained some exact traveling wave solutions of the quantum Zakharov system by using the hyperbolic tangent function expansion, hyperbolic secant function expansion, and Jacobi elliptic functions expansion. In [15], the existence of weak global solutions to quantum Zakharov system was obtained by using the Arzela–Ascoli theorem and the Faedo–Galerkin method.

Recently, fractional calculus [16–20] has been playing more and more important roles in quantum mechanics. In particular, many numerical methods such as finite difference scheme, Fourier spectral scheme, finite element scheme, etc. have been developed for the space fractional Schrödinger equations [21–32], space fractional Klein–Gordon–Schrödinger equations [33–36], and space fractional Klein–Gordon–Zakharov equations [37] with zeros boundary condition or periodic boundary condition. In this paper, we consider the modified Zakharov system with high-order space fractional quantum correction [38]

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial^2 x} - H^2(-\triangle)^{\alpha} E - NE = 0, \quad x \in (-L/2, L/2), t \geq 0, \tag{3}$$

$$\frac{\partial^2 N}{\partial^2 t} - \frac{\partial^2 N}{\partial^2 x} + H^2(-\triangle)^{\beta} N - \frac{\partial^2}{\partial^2 x}\left(|E|^2\right) = 0, \quad x \in (-L/2, L/2), t \geq 0, \tag{4}$$

$$E(x,0) = E_0(x), \qquad N(x,0) = N_0(x), \qquad N_t(x,0) = N_1(x), \quad x \in (-L/2, L/2), \tag{5}$$

$$E(x+L/2,t) = E(x-L/2,t), \qquad N(x+L/2,t) = N(x-L/2,t), \quad t \geq 0, \tag{6}$$

where $1 < \alpha \leq \beta \leq 2$. The global existence and uniqueness of the solution of system (3)–(4) are shown in [38]. In [39], the finite difference scheme is used to solve the fractional modified Zakharov system (3)–(6) with periodic boundary condition, and strict theoretical analysis such as existence, uniqueness, convergence, and stability of the scheme is also shown. Moreover, the scheme can exactly preserve the mass and energy conservation laws. When $H = 0$, system (3)–(4) reduces to the classical Zakharov system, which has been studied extensively in [1–9]. When $H \neq 0$, $\alpha = \beta = 2$, system (3)–(4) reduces to the quantum Zakharov system (1)–(2). It is easy to show that system (3)–(6) satisfies the mass and energy conserved laws [38, 39]

$$\frac{d}{dt}\int_{-L/2}^{L/2} \left|E(x,t)\right| dx = 0, \tag{7}$$

$$\begin{aligned} &\frac{d}{dt}\int_{-L/2}^{L/2} \biggl(|\partial_x E|^2 + \frac{1}{2}\left(|\partial_x u|^2 + N^2\right) + H^2\left|(-\triangle)^{\frac{\alpha}{2}} E\right|^2 \\ &\quad + \frac{H^2}{2}\left|(-\triangle)^{\frac{\beta-1}{2}} N\right|^2 + N|E|^2 \biggr)\, dx = 0, \end{aligned} \tag{8}$$

where $\partial_x^2 u = \partial_t N$.

Some numerical schemes have been designed to discrete fractional Laplace operator, including finite difference scheme, finite element scheme etc. Under the periodic boundary condition, the fractional Laplacian operator is defined as [17]

$$-(-\Delta)^{\alpha}E(x,t) = -\sum_{k=-\infty}^{+\infty}|k|^{2\alpha}\hat{E}(k,t)e^{ik(x+L/2)},$$

where $\hat{E}$ is the Fourier transform, and $(-\Delta)^{\beta}N(x,t)$ can be defined in the same way. As a class of high accuracy methods, Fourier spectral methods are often chosen to solve differential equations with periodic boundary condition. To the best of the authors' knowledge, there exist few reports on Fourier spectral method for the fractional quantum Zakharov system (3)–(6). The first aim of this paper is to develop a Fourier spectral method to solve the fractional modified Zakharov system (3)–(6) with periodic boundary condition. To deal with the nonlinear term, we also introduce the Fourier interpolation operator. It is very important to construct a numerical method for the nonlinear partial differential equations. In addition, a large number of numerical experiments show that the conservation numerical schemes are superior to the traditional numerical schemes. The second aim of this paper is to develop a conservative numerical scheme to solve the fractional modified Zakharov system (3)–(6) with periodic boundary condition.

The outline of the paper is as follows. In Sect. 2, we give some useful lemmas. In Sect. 3, the fully discrete Fourier spectral method is proposed by the Fourier spectral scheme in space, as well as Crank–Nicolson and leap-frog schemes in time, and the conservativeness of the scheme is shown. In Sect. 4, the stability of the fully discrete Fourier spectral scheme is analyzed. In Sect. 5, the convergence and error estimate for the fully discrete Fourier spectral scheme are presented. In Sect. 6, some numerical experiments are given to show the efficiency of the conservative scheme. Finally, a conclusion is given in Sect. 7.

2 A useful lemma

In this paper, we select periodic boundary condition, the solution $E(x,t)$, $N(x,t)$ can be expressed as

$$E(x,t) = \sum_{l=-\infty}^{\infty}\hat{E}_l e^{il(x+L/2)}, \hat{E}_l = \frac{1}{2\pi}\int_{\Omega}E(x,t)e^{-il(x+L/2)}\,dx,$$

$$N(x,t) = \sum_{l=-\infty}^{\infty}\hat{N}_l e^{il(x+L/2)}, \hat{N}_l = \frac{1}{2\pi}\int_{\Omega}N(x,t)e^{-il(x+L/2)}\,dx,$$

where $\Omega = [-\frac{L}{2},\frac{L}{2}]$. Let $r > 0$. Then $H_p^r(\Omega)$ represents the Sobolev space in which the periodic function is formed, and Sobolev norm and seminorm are as follows:

$$\|E\|_r = \left(\sum_{|k|<\infty}(1+k)^{2r}|\hat{E}_k|^2\right)^{1/2}, \qquad |E|_r = \left(\sum_{0<|k|<\infty}|k|^{2r}|\hat{E}_k|^2\right)^{1/2}.$$

Let $V_M = \{E(x)|E(x) = \sum_{|k|\le M/2}\hat{E}_k e^{-ik(x+L/2)}, M/2 \in Z^+\}$. Then define the orthogonal projector

$$P_M : L^2(\Omega) \to V_M, \qquad P_M E(x,t) = \sum_{|k|\le M/2}\hat{E}_k e^{-ik(x+L/2)}$$

to approximate the function $E(x,t)$. From the definition of the orthogonal projector operator, we get

$$(P_M E - E, v) = 0, \quad v \in V_M,$$
$$-(-\Delta)^\alpha \big(P_M E(x,t)\big) = P_M\big(-(-\Delta)^\alpha E(x,t)\big),$$

where the inner product $(\cdot,\cdot)$ can be expressed as

$$(f,g) = \int_\Omega f(x)\overline{g(x)}\,dx,$$

where $\overline{g(x)}$ is the conjugate complex function of $g(x)$.

Lemma 1 [40] *Let $r > 0$, $E \in H_p^r(\Omega)$. Then there exists a constant C independent of E and M such that*

$$\|P_M E - E\|_l \le C M^{l-r}|E|_r, \quad 0 \le l \le r.$$

3 A conservative fully discrete scheme for the fractional quantum Zakharov system

First, we introduce some finite difference operators

$$E_t^n = \frac{1}{\tau}\big(E^{n+1} - E^n\big), \qquad E_{\bar t}^n = \frac{1}{\tau}\big(E^n - E^{n-1}\big), \qquad E_{\hat t}^n = \frac{1}{2\tau}\big(E^{n+1} - E^{n-1}\big),$$
$$E^{\bar n} = \frac{1}{2}\big(E^{n+1} + E^{n-1}\big), \qquad E^{n+\frac{1}{2}} = \frac{1}{2}\big(E^{n+1} + E^n\big).$$

In the following sections, C represents a general constant, and it may have different values in different places.

We apply the Crank–Nicolson/ leap-frog methods in time and the Fourier spectral method in space and obtain the three-level scheme

$$i\big(E_{Mt}^n, \varphi\big) + \big(\partial_x^2 E_M^{n+\frac{1}{2}}, \varphi\big) - H^2\big((-\triangle)^{\alpha-1}\partial_x^2 E_M^{n+\frac{1}{2}}, \varphi\big)$$
$$= \big(P_M\big(N_M^{n+\frac{1}{2}} E_M^{n+\frac{1}{2}}\big), \varphi\big), \quad \forall \varphi \in V_M, \tag{9}$$

$$\big(N_{Mt\bar t}^n, \varphi\big) - \big(\partial_x^2 N_M^{\bar n}, \varphi\big) + H^2\big((-\triangle)^{\beta-1}\partial_x^2 N_M^{\bar n}, \varphi\big) = \big(\partial_x^2 P_M\big(\big|E_M^n\big|^2\big), \varphi\big), \quad \forall \varphi \in V_M, \tag{10}$$

$$\big(E_M^0, \varphi\big) = (P_M E_0, \varphi), \quad \forall \varphi \in V_M, \tag{11}$$

$$\big(N_M(x,0), \varphi\big)$$
$$= \bigg(N_M^0 + \tau P_M N_1 + \frac{\tau^2}{2} P_M\big(\partial_x^2 N_M^0 - H^2(-\triangle)^{\beta-1} N_M^0 + \partial_x^2\big(\big|E_M^0\big|^2\big)\big), \varphi\bigg),$$
$$\forall \varphi \in V_M, \tag{12}$$

$$\big(\partial_x^2 u_M^{n+\frac{1}{2}}, \varphi\big) = \big(N_{Mt}^n, \varphi\big), \quad \forall \psi \in V_M, \tag{13}$$

where

$$E_M = \sum_{l=-M}^{M} \hat{E}_l e^{il(x+L/2)}, \qquad N_M = \sum_{l=-M}^{M} \hat{N}_l e^{il(x+L/2)},$$
$$E_{Mt}^n = \frac{1}{\tau}\big(E_M^{n+1} - E_M^n\big), \qquad N_{Mt\bar{t}}^n = \frac{1}{\tau^2}\big(N_M^{n+1} - 2N_M^n - N_M^{n-1}\big),$$
$$E_M^{n+\frac{1}{2}} = \frac{1}{2}\big(E_M^{n+1} + E_M^n\big), \qquad N_M^{\bar{n}} = \frac{1}{2}\big(N_M^{n+1} + N_M^{n-1}\big).$$

Theorem 1 *The Fourier spectral scheme* (9)–(13) *is conservative in the sense*

$$\begin{aligned}
&\|E_M^n\|^2 = \|E_M^0\|^2,\\
&\|\Lambda^{n+1}\|^2 = \|\partial_x E_M^{n+1}\|^2 + \frac{1}{2}\|\partial_x u^{n+\frac{1}{2}}\|^2 + \frac{1}{4}\big(\|N_M^n\|^2 + \|N_M^{n+1}\|^2\big)\\
&\qquad + H^2\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x E_M^{n+1}\|^2 + \frac{1}{4}H^2\big(\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\|^2\\
&\qquad + \|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\|^2\big) + \frac{1}{2}\big(N_M^{n+1} + N_M^n, |E_M^{n+1}|^2\big) = C.
\end{aligned}$$

Proof Let $\varphi = E_M^{n+1} + E_M^n$ in (9). Then taking the imaginary part of equation (9) yields

$$\|E_M^{n+1}\|^2 = \|E_M^n\|^2 = \cdots = \|E_M^0\|^2.$$

Let $\varphi = \frac{2}{\tau}(E_M^{n+1} - E_M^n)$ in (9). Then taking the real part of equation (9) yields

$$\begin{aligned}
&\frac{1}{\tau}\big(\|\partial_x E_M^{n+1}\|^2 - \|\partial_x E_M^n\|^2\big) + \frac{1}{\tau}H^2\big(\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x E_M^{n+1}\|^2 - \|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x E_M^n\|^2\big)\\
&\quad + \frac{1}{2\tau}\big(N_M^{n+1} + N_M^n, |E_M^{n+1}|^2 - |E_M^n|^2\big) = 0.
\end{aligned} \tag{14}$$

Taking $\varphi = \frac{1}{2}(u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}})$ in (10) yields

$$\begin{aligned}
&\frac{1}{2}\left(\frac{N_M^{n+1} - 2N_M^n + N_M^{n-1}}{\tau^2}, u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\right) - \frac{1}{4}\big(\partial_x^2 N_M^{n+1} + \partial_x^2 N_M^{n-1}, u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big)\\
&\quad + \frac{1}{4}H^2\big((-\triangle)^{\beta-1}\big(\partial_x^2 N_M^{n+1} + \partial_x^2 N_M^{n-1}\big), u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big) = \frac{1}{2}\big(\partial_x^2 P_M\big(|E_M^n|^2\big), u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big).
\end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{1}{2}\left(\frac{N_M^{n+1} - 2N_M^n + N_M^{n-1}}{\tau^2}, u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\right) = -\frac{1}{2\tau}\big(\|\partial_x u^{n+\frac{1}{2}}\|^2 - \|\partial_x u^{n-\frac{1}{2}}\|^2\big),\\
&-\frac{1}{4}\big(\partial_x^2 N_M^{n+1} + \partial_x^2 N_M^{n-1}, u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big) = -\frac{1}{4\tau}\big(\|N_M^{n+1}\|^2 + \|N_M^{n-1}\|^2\big),\\
&\frac{1}{4}H^2\big((-\triangle)^{\beta-1}\big(\partial_x^2 N_M^{n+1} + \partial_x^2 N_M^{n-1}\big), u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big)\\
&\quad = \frac{1}{4\tau}H^2\big(\|(-\triangle)^{\frac{\beta-1}{2}}\big(N_M^{n+1}\big)\|^2 - \|(-\triangle)^{\frac{\beta-1}{2}}\big(N_M^{n-1}\big)\|^2\big),\\
&\frac{1}{2}\big(\partial_x^2 P_M\big(|E_M^n|^2\big), u^{n+\frac{1}{2}} + u^{n-\frac{1}{2}}\big) = \frac{1}{2\tau}\big(|E_M^n|^2, N_M^{n+1} - N_M^{n-1}\big),
\end{aligned}$$

we obtain

$$\frac{1}{2\tau}\left(\left\|\partial_x u^{n+\frac{1}{2}}\right\|^2-\left\|\partial_x u^{n-\frac{1}{2}}\right\|^2\right)+\frac{1}{4\tau}H^2\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}\left(N_M^{n+1}\right)\right\|^2-\left\|(-\triangle)^{\frac{\beta-1}{2}}\left(N_M^{n-1}\right)\right\|^2\right)$$
$$+\frac{1}{4\tau}\left(\left\|N_M^{n+1}\right\|^2+\left\|N_M^{n-1}\right\|^2\right)+\frac{1}{2\tau}\left(\left|E_M^n\right|^2,N_M^{n+1}-N_M^{n-1}\right)=0.$$

The above equation and (14) yield

$$\begin{aligned}
&\left\|\partial_x E_M^{n+1}\right\|^2+\frac{1}{2}\left\|\partial_x u^{n+\frac{1}{2}}\right\|^2+\frac{1}{4}\left(\left\|N_M^n\right\|^2+\left\|N_M^{n+1}\right\|^2\right)\\
&\quad+H^2\left\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x E_M^{n+1}\right\|^2+\frac{1}{4}H^2\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\right\|^2+\left\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\right\|^2\right)\\
&\quad+\frac{1}{2}\left(N_M^{n+1}+N_M^n,\left|E_M^{n+1}\right|^2\right)\\
&=\left\|\partial_x E_M^n\right\|^2+\frac{1}{2}\left\|\partial_x u^{n-\frac{1}{2}}\right\|^2+\frac{1}{4}\left(\left\|N_M^{n-1}\right\|^2+\left\|N_M^n\right\|^2\right)\\
&\quad+H^2\left\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x E_M^n\right\|^2+\frac{1}{4}H^2\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\right\|^2+\left\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n-1}\right\|^2\right)\\
&\quad+\frac{1}{2}\left(N_M^n+N_M^{n-1},\left|E_M^n\right|^2\right).
\end{aligned}$$

Thus $\Lambda^{n+1}=\Lambda^n=\cdots=\Lambda^1=C$. This ends the proof. □

Because of the nonlinear term of the fractional modified Zakharov system (3)–(6), numerical scheme (9)–(13) takes a lot of calculation time. To improve the efficiency of calculation, we introduce the interpolation operator $I_M: L^2(\Omega)\to V_J''$ by

$$I_M u(x,t)=\sum_{j=0}^{M-1}u_j g_j(x),$$

where

$$V_M''=\left\{u(x)=\sum_{|l|\le M/2}\tilde{u}_l e^{il\mu(x-a)},\tilde{u}_{M/2}=\tilde{u}_{-M/2}\right\},$$

$$\tilde{u}_l=\frac{1}{Mc_l}\sum_{j=0}^{M-1}u(x_j)e^{-ik(x_j-a)},$$

$$g_j(x)=\frac{1}{M}\sum_{l=-\frac{M}{2}}^{\frac{M}{2}-1}\frac{1}{c_l}e^{il\mu(x-x_j)},\quad c_l=1\left(|l|\ne\frac{M}{2}\right),c_{\frac{M}{2}}=c_{-\frac{M}{2}}=2.$$

Applying the above interpolation operator to the nonlinear term of the fractional modified Zakharov system (3)–(6), we can obtain the following numerical scheme:

$$\begin{aligned}
&i\left(E_{Mt}^n,\varphi\right)+\left(\partial_x^2 E_M^{n+\frac{1}{2}},\varphi\right)-H^2\left((-\triangle)^{\alpha-1}\partial_x^2 E_M^{n+\frac{1}{2}},\varphi\right)\\
&\quad=\left(I_M\left(N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}\right),\varphi\right),\quad\forall\varphi\in V_M,
\end{aligned}\tag{15}$$

$$\begin{aligned}&\big(N_{Mt\bar{t}}^{n},\varphi\big)-\big(\partial_x^2N_M^{\bar{n}},\varphi\big)+H^2\big((-\triangle)^{\beta-1}\partial_x^2N_M^{\bar{n}},\varphi\big)\\&\quad=\big(\partial_x^2I_M\big(\big|E_M^n\big|^2\big),\varphi\big),\quad\forall\varphi\in V_M.\end{aligned}\tag{16}$$

4 Stability analysis for the fully discrete Fourier spectral scheme

Lemma 2 *There exists a constant C depending only on the initial and boundary values such that the solution of the Fourier spectral scheme* (9)–(13) *satisfies*

$$\begin{aligned}&\frac{3}{4}\big\|\partial_xE_M^{n+1}\big\|^2+\frac{1}{2}\big\|\partial_xu^{n+\frac{1}{2}}\big\|^2+\frac{1}{8}\big(\big\|N_M^n\big\|^2+\big\|N_M^{n+1}\big\|^2\big)\\&\quad+H^2\big\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_xE_M^{n+1}\big\|^2+\frac{1}{4}H^2\big(\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\big\|^2+\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\big\|^2\big)\le C.\end{aligned}$$

Proof It follows from the Young inequality that

$$\begin{aligned}&\big\|\partial_xE_M^{n+1}\big\|^2+\frac{1}{2}\big\|\partial_xu^{n+\frac{1}{2}}\big\|^2+\frac{1}{4}\big(\big\|N_M^n\big\|^2+\big\|N_M^{n+1}\big\|^2\big)+H^2\big\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_xE_M^{n+1}\big\|^2\\&\qquad+\frac{1}{4}H^2\big(\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\big\|^2+\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\big\|^2\big)\\&\quad=C-\frac{1}{2}\big(N_M^{n+1}+N_M^n,\big|E_M^{n+1}\big|^2\big)\\&\quad\le\frac{1}{2}\big(\big|N_M^{n+1}\big|+\big|N_M^n\big|,\big|E_M^{n+1}\big|^2\big)+C\\&\quad\le\frac{\varepsilon}{4}\big(\big\|N_M^{n+1}\big\|^2+\big\|N_M^n\big\|^2\big)+\frac{1}{2\varepsilon}\big\|E_M^{n+1}\big\|_4^2+C,\end{aligned}$$

where $\varepsilon>0$ is the Young inequality parameter.

Taking $\varepsilon=\frac{1}{2}$ can yield

$$\begin{aligned}&\big\|\partial_xE_M^{n+1}\big\|^2+\frac{1}{2}\big\|\partial_xu^{n+\frac{1}{2}}\big\|^2+\frac{1}{4}\big(\big\|N_M^n\big\|^2+\big\|N_M^{n+1}\big\|^2\big)+H^2\big\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_xE_M^{n+1}\big\|^2\\&\qquad+\frac{1}{4}H^2\big(\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\big\|^2+\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\big\|^2\big)\\&\quad\le\frac{1}{8}\big(\big\|N_M^{n+1}\big\|^2+\big\|N_M^n\big\|^2\big)+\big\|E_M^{n+1}\big\|_4^2+C.\end{aligned}$$

According to the Sobolev and Young inequalities, we obtain

$$\begin{aligned}\big\|E_M^{n+1}\big\|_4^2=\big(\big|E_M^{n+1}\big|^2,\big|E_M^{n+1}\big|^2\big)&\le\big\|E_M^{n+1}\big\|_\infty^2\big\|E_M^{n+1}\big\|^2\\&\le C\big\|E_M^{n+1}\big\|\big\|\partial_xE_M^{n+1}\big\|\\&\le\frac{1}{4}\big\|\partial_xE_M^{n+1}\big\|^2+C\big\|E_M^{n+1}\big\|^2\\&\le\frac{1}{4}\big\|\partial_xE_M^{n+1}\big\|^2+C.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{3}{4}\big\|\partial_xE_M^{n+1}\big\|^2+\frac{1}{2}\big\|\partial_xu^{n+\frac{1}{2}}\big\|^2+\frac{1}{8}\big(\big\|N_M^n\big\|^2+\big\|N_M^{n+1}\big\|^2\big)\\&\quad+H^2\big\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_xE_M^{n+1}\big\|^2+\frac{1}{4}H^2\big(\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^{n+1}\big\|^2+\big\|(-\triangle)^{\frac{\beta-1}{2}}N_M^n\big\|^2\big)\le C.\end{aligned}$$

This ends the proof. □

Theorem 2 *The Fourier spectral scheme* (9)–(13) *is bounded in the discrete* l^2 *and* l^∞ *norms, and*

$$\begin{aligned} &\|E_M^n\| \le C, \qquad \|N_M^n\| \le C, \qquad \|\partial_x u^{n+\frac{1}{2}}\| \le C, \\ &\|E_M^n\|_\infty \le C, \qquad \|N_M^n\|_\infty \le C. \end{aligned}$$

Proof It follows from Lemma 2 and Theorem 1 that

$$\begin{aligned} &\|E_M^n\| \le C, \qquad \|N_M^n\| \le C, \qquad \|\partial_x u^{n+\frac{1}{2}}\| \le C, \\ &\|\partial_x E_M^n\| \le C, \qquad \|(-\triangle)^{\frac{\beta-1}{2}} N_M^n\| \le C. \end{aligned}$$

According to the Sobolev inequality, it holds that

$$\begin{aligned} &\|E_M^n\|_\infty \le C\|E_M^n\|_1^{\frac{1}{2}}\|E_M^n\|^{\frac{1}{2}} \le C, \\ &\|N_M^n\|_\infty \le C\|N_M^n\|_{H^{\frac{\beta-1}{2}}} = C\big(\|N_M^n\|^2 + \|(-\triangle)^{\frac{\beta-1}{2}} N_M^n\|^2\big)^{\frac{1}{2}} \le C. \end{aligned}$$

This ends the proof. □

5 Convergence and error estimates

Let

$$e^n = E^n - E_M^n = E^n - P_M E^n + P_M E^n - E_M^n = \tilde{e}^n + e_M^n, \tag{17}$$

$$\eta^n = N^n - N_M = N^n - P_M N^n + P_M N^n - N_M^n = \tilde{\eta}^n + \eta_M^n, \tag{18}$$

where $\tilde{e}^n = E^n - P_M E^n$, $\tilde{\eta} = N^n - P_M N^n$, $e_M^n = P_M E^n - E_M^n$, $\eta_M^n = P_M N^n - N_M^n$. Substituting the solutions $E(x,t_n), N(x,t_n)$ into equations (3)–(4) and subtracting (11) from (3) and (12) from (4) respectively, we have

$$\begin{aligned} &\big(ie_{Mt}^n, \varphi\big) + \big(\partial_x^2 e_M^{n+\frac{1}{2}}, \varphi\big) + H^2\big((-\triangle)^{\alpha-1}\partial_x^2 e_M^{n+\frac{1}{2}}, \varphi\big) \\ &\quad - \big(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, \varphi\big) = \big(R_1^n, \varphi\big), \end{aligned} \tag{19}$$

$$\begin{aligned} &\big(\eta_{Mt\bar{t}}^n, \varphi\big) - \big(\partial_x^2 \eta_M^{\bar{n}}, \varphi\big) - H^2\big((-\triangle)^{\beta-1}\partial_x^2 \eta_M^{\bar{n}}, \varphi\big) - \big(\partial_x^2\big(|E^n|^2\big) \\ &\quad - \partial_x^2\big(|E_M^n|^2\big), \varphi\big) = \big(R_2^n, \varphi\big), \end{aligned} \tag{20}$$

where

$$\begin{aligned} &R_1^n = i\big(E_t^n - \partial_t E^{n+\frac{1}{2}}, \varphi\big) + \big((NE)^{n+\frac{1}{2}} - N^{n+\frac{1}{2}}E^{n+\frac{1}{2}}, \varphi\big), \\ &R_2^n = \big(N_{t\bar{t}}^n - \partial_t^2 N^{\bar{n}}, \varphi\big) + \big(\partial_x^2\big(|E^{\bar{n}}|\big)^2 - \partial_x^2\big(|E^n|^2\big), \varphi\big). \end{aligned}$$

It follows from Theorem 2 that we can obtain the following lemma easily.

Lemma 3 *Assume that* E, N *are a solution of the fractional quantum Zakharov system* (3)–(6), *and the initial values* $E_0 \in H_P^1, N_0, N_1 \in L_p^2$. *Then there exists the unique solution*

E_M, N_M *of the Fourier spectral scheme* (9)–(13). *Moreover, we have the following estimate*:

$$\begin{aligned}&\operatorname{Im}\bigl(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}}-N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\\&\quad\le C\bigl(\|\eta_M^{n+1}\|^2+\|\eta_M^n\|^2+\|e_M^{n+1}\|^2+\|e_M^n\|^2+\|\tilde{\eta}^{n+1}\|^2+\|\tilde{\eta}^n\|^2+\|\tilde{e}^{n+1}\|^2+\|\tilde{e}^n\|^2\bigr).\end{aligned}$$

Proof It follows from (17)–(18) that

$$\begin{aligned}&\operatorname{Im}\bigl(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}}-N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\\&\quad=\operatorname{Im}\bigl(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}}-N^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}+N^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}-N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\\&\quad=\operatorname{Im}\left(\frac{1}{2}N^{n+\frac{1}{2}}\bigl(\tilde{e}^{n+1}+e_M^{n+1}+\tilde{e}^n+e_M^n\bigr),e_M^{n+\frac{1}{2}}\right)\\&\qquad+\operatorname{Im}\left(\frac{1}{2}E_M^{n+\frac{1}{2}}\bigl(\tilde{\eta}^{n+1}+\eta_M^{n+1}+\tilde{\eta}^n+\eta_M^n\bigr),e_M^{n+\frac{1}{2}}\right)\\&\quad=\operatorname{Im}\bigl(N^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)+\operatorname{Im}\bigl(N^{n+\frac{1}{2}}e_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)+\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\tilde{\eta}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\\&\qquad+\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\eta_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\\&\quad=\operatorname{Im}\left(\frac{1}{2}N^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\right)+\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\tilde{\eta}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)+\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\eta_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr).\end{aligned}$$

Noting that

$$\begin{aligned}\operatorname{Im}\bigl(N^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)&\le\bigl|\bigl(N^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\bigr|\\&\le\frac{1}{8}\bigl(\bigl(|N^{n+1}|+|N^n|\bigr)\bigl(|\tilde{e}^{n+1}|+|\tilde{e}^n|\bigr),|e_M^{n+1}|+|e_M^n|\bigr)\\&\le\frac{1}{8}\bigl(\|N^{n+1}\|_\infty+\|N^n\|_\infty\bigr)\bigl(|\tilde{e}^{n+1}|+|\tilde{e}^n|,|e_M^{n+1}|+|e_M^n|\bigr)\\&\le C\bigl(\|\tilde{e}^{n+1}\|^2+\|\tilde{e}^n\|^2+\|e_M^{n+1}\|^2+\|e_M^n\|^2\bigr),\\\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\tilde{\eta}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)&\le\bigl|\bigl(E_M^{n+\frac{1}{2}}\tilde{\eta}^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\bigr|\\&\le\frac{1}{8}\bigl(\bigl(|\tilde{\eta}^{n+1}|+|\tilde{\eta}^n|\bigr)\bigl(|E_M^{n+1}|+|E_M^n|\bigr),|e_M^{n+1}|+|e_M^n|\bigr)\\&\le\frac{1}{8}\bigl(\|E_M^{n+1}\|_\infty+\|E_M^n\|_\infty\bigr)\bigl(|\tilde{\eta}^{n+1}|+|\tilde{\eta}^n|,|e_M^{n+1}|+|e_M^n|\bigr)\\&\le C\bigl(\|\tilde{\eta}^{n+1}\|^2+\|\tilde{\eta}^n\|^2+\|e_M^{n+1}\|^2+\|e_M^n\|^2\bigr),\\\operatorname{Im}\bigl(E_M^{n+\frac{1}{2}}\eta_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)&\le\bigl|\bigl(E_M^{n+\frac{1}{2}}\eta_M^{n+\frac{1}{2}},e_M^{n+\frac{1}{2}}\bigr)\bigr|\\&\le\frac{1}{8}\bigl(\bigl(|\eta_M^{n+1}|+|\eta_M^n|\bigr)\bigl(|E_M^{n+1}|+|E_M^n|\bigr),|e_M^{n+1}|+|e_M^n|\bigr)\\&\le\frac{1}{8}\bigl(\|E_M^{n+1}\|_\infty+\|E_M^n\|_\infty\bigr)\bigl(|\eta_M^{n+1}|+|\eta_M^n|,|e_M^{n+1}|+|e_M^n|\bigr)\\&\le C\bigl(\|\eta_M^{n+1}\|^2+\|\eta_M^n\|^2+\|e_M^{n+1}\|^2+\|e_M^n\|^2\bigr),\end{aligned}$$

we obtain

$$\begin{aligned}&\operatorname{Im}\big(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big)\\ &\quad\le C\big(\|\eta_M^{n+1}\|^2 + \|\eta_M^n\|^2 + \|e_M^{n+1}\|^2 + \|e_M^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2\big).\end{aligned}$$

This ends the proof. □

Accordingly, Lemma 3 can yield the following lemma.

Lemma 4 *There exists a constant C depending only on the initial and boundary values such that the solution of the discrete scheme satisfies*

$$\begin{aligned}\|e_M^{n+1}\|^2 \le{}& \frac{1+C\tau}{1-C\tau}\|e_M^n\|^2 + \frac{C\tau}{1-C\tau}\big(\|\eta_M^{n+1}\|^2 + \|\eta_M^n\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2\\ &+ \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \operatorname{Im}\big(R_1^n, e_M^{n+\frac{1}{2}}\big)\big).\end{aligned}$$

Proof Let $\varphi = e_M^{n+\frac{1}{2}}$. Then taking the imaginary part of equation (19) yields

$$\begin{aligned}&\operatorname{Im}\big(ie_{Mt}^n, e_M^{n+\frac{1}{2}}\big) + \operatorname{Im}\big(\partial_x^2 e_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big) + \operatorname{Im}H^2\big((-\triangle)^{\alpha-1}\partial_x^2 e_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big)\\ &\quad - \operatorname{Im}\big(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big) = \operatorname{Im}\big(R_1^n, e_M^{n+\frac{1}{2}}\big).\end{aligned}$$

Noting that

$$\begin{aligned}&\operatorname{Im}\big(ie_{Mt}^n, e_M^{n+\frac{1}{2}}\big) = \operatorname{Re}\left(\frac{e_M^{n+1}-e_M^n}{\tau}, e_M^{n+\frac{1}{2}}\right) = \frac{1}{\tau}\big(\|e_M^{n+1}\|^2 - \|e_M^n\|^2\big),\\ &\operatorname{Im}\big(\partial_x^2 e_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big) = 0,\\ &\operatorname{Im}H^2\big((-\triangle)^{\alpha-1}\partial_x^2 e_M^{n+\frac{1}{2}}, e_M^{n+\frac{1}{2}}\big) = 0\end{aligned}$$

and using Lemma 3, we obtain

$$\begin{aligned}\|e_M^{n+1}\|^2 \le{}& \frac{1+C\tau}{1-C\tau}\|e_M^n\|^2 + \frac{C\tau}{1-C\tau}\big(\|\eta_M^{n+1}\|^2 + \|\eta_M^n\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2\\ &+ \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \operatorname{Im}\big(R_1^n, e_M^{n+\frac{1}{2}}\big)\big).\end{aligned}$$

This ends the proof. □

Now, we consider the energy modulus estimate of $E_M^{n+\frac{1}{2}}$. First, we give the following lemma.

Lemma 5 *Suppose that* $E_0 \in H_p^1, N_0, N_1 \in L_p^2$. *Then we have the following estimate*:

$$\begin{aligned}&-\frac{1}{\tau}\operatorname{Re}\big(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+1} - e_M^n\big)\\ &\quad\le \tau\big(-A^n + A^{n-1} - B^n + B^{n-1} - \tilde{A}^n + \tilde{A}^{n-1} - \tilde{B}^n + \tilde{B}^{n-1}\big)\end{aligned}$$

$$+ C\tau\left(\theta^{n}+\theta^{n-1}\right) + C\tau\left(\left\|\tilde{e}^{n+1}\right\|^2 + \left\|\tilde{e}^{n}\right\|^2 + \left\|\tilde{\eta}^{n+1}\right\|^2 + \left\|\tilde{\eta}^{n}\right\|^2 + \left\|\partial_x\tilde{e}^{n}\right\|^2 + \left\|\partial_t\tilde{e}^{n+1}\right\|^2 + \left\|\partial_t\tilde{e}^{n}\right\|^2 + \left\|\partial_t\tilde{\eta}^{n+1}\right\|^2 + \left\|\partial_t\tilde{\eta}^{n}\right\|^2\right),$$

where

$$\theta^n = \frac{1}{2}\left[\left\|e_M^{n+1}\right\|^2 + \left\|\partial_x e_M^{n+1}\right\|^2 + \left\|\partial_x U^{n+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left(\left\|\eta_M^{n+1}\right\|^2 + \left\|\eta_M^{n}\right\|^2\right)\right] + \frac{H^2}{4}\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n+1}\right\|^2 + \left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n}\right\|^2\right) + \frac{H^2}{2}\left\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^{n+1}\right\|^2.$$

Proof It follows from (17)–(18) that

$$\begin{aligned} &-\frac{1}{\tau}\operatorname{Re}\left(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right) \\ &= -\frac{1}{\tau}\operatorname{Re}\left(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E^{n+\frac{1}{2}} + N_M^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right) \\ &= -\frac{1}{\tau}\operatorname{Re}\left(\tilde{\eta}^{n+\frac{1}{2}}E^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right) - \frac{1}{\tau}\operatorname{Re}\left(\eta_M^{n+\frac{1}{2}}E^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right) \\ &\quad - \frac{1}{\tau}\operatorname{Re}\left(N_M^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right) - \frac{1}{\tau}\operatorname{Re}\left(N_M^{n+\frac{1}{2}}, \left|e_M^{n+1}\right|^2 - \left|e_M^{n}\right|^2\right). \end{aligned} \tag{21}$$

Let

$$\begin{aligned} I &= -\frac{1}{\tau}\operatorname{Re}\left(\tilde{\eta}^{n+\frac{1}{2}}E^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right), \\ II &= -\frac{1}{\tau}\operatorname{Re}\left(\eta_M^{n+\frac{1}{2}}E^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right), \\ III &= -\frac{1}{\tau}\operatorname{Re}\left(N_M^{n+\frac{1}{2}}\tilde{e}^{n+\frac{1}{2}}, e_M^{n+1}-e_M^{n}\right), \\ VI &= -\frac{1}{\tau}\operatorname{Re}\left(N_M^{n+\frac{1}{2}}, \left|e_M^{n+1}\right|^2 - \left|e_M^{n}\right|^2\right), \\ \omega^n &= E^{n+\frac{1}{2}}, \tilde{\omega}^n = \tilde{e}^{n+\frac{1}{2}}. \end{aligned}$$

For the first term of equation (21), it holds that

$$\begin{aligned} I &= -\frac{1}{2\tau}\operatorname{Re}\left(\left(\tilde{\eta}^{n+1}+\tilde{\eta}^{n}\right)\omega^n, e_M^{n+1}-e_M^{n}\right) \\ &= -\frac{1}{2\tau}\operatorname{Re}\left\{\left(\tilde{\eta}^{n+1}\omega^n, e_M^{n+1}\right) + \left(\tilde{\eta}^{n}\omega^n, e_M^{n+1}\right) - \left(\tilde{\eta}^{n}\omega^{n-1}, e_M^{n}\right) - \left(\tilde{\eta}^{n-1}\omega^{n-1}, e_M^{n}\right)\right. \\ &\quad \left. + \left(\tilde{\eta}^{n-1}\omega^{n-1}, e_M^{n}\right) - \left(\tilde{\eta}^{n}\left(\omega^n-\omega^{n-1}\right), e_M^{n}\right) - \left(\tilde{\eta}^{n+1}\omega^n, e_M^{n}\right)\right\} \\ &= -\tilde{A}^n + \tilde{A}^{n-1} + \frac{1}{2\tau}\operatorname{Re}\left(\left(\tilde{\eta}^{n+1}+\tilde{\eta}^{n}\right)\left(\omega^n-\omega^{n-1}\right), e_M^{n}\right) + \frac{1}{2\tau}\operatorname{Re}\left(\omega^{n-1}\left(\eta^{n+1}-\eta^{n-1}\right), e_M^{n}\right) \\ &\le -\tilde{A}^n + \tilde{A}^{n-1} + \frac{1}{4}\left(\left\|E_t^{n+1}\right\|_\infty + \left\|E_t^{n}\right\|_\infty\right)\left(\left|\tilde{\eta}^{n+1}\right| + \left|\tilde{\eta}^{n}\right|, \left|e_M^{n}\right|\right) \\ &\quad + \frac{1}{2\tau}\left|\left(\omega^{n-1}\left(\eta^{n+1}-\eta^{n-1}\right), e_M^{n}\right)\right| \\ &\le -\tilde{A}^n + \tilde{A}^{n-1} + C\left(\theta^{n-1}+\theta^{n-1} + \left\|\tilde{\eta}^{n+1}\right\|^2 + \left\|\tilde{\eta}^{n}\right\|^2 + \left\|\tilde{\eta}_t^{n+1}\right\|^2 + \left\|\tilde{\eta}_t^{n}\right\|^2\right), \end{aligned} \tag{22}$$

here $\tilde{A}^n = \frac{1}{2\tau}\operatorname{Re}((\tilde{\eta}^{n+1}+\tilde{\eta}^{n})\omega^n, e_M^{n+1})$.

For the second term of equation (21), it holds that

$$
\begin{aligned}
II &= -\frac{1}{2\tau}\operatorname{Re}\big(\big(\eta_M^{n+1}+\eta_M^{n-1}\big)\omega^n, e_M^{n+1}-e_M^n\big) \\
&= -\frac{1}{2\tau}\operatorname{Re}\big\{\big(\eta_M^{n+1}\omega^n, e_M^{n+1}\big)+\big(\eta_M^n\omega^n, e_M^{n+1}\big)-\big(\eta_M^n\omega^{n-1}, e_M^n\big)-\big(\eta_M^{n-1}\omega^{n-1}, e_M^n\big)\big] \\
&\quad+\big(\eta_M^{n-1}\omega^{n-1}, e_M^n\big)-\big(\eta_M^n\big(\omega^n-\omega^{n-1}\big), e_M^n\big)-\big(\eta_M^{n+1}\omega^n, e_M^n\big)\big\} \\
&= -A^n+A^{n-1}+\frac{1}{2\tau}\operatorname{Re}\{\big(\big(\eta_M^{n+1}+\eta_M^n\big)\big(\omega^n-\omega^{n-1}\big), e_M^n\big) \\
&\quad+\frac{1}{2\tau}\operatorname{Re}\{\big(\omega^{n-1}\big(\omega_M^{n+1}-\omega_M^{n-1}\big), e_M^n\big) \qquad (23)\\
&\le -A^n+A^{n-1}\frac{1}{2\tau}\big(\big\|E_t^{n+1}\big\|_\infty+\big\|E_t^n\big\|_\infty\big)\big(\big|\eta_M^{n+1}\big|+\big|\eta_M^n\big|,\big|e_M^n\big|\big) \\
&\quad+\frac{1}{2}\big\|\omega^{n-1}\big\|_\infty\big(\big|\partial_x U^{n+1}\big|+\big|\partial_x U^n\big|,\big|\partial_x e_M^n\big|\big) \\
&\le -A^n+A^{n-1}+C\big(\theta^n+\theta^{n-1}\big),
\end{aligned}
$$

here $A^n=\frac{1}{2\tau}\operatorname{Re}((\eta_M^{n+1}+\eta_M^n)\omega^n, e_M^{n+1})$.

For the third term of equation (21), it holds that

$$
\begin{aligned}
III &= -\frac{1}{2\tau}\operatorname{Re}\big\{\big(N_M^{n+1}\tilde\omega^n, e_M^{n+1}\big)+\big(N_M^n\tilde\omega^n, e_M^{n+1}\big)-\big(N_M^n\tilde\omega^{n-1}, e_M^n\big)-\big(N_M^{n-1}\tilde\omega^{n-1}, e_M^n\big) \\
&\quad+\big(N_M^{n-1}\tilde\omega^{n-1}, e_M^n\big)-\big(N_M^n\big(\tilde\omega^n-\tilde\omega^{n-1}\big), e_M^n\big)-\big(N_M^{n+1}\tilde\omega^n, e_M^n\big)\big\} \\
&= -\tilde B^n+\tilde B^{n-1}+\frac{1}{2\tau}\operatorname{Re}\big(\big(N_M^{n+1}+N_M^n\big)\big(\tilde\omega^n-\tilde\omega^{n-1}\big), e_M^n\big) \\
&\quad+\frac{1}{2\tau}\operatorname{Re}\big(\big(N_M^{n+1}-N_M^n\big)\tilde\omega^{n-1}, e_M^n\big) \qquad (24)\\
&\le -\tilde B^n+\tilde B^{n-1}+C\big(\big\|\tilde e_t^{n+1}\big\|^2+\big\|\tilde e_t^n\big\|^2+\big\|N_M^{n+1}\big\|^2+\big\|N_M^{n+1}\big\|^2+\big\|e_M^n\big\|^2\big) \\
&\quad+C\big(\big\|\tilde e^{n+1}\big\|^2+\big\|\tilde e^n\big\|^2+\big\|\partial_x\tilde e^{n+1}\big\|^2+\big\|\partial_x\tilde e^n\big\|^2\big\|\partial_x e_M^n\big\|^2\big) \\
&\le -\tilde B^n+\tilde B^{n-1}+C\big(\theta^n+\theta^{n-1}\big\|\tilde e_t^{n+1}\big\|^2+\big\|\tilde e_t^n\big\|^2+\big\|\tilde e^{n+1}\big\|^2 \\
&\quad+\big\|\tilde e^n\big\|^2+\big\|\partial_x\tilde e^{n+1}\big\|^2+\big\|\partial_x\tilde e^n\big\|^2\big),
\end{aligned}
$$

here $\tilde B^n=\frac{1}{2\tau}\operatorname{Re}((N_M^{n+1}+N_M^n)\tilde\omega^n, e_M^{n+1})$.

For the fourth term of equation (21), it holds that

$$
\begin{aligned}
VI &= -\frac{1}{2\tau}\operatorname{Re}\big\{\big(N_M^{n+1}+N_M^n,\big|e_M^{n+1}\big|^2\big)-\big(N_M^n+N_M^{n-1},\big|e_M^n\big|^2\big)-\big(N_M^{n+1}-N_M^{n-1},\big|e_M^n\big|^2\big)\big\} \qquad (25)\\
&\le B^n-B^{n-1}+C\theta^{n-1},
\end{aligned}
$$

here $B^n=\frac{1}{\tau}\operatorname{Re}(N_M^{n+\frac{1}{2}},|e_M^{n+1}|^2)$.

Finally, noting equations (22)–(25), we get

$$
\begin{aligned}
&-\frac{1}{\tau}\operatorname{Re}\big(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}}-N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, e_M^{n+1}-e_M^n\big) \\
&\quad\le \tau\big(-A^n+A^{n-1}-B^n+B^{n-1}-\tilde A^n+\tilde A^{n-1}-\tilde B^n+\tilde B^{n-1}\big)
\end{aligned}
$$

$$+ C\tau\left(\theta^n + \theta^{n-1}\right) + C\tau\left(\|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_x \tilde{e}^n\|^2 + \|\partial_t \tilde{e}^{n+1}\|^2 + \|\partial_t \tilde{e}^n\|^2 + \|\partial_t \tilde{\eta}^{n+1}\|^2 + \|\partial_t \tilde{\eta}^n\|^2\right).$$

This ends the proof. □

It follows from Lemma 5 that we can obtain the following lemma.

Lemma 6 *There exists a constant C depending only on the initial and boundary values such that the solution of the discrete scheme satisfies*

$$\begin{aligned}
&\frac{1}{2}\|\partial_x e_M^{n+1}\|^2 - \frac{1}{2}\|\partial_x e_M^n\|^2 + \frac{H^2}{2}\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^{n+1}\|^2 - \frac{H^2}{2}\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^n\|^2 \\
&\quad + \tau\left(A^n - A^{n-1} + B^n - B^{n-1} + \tilde{A}^n - \tilde{A}^{n-1} + \tilde{B}^n - \tilde{B}^{n-1}\right) \\
&\leq C\tau\left(\theta^n + \theta^{n-1}\right) + C\tau\left(\|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_x \tilde{e}^n\|^2\right. \\
&\quad \left. + \|\partial_t \tilde{e}^{n+1}\|^2 + \|\partial_t \tilde{e}^n\|^2 + \|\partial_t \tilde{\eta}^{n+1}\|^2 + \|\partial_t \tilde{\eta}^n\|^2 + \operatorname{Re}\left(R_1^n, e_M^{n+1} - e_M^n\right)\right).
\end{aligned}$$

Proof Let $\varphi = -\frac{1}{\tau}(e_M^{n+1} - e_M^n)$ in (19). Then taking the real part of equation (19) yields

$$\begin{aligned}
&-\frac{1}{\tau}\operatorname{Re} i\left(\frac{e_M^{n+1} - e_M^n}{\tau}, e_M^{n+1} - e_M^n\right) - \frac{1}{\tau}\operatorname{Re}\left(e_{Mxx}^{n+\frac{1}{2}}, e_M^{n+1} - e_M^n\right) \\
&\quad - \frac{1}{\tau}\operatorname{Re} H^2\left((-\triangle)^{\alpha-1} e_{Mxx}^{n+\frac{1}{2}}, e_M^{n+1} - e_M^n\right) \\
&= -\frac{1}{\tau}\operatorname{Re}\left(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, \varphi\right) - \frac{1}{\tau}\operatorname{Re}\left(R_1^n, \varphi\right).
\end{aligned}$$

Noting that

$$\begin{aligned}
&-\frac{1}{\tau}\operatorname{Re} i\left(\frac{e_M^{n+1} - e_M^n}{\tau}, e_M^{n+1} - e_M^n\right) = -\frac{1}{\tau}\left(\|e_M^{n+1} - e_M^n\|^2\right) = 0, \\
&-\frac{1}{\tau}\operatorname{Re}\left(e_{Mxx}^{n+\frac{1}{2}}, e_M^{n+1} - e_M^n\right) = \frac{1}{\tau}\operatorname{Re}\left(e_{Mx}^{n+\frac{1}{2}}, e_{Mx}^{n+1} - e_{Mx}^n\right) = \frac{1}{2\tau}\left(\|e_{Mx}^{n+1}\|^2 - \|e_{Mx}^n\|^2\right), \\
&-\frac{1}{\tau}\operatorname{Re} H^2\left((-\triangle)^{\alpha-1} e_{Mxx}^{n+\frac{1}{2}}, e_M^{n+1} - e_M^n\right) \\
&\quad = \frac{1}{\tau}\operatorname{Re} H^2\left((-\triangle)^{\alpha-1} e_{Mx}^{n+\frac{1}{2}}, e_{Mx}^{n+1} - e_{Mx}^n\right) \\
&\quad = \frac{1}{\tau}\operatorname{Re} H^2\left((-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^{n+\frac{1}{2}}, (-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^{n+1} - (-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^n\right) \\
&\quad = \frac{1}{2\tau}H^2\left(\|(-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^{n+1}\|^2 - \|(-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^n\|^2\right),
\end{aligned}$$

we can obtain

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|e_{Mx}^{n+1}\|^2 - \|e_{Mx}^n\|^2\right) + \frac{1}{2\tau}H^2\left(\|(-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^{n+1}\|^2 - \|(-\triangle)^{\frac{\alpha-1}{2}} e_{Mx}^n\|^2\right) \\
&\quad = -\frac{1}{\tau}\operatorname{Re}\left(N^{n+\frac{1}{2}}E^{n+\frac{1}{2}} - N_M^{n+\frac{1}{2}}E_M^{n+\frac{1}{2}}, \varphi\right) - \frac{1}{\tau}\operatorname{Re}\left(R_1^n, \varphi\right).
\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}
&\frac{1}{2}\|\partial_x e_M^{n+1}\|^2 - \frac{1}{2}\|\partial_x e_M^n\|^2 + \frac{H^2}{2}\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^{n+1}\|^2 - \frac{H^2}{2}\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^n\|^2 \\
&\quad + \tau\big(A^n - A^{n-1} + B^n - B^{n-1} + \tilde{A}^n - \tilde{A}^{n-1} + \tilde{B}^n - \tilde{B}^{n-1}\big) \\
&\quad \le C\tau\big(\theta^n + \theta^{n-1}\big) + C\tau\big(\|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_x\tilde{e}^n\|^2 \\
&\quad + \|\partial_t\tilde{e}^{n+1}\|^2 + \|\partial_t\tilde{e}^n\|^2 + \|\partial_t\tilde{\eta}^{n+1}\|^2 + \|\partial_t\tilde{\eta}^n\|^2 + \operatorname{Re}\big(R_1^n, e_M^{n+1} - e_M^n\big)\big).
\end{aligned}$$

This ends the proof. □

Lemma 7 *There exists a constant C depending only on the initial and boundary values such that the solution of the discrete scheme satisfies*

$$\begin{aligned}
&\frac{1}{2}\|\partial_x U_M^{n+\frac{1}{2}}\|^2 - \frac{1}{2}\|\partial_x U_M^{n-\frac{1}{2}}\|^2 + \frac{1}{4}\big(\|\eta_M^{n+1}\|^2 + \|\eta_M^n\|^2\big) \\
&\quad - \frac{1}{4}\big(\|\eta_M^n\|^2 + \|\eta_M^{n-1}\|^2\big) + \frac{H^2}{4}\big(\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n+1}\|^2 + \|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^n\|^2\big) \\
&\quad - \frac{H^2}{4}\big(\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^n\|^2 + \|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n-1}\|^2\big) \\
&\le C\tau\big(\theta^n + \theta^{n-1}\big) + C\tau\left(\|\partial_x\tilde{e}^n\|^2 + \|\tilde{e}^n\|^2 - \frac{1}{2}\big(R_2^n, U^n + U^{n-1}\big)\right).
\end{aligned}$$

Proof Let $\varphi = -\frac{1}{2}(U^n + U^{n-1})$ in (20). Then we obtain

$$\begin{aligned}
&\bigg(-\frac{1}{2}\frac{\eta_M^{n+1} - 2\eta_M^n + \eta_M^{n-1}}{\tau^2} + \frac{1}{4}\big(\partial_x^2\eta_M^{n+1} + \partial_x^2\eta_M^{n-1}\big) \\
&\quad + \frac{H^2}{4}(-\triangle)^{\beta-1}\big(\partial_x^2\eta_M^{n+1} + \partial_x^2\eta_M^{n-1}\big), U^n + U^{n-1}\bigg) \\
&\quad = -\frac{1}{2}\big(\partial_x^2\big(|E^n|^2\big) - \partial_x^2\big(|E_M^n|^2\big), U^n + U^{n-1}\big) - \frac{1}{2}\big(R_2^n, U^n + U^{n-1}\big).
\end{aligned}$$

It follows from integration by parts that

$$\begin{aligned}
&-\frac{1}{2}\big(\partial_x^2\big(|E^n|^2\big) - \partial_x^2\big(|E_M^n|^2\big), U^n + U^{n-1}\big) \\
&\quad = \frac{1}{2}\big(\partial_x\big[\big(E^n - E_M^n\big)\big(\bar{E}^n + \bar{E}_M^n\big], \partial_x U^n + \partial_x U^{n-1}\big) \\
&\quad = \frac{1}{2}\big(\partial_x\big(\tilde{e}^n + e_M^n\big)\big(\bar{E}^n + \bar{E}_M^n\big), \partial_x U^n + \partial_x U^{n-1}\big) \\
&\qquad + \frac{1}{2}\big(\big(\tilde{e}^n + e_M^n\big)\partial_x\big(\bar{E}^n + \bar{E}_M^n\big), \partial_x U^n + \partial_x U^{n-1}\big) \\
&\quad = \frac{1}{2}\big(\partial_x\tilde{e}^n\big(\bar{E}^n + \bar{E}_M^n\big), \partial_x U^n + \partial_x U^{n-1}\big) + \frac{1}{2}\big(\partial_x e_M^n\big(\bar{E}^n + \bar{E}_M^n, \partial_x U^n + \partial_x U^{n-1}\big) \\
&\qquad + \frac{1}{2}\big(\tilde{e}^n\partial_x\big(\bar{E}^n + \bar{E}_M^n\big), \partial_x U^n + \partial_x U^{n-1}\big)\frac{1}{2}\big(e_M^n\partial_x\big(\bar{E}^n + \bar{E}_M^n\big), \partial_x U^n + \partial_x U^{n-1}\big) \\
&\quad \le C\big(\theta^n + \theta^{n-1} + \|\partial_x\tilde{e}^n\|^2\big) + C\big(\theta^n + \theta^{n-1}\big) + C\big(\theta^n + \theta^{n-1}\big) \\
&\qquad + C\big(\|\tilde{e}^n\|^2 + \|\partial_x\tilde{e}^n\|^2\big) + C\big(\theta^n + \theta^{n-1}\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\left\|\partial_x U_M^{n+1}\right\|^2-\frac{1}{2}\left\|\partial_x U_M^{n}\right\|^2+\frac{1}{4}\left(\left\|\eta_M^{n+1}\right\|^2+\left\|\eta_M^{n}\right\|^2\right)-\frac{1}{4}\left(\left\|\eta_M^{n}\right\|^2+\left\|\eta_M^{n-1}\right\|^2\right)\\
&\quad\frac{H^2}{4}\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n+1}\right\|^2+\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n}\right\|^2\right)\\
&\quad-\frac{H^2}{4}\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n}\right\|^2+\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n-1}\right\|^2\right)\\
&\quad\le C\tau\left(\theta^n+\theta^{n-1}\right)+C\tau\left(\left\|\partial_x\tilde{e}^n\right\|^2+\left\|\tilde{e}^n\right\|^2-\frac{1}{2}\left(R_2^n,U^n+U^{n-1}\right)\right).
\end{aligned}$$

This ends the proof. □

Theorem 3 *Suppose that* E, N *satisfy* $N,\partial_t N,\partial_t E\in L^2[0,T;H_p^\sigma(I)]$, $\partial_x E\in H_p^{\sigma-1}(I)$, $\sigma>1$,

$$\partial_{tt}E,\partial_{ttt}E,\partial_{tt}N,\partial_{tttt}N,\partial_{xxtt}E,\partial_{xxtt}N,(-\triangle)^{\alpha-1}\partial_{xxtt}E,(-\triangle)^{\beta-1}\partial_{xxtt}N\in L^2(I).$$

There exists a constant C *depending only on the initial and boundary values such that*

$$\theta^n\le C\left(M^{-2\sigma}+\tau^4\right).$$

Proof Let

$$\begin{aligned}
\rho^{n-1}&=\frac{1}{2}\left\|\partial_x e_M^n\right\|^2+\frac{1}{2}\left\|\partial_x U^{n-\frac{1}{2}}\right\|^2+\frac{1}{4}\left(\left\|\eta_M^{n}\right\|^2+\left\|\eta_M^{n-1}\right\|^2\right)\\
&\quad+\frac{H^2}{4}\left(\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n+1}\right\|^2+\left\|(-\triangle)^{\frac{\beta-1}{2}}\eta_M^{n}\right\|^2\right)+\frac{H^2}{2}\left\|(-\triangle)^{\frac{\alpha-1}{2}}\partial_x e_M^{n+1}\right\|^2.
\end{aligned}$$

It follows from Lemmas 5 and 6 that

$$\begin{aligned}
&\rho^n+\tau\left(A^n+B^n+\tilde{A}^n+\tilde{B}^n\right)\\
&\quad\le\rho^{n-1}+\tau\left(A^{n-1}+B^{n-1}+\tilde{A}^{n-1}+\tilde{B}^{n-1}\right)+C\tau\left(\left\|\partial_x\tilde{e}^{n+1}\right\|^2+\left\|\tilde{e}^{n+1}\right\|^2+\left\|\tilde{e}^{n+1}\right\|^2\right.\\
&\quad\quad\left.+\left\|\tilde{\eta}^{n+1}\right\|^2+\left\|\tilde{\eta}^{n}\right\|^2+\left\|\partial_t\tilde{e}^{n+1}\right\|^2+\left\|\partial_t\tilde{e}^{n}\right\|^2+\left\|\partial_t\tilde{\eta}^{n+1}\right\|^2+\left\|\partial_t\tilde{\eta}^{n}\right\|^2\right)\\
&\quad\quad+C\tau\left(\theta^n+\theta^{n-1}\right)+C\tau\left(\operatorname{Re}\left(R_1^n,e_M^{n+1}-e_M^n\right)-\frac{1}{2}\left(R_2^n,U^n+U^{n-1}\right)\right).
\end{aligned}$$

Let

$$\hat{\theta}^n=\gamma\left\|e_M^{n+1}\right\|^2+\rho^n+\tau A^n+\tau B^n+\tau\tilde{A}^n+\tau\tilde{B}^n.$$

Then it follows from Lemma 4 that

$$\begin{aligned}
\hat{\theta}^n&\le\gamma\frac{1+C\tau}{1-C\tau}\left\|e_M^n\right\|^2+\gamma\frac{C\tau}{1-C\tau}\left(\left\|\eta_M^{n+1}\right\|^2+\left\|\eta_M^{n}\right\|^2\right)+\rho^{n-1}\\
&\quad+\tau A^{n-1}+\tau B^{n-1}+\tau\tilde{A}^{n-1}+\tau\tilde{B}^{n-1}+C\tau\left(\theta^n+\theta^{n-1}\right)\\
&\quad+C\tau\left(\left\|\partial_x\tilde{e}^{n+1}\right\|^2+\left\|\tilde{e}^{n+1}\right\|^2+\left\|\tilde{e}^{n}\right\|^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_t\tilde{e}^{n+1}\|^2 + \|\partial_t\tilde{e}^n\|^2 + \|\partial_t\tilde{\eta}^{n+1}\|^2 + \|\partial_t\tilde{\eta}^n\|^2\big) \\
&+ C\tau\left(\operatorname{Im}\big(R_1^n, e_M^{n+\frac{1}{2}}\big) + \operatorname{Re}\big(R_1, e_M^{n+1} - e_M^n\big) - \frac{1}{2}\big(R_2^n, U^n + U^{n-1}\big)\right).
\end{aligned}
$$

When $\gamma > 1$, we can obtain

$$
\hat{\theta}^n \geq C\|e_M^{n+1}\|^2 + \frac{3}{4}\rho^n + \tau A^n + \tau B^n + \tau\tilde{A}^n + \tau\tilde{B}^n.
$$

Noting that

$$
\begin{aligned}
&\tau|A^n| + \tau|B^n| + |\tau\tilde{A}^n| + |\tau\tilde{B}^n| \\
&\quad\leq \frac{3}{4}\rho^n + C\big(\|e_M^{n+1}\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\tilde{e}_x^{n+1}\|^2 + \|\tilde{e}^{n+1}\|^2\big), \\
&\tau\left(\operatorname{Im}\big(R_1^n, e_M^{n+\frac{1}{2}}\big) + \operatorname{Re}\big(R_1, e_M^{n+1} - e_M^n\big) - \frac{1}{2}\big(R_2^n, U^n + U^{n-1}\big)\right) \\
&\quad\leq C\tau\big(\theta^n + \theta^{n-1}\big) + C\tau^4\int_{t_{n-1}}^{t_{n+1}}\big(\|\partial_{ttt}E(s)\|^2 + \|\partial_{tt}E(s)\|^2 + \|\partial_{tt}N(s)\|^2 + \|\partial_{tttt}N(s)\|^2 \\
&\qquad+ \|\partial_{xxtt}E(s)\|^2 + \|\partial_{xxtt}N(s)\|^2 + \|(-\Delta)^{\alpha-1}\partial_{xxtt}E(s)\|^2 + \|(-\Delta)^{\beta-1}\partial_{xxtt}N(s)\|^2\big)\,ds,
\end{aligned}
$$

we get

$$
\begin{aligned}
\hat{\theta}^n &\leq \hat{\theta}^{n-1} + \gamma C\tau\big(\theta^n + \theta^{n-1}\big) + C\tau\big(\|\partial_x\tilde{e}^{n+1}\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 \\
&\quad+ \|\tilde{\eta}^n\|^2 + \|\partial_t\tilde{e}^{n+1}\|^2 + \|\partial_t\tilde{e}^n\|^2 + \|\partial_t\tilde{\eta}^{n+1}\|^2 + \|\partial_t\tilde{\eta}^n\|^2\big) \\
&\quad+ C\tau^4\int_{t_{n-1}}^{t_{n+1}}\big(\|\partial_{ttt}E(s)\|^2 + \|\partial_{tt}E(s)\|^2 + \|\partial_{tt}N(s)\|^2 + \|\partial_{tttt}N(s)\|^2 \\
&\quad+ \|\partial_{xxtt}E(s)\|^2 + \|\partial_{xxtt}N(s)\|^2 + \|(-\Delta)^{\alpha-1}\partial_{xxtt}E(s)\|^2 + \|(-\Delta)^{\beta-1}\partial_{xxtt}N(s)\|^2\big)\,ds.
\end{aligned}
\tag{26}
$$

Based on the definition of θ^n, we have

$$
\theta^n = \frac{1}{2}\|e_M^{n+1}\|^2 + \rho^n \leq \hat{\theta}^n + \frac{1}{4}\theta^n + C\|e_M^{n+1}\|^2, \tag{27}
$$

$$
\frac{3}{4}\theta^n \leq \hat{\theta}^n + C\|e_M^{n+1}\|^2 \leq C_\gamma\hat{\theta}^n. \tag{28}
$$

It follows from (26)–(28) that

$$
\begin{aligned}
\hat{\theta}^n &\leq \frac{1 + C_\gamma\tau}{1 - C_\gamma\tau}\hat{\theta}^{n-1} + \frac{C\tau}{1 - C_\gamma\tau}\big(\|\partial_x\tilde{e}^{n+1}\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 \\
&\quad+ \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_t\tilde{e}^{n+1}\|^2 + \|\partial_t\tilde{e}^n\|^2 + \|\partial_t\tilde{\eta}^{n+1}\|^2 \\
&\quad+ \|\partial_t\tilde{\eta}^n\|^2\big) + \frac{C\tau^4}{1 - C_\gamma\tau}\int_{t_{n-1}}^{t_{n+1}}\big(\|\partial_{ttt}E(s)\|^2 + \|\partial_{tt}E(s)\|^2 + \|\partial_{tt}N(s)\|^2 + \|\partial_{tttt}N(s)\|^2 \\
&\quad+ \|\partial_{xxtt}E(s)\|^2 + \|\partial_{xxtt}N(s)\|^2 + \|(-\Delta)^{\alpha-1}\partial_{xxtt}E(s)\|^2 + \|(-\Delta)^{\beta-1}\partial_{xxtt}N(s)\|^2\big)\,ds.
\end{aligned}
$$

Let

$$s = \frac{1 + C_\gamma \tau}{1 - C_\gamma \tau},$$
$$q^n = \left(\|\partial_x \tilde{e}^{n+1}\|^2 + \|\tilde{e}^{n+1}\|^2 + \|\tilde{e}^n\|^2 + \|\tilde{\eta}^{n+1}\|^2 + \|\tilde{\eta}^n\|^2 + \|\partial_t \tilde{e}^{n+1}\|^2 + \|\partial_t \tilde{e}^n\|^2 + \|\partial_t \tilde{\eta}^{n+1}\|^2 + \|\partial_t \tilde{\eta}^n\|^2\right).$$

Then, by the similarity analysis in [41], we have

$$\begin{aligned}
\hat{\theta}^n &\le s^n \hat{\theta}^0 + C\tau \sum_{i=1}^n s^{n-i} q^i \\
&\quad + C\tau^4 \sum_{i=1}^n s^{n-i} \int_{t_{n-1}}^{t_{n+1}} \Big(\|\partial_{ttt} E(s)\|^2 + \|\partial_{tt} E(s)\|^2 + \|\partial_{tt} N(s)\|^2 + \|\partial_{tttt} N(s)\|^2 \\
&\quad + \|\partial_{xxtt} E(s)\|^2 + \|\partial_{xxtt} N(s)\|^2 + \|(-\Delta)^{\alpha-1} \partial_{xxtt} E(s)\|^2 + \|(-\Delta)^{\alpha-1} \partial_{xxtt} N(s)\|^2\Big)\, ds \\
&\le C\hat{\theta}^0 + CM^{-2\sigma} + C\tau^4.
\end{aligned}$$

Noting that $e_M^0 = \eta_M^0 = \eta_M^1 = 0$, we get $\theta^n \le C\hat{\theta}^n \le C(M^{-2\sigma} + \tau^4)$. This ends the proof. □

6 Numerical experiments

In this section, we use the Fourier spectral method in space and the Crank–Nicolson/leap-frog in time to solve the fractional modified Zakharov system (3)–(6) with periodic boundary condition. We report the numerical accuracy, CPU time, invariants-preserving properties, and solitary wave graph for the fractional modified Zakharov system (3)–(6).

6.1 Experiment A ($H = 0$)

When $H = 0$, system (3)–(6) becomes the classical Zakharov system and has the exact solitary wave solutions [42]

$$E(x,t) = i\sqrt{2B^2(1-v^2)}\,\mathrm{sech}\big(B(x - x_0 - vt)\big) e^{i((x-x_0)/2 - (v^2/4 - B^2)t},$$
$$N(x,t) = -2B^2 \,\mathrm{sech}^2\big(B(x - x_0 - vt)\big).$$

Take the initial values with $B = 1$, $x_0 = 0$, $v = 0.5$,

$$E_0(x) = i\sqrt{1.5}\,\mathrm{sech}(Bx) e^{ix/4}, \qquad N_0(x) = -2\,\mathrm{sech}^2(x).$$

Table 1 Errors and orders in time for the classical Zakharov system

$E(x,t)$	M	N	$\Vert E - E_M\Vert_\infty$	Order	$\Vert E - E_M\Vert_2$	Order
	4096	16	7.4783e–07		2.3871e–06	
	4096	32	1.9320e–07	1.9526	6.1155e–07	1.9647
	4096	64	4.9076e–08	1.9770	1.5562e–07	1.9744
	4096	128	1.2366e–08	1.9886	4.2652e–08	1.8674
$N(x,t)$	M	N	$\Vert N - N_M\Vert_\infty$	Order	$\Vert N - N_M\Vert_2$	Order
	4096	16	8.8471e–07		1.9594e–06	
	4096	32	2.3678e–07	1.9017	5.1050e–07	1.9404
	4096	64	6.1223e–08	1.9514	1.3026e–07	1.9705
	4096	128	1.5561e–08	1.9761	3.2899e–08	1.9853

Table 2 Errors and orders in space for the classical Zakharov system

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	16	4096	7.5463e–02		3.1923e–01	
	32	4096	2.0507e–02	1.8796	6.6572e–02	2.2616
	64	4096	6.9476e–04	4.8835	2.3191e–03	4.8433
	128	4096	1.4839e–07	12.193	5.7162e–07	11.986
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	16	4096	3.7073e–03		1.1998e–02	
	32	4096	1.4899e–03	1.3152	3.4423e–03	1.8014
	64	4096	1.1068e–04	3.7507	2.2889e–04	3.9106
	128	4096	1.8260e–07	9.2435	3.9745e–07	9.1697

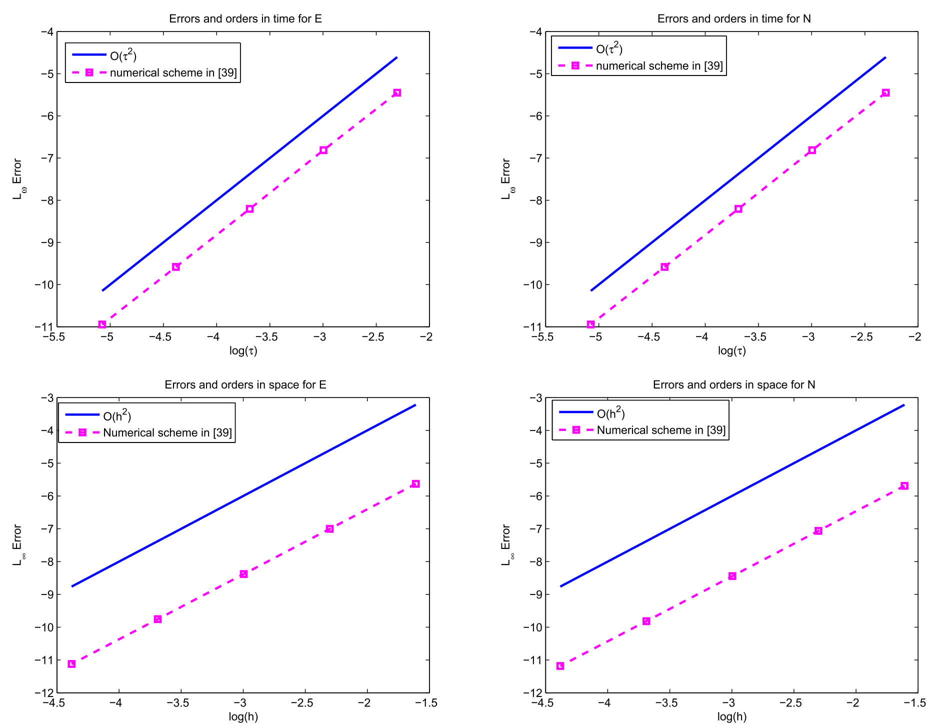

Figure 1 Errors and orders for the classical Zakharov system

We first calculate the convergence orders in time by using the conservative Fourier spectral scheme (9)–(13) with $H = 0$. Table 1 displays the numerical orders of time accuracy by scheme (9)–(13) when $t = 1$ and $M = 4096$. It clearly indicates that the proposed scheme (9)–(13) is of order 2 in time. Secondly, we test the space errors and the convergence orders by using the conservative Fourier spectral scheme (9)–(13) when $\tau = 1/4096$, $t = 1$. Table 2 displays the numerical orders of space accuracy by scheme (9)–(13). It can be seen that the Fourier spectral scheme (9)–(13) achieves spectral convergence up to machine precision. In addition, we display also the convergence orders and errors of time and space accuracy by the numerical scheme in [39]. Figure 1 displays the numerical orders of time and space accuracy by the numerical scheme in [39] when $t = 1$. It clearly indicates that the proposed scheme is of order 2 in time and space. Then, we choose the parameters $t \in [0, 20]$, $\tau = 0.001$, $M = 512$. The waveform diagrams of numerical solution are given by Fig. 2.

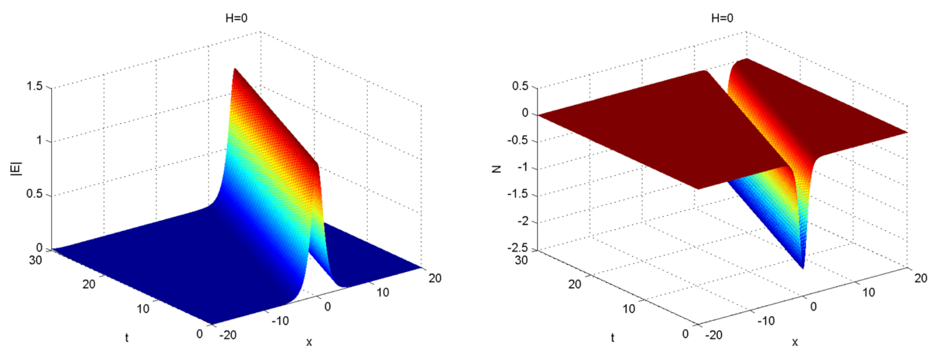

Figure 2 The module of numerical E and numerical N for classical Zakharov system

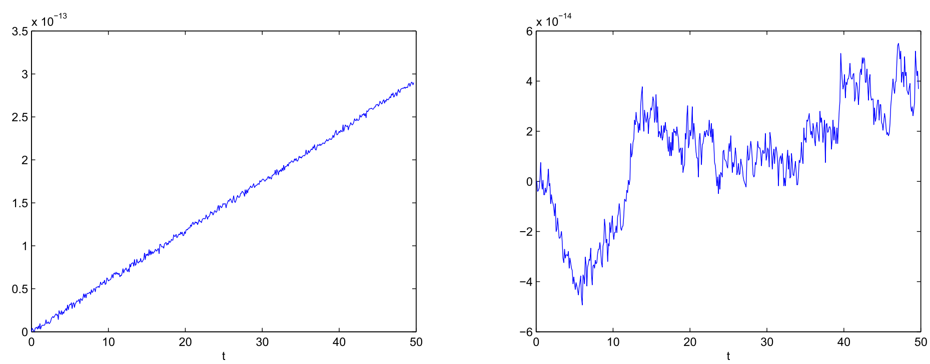

Figure 3 The errors of discrete mass and energy for the classical Zakharov system(Left:discrete mass; Right: discrete energy)

Table 3 Errors and orders in time for the quantum Zakharov system

$E(x,t)$	M	N	$\|E-E_M\|_\infty$	Order	$\|E-E_M\|_2$	Order
	4096	16	7.4781e–07		2.3870e–06	
	4096	32	1.9321e–07	1.9525	6.1135e–07	1.9652
	4096	64	4.9212e–08	1.9731	1.5481e–07	1.9815
	4096	128	1.2492e–08	1.9780	3.9451e–08	1.9723
$N(x,t)$	M	N	$\|N-N_M\|_\infty$	Order	$\|N-N_M\|_2$	Order
	4096	16	8.8454e–07		1.9595e–06	
	4096	32	2.3661e–07	1.9024	5.1067e–07	1.9401
	4096	64	6.1063e–08	1.9542	1.3052e–07	1.9681
	4096	128	1.5403e–08	1.9870	3.3441e–08	1.9646

Finally, we will test the conservative property of the conservative Fourier spectral method (9)–(13). We choose the parameters $t \in [0,50]$, $\tau = 0.001$, $M = 512$. Figure 3 displays the errors in the total mass M^n and energy E^n. It clearly indicates that the conservative Fourier spectral scheme (9)–(13) preserves the mass and energy conservation laws very well simultaneously.

6.2 Experiment B ($H > 0$)

When $H > 0$, system (3)–(6) becomes the quantum Zakharov system or fractional quantum Zakharov system. We take the same initial values with Example A. When $H > 0$, the

Table 4 Errors and orders in space for the quantum Zakharov system

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	16	4096	7.5463e–02		3.1923e–01	
	32	4096	2.0508e–02	1.8796	6.6572e–02	2.2616
	64	4096	6.9476e–04	4.8835	1.6399e–04	4.8433
	128	4096	1.4839e–07	12.193	4.0414e–8	11.986
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	16	4096	3.7073e–03		1.1998e–02	
	32	4096	1.4899e–03	1.3152	3.4423e–03	1.8014
	64	4096	1.1068e–04	3.7507	2.2889e–04	3.9106
	128	4096	1.8260e–07	9.2435	3.9745e–07	9.1697

Table 5 Errors and orders in time for the fractional quantum Zakharov system with $\alpha = 1.6$, $\beta = 1.7$

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	4096	32	1.9321e–07		1.9354e–06	
	4096	64	4.9212e–08	1.9731	1.5481e–07	1.9815
	4096	128	1.2492e–08	1.9780	3.9451e–08	1.9723
	4096	256	3.2661e–09	1.9354	1.1444e–08	1.7855
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	4096	32	2.3661e–07		5.1067e–07	
	4096	64	6.1063e–08	1.9542	1.3052e–07	1.9681
	4096	128	1.5403e–08	1.9870	3.3441e–08	1.9646
	4096	256	3.7589e–09	2.0349	9.8035e–09	1.7703

Table 6 Errors and orders in space for the fractional quantum Zakharov system with $\alpha = 1.6$, $\beta = 1.7$

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	16	4096	7.5463e–02		3.1923e–01	
	32	4096	2.0507e–02	1.8796	6.6572e–02	2.2616
	64	4096	6.9476e–04	4.8835	2.3191e–03	4.8433
	128	4096	1.4839e–07	12.193	5.7156e–07	11.986
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	16	4096	3.7073e–03		1.1998e–02	
	32	4096	1.4899e–03	1.3152	3.4423e–03	1.8014
	64	4096	1.1068e–04	3.7507	2.2889e–04	3.9106
	128	4096	1.8260e–07	9.2435	3.9745e–07	9.1697

Table 7 Errors and orders in time for the fractional quantum Zakharov system with $\alpha = 1.7$, $\beta = 1.8$

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	4096	32	8.9082e–01		3.7028e+00	
	4096	64	1.4329e–01	2.6361	6.4546e–01	2.5202
	4096	128	3.7257e–02	1.9434	1.6770e–01	1.9445
	4096	256	9.4414e–03	1.9804	4.2466e–02	1.9815
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	4096	32	1.7973e+00		6.5644e+00	
	4096	64	3.4486e–01	2.3817	1.2186e+00	2.4294
	4096	128	7.6059e–02	2.1808	2.7550e–01	2.1451
	4096	256	1.8461e–02	2.0427	6.7274e–02	2.0339

numerical exact solutions E, N are obtained by $M = 4096$, $\tau = 1/4096$. First, calculate the convergence orders in time and space by using the conservative Fourier spectral scheme (9)–(13) for the quantum Zakharov system with $H = 0.0001$. The results are listed in Ta-

Table 8 Errors and orders in space for the fractional quantum Zakharov system with $\alpha = 1.7$, $\beta = 1.8$

$E(x,t)$	M	N	$\|E - E_M\|_\infty$	Order	$\|E - E_M\|_2$	Order
	16	4096	1.2348e+00		1.0760e+01	
	32	4096	1.9619e+00	−0.6677	9.8699e+00	0.1245
	64	4096	6.5951e−03	8.2166	2.6440e−02	8.5441
	128	4096	1.4866e−07	15.437	6.3216e−07	15.352
$N(x,t)$	M	N	$\|N - N_M\|_\infty$	Order	$\|N - N_M\|_2$	Order
	16	4096	2.0552e+00		1.3383e+01	
	32	4096	1.7036e+00	2.7070	6.3425e+00	1.0772
	64	4096	1.1000e−02	7.2749	9.6132e−02	6.0439
	128	4096	1.7211e−06	12.642	1.1154e−05	13.073

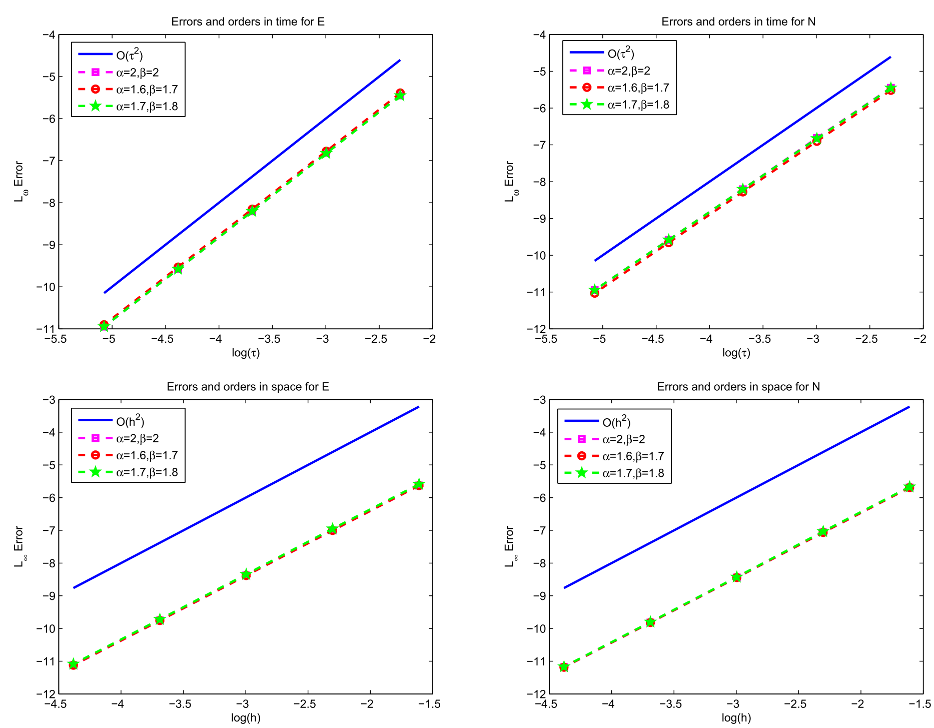

Figure 4 Errors and orders for the fractional quantum Zakharov system

Table 9 CPU time for difference numerical schemes with $M = 128$, $\tau = 0.01$

		Numerical scheme (15)–(16)	Numerical scheme (9)–(13)	Finite difference scheme
$\alpha = 1.4$	$T = 10$	60.13	406.23	828.34
	$T = 50$	278.18	1859.34	3887.57
	$T = 80$	679.23	3845.78	5956.8
$\alpha = 1.6$	$T = 10$	62.42	425.56	889.24
	$T = 50$	312.15	2134.12	4122.32
	$T = 80$	769.34	4238.35	6348.9

bles 3–4. Secondly, we verify the time and space convergence orders by scheme (9)–(13) for the fractional quantum Zakharov system with $H = 0.0001$ and different α, β. The results are listed in Tables 5–8. From Tables 3–8, it is found that scheme (9)–(13) is also of

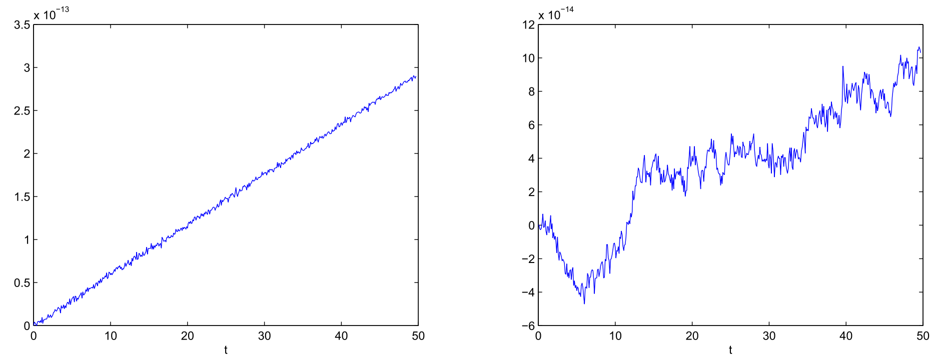

Figure 5 The errors of discrete mass and energy of the quantum Zakharov system (Left:discrete mass; Right: discrete energy)

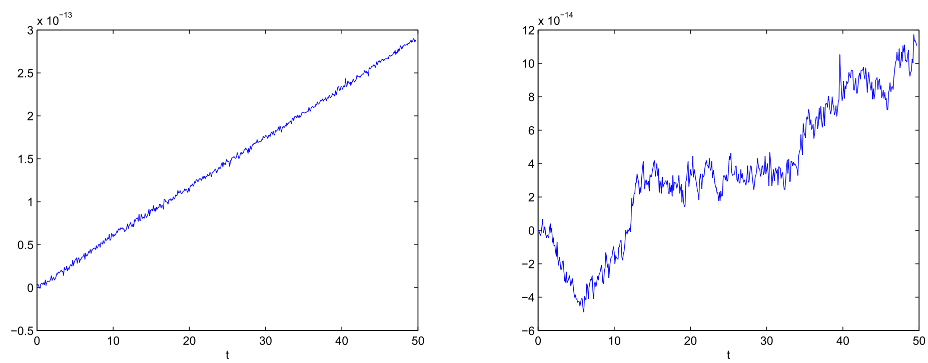

Figure 6 The errors of discrete mass and energy of the fractional quantum Zakharov system with $\alpha = 1.6$, $\beta = 1.7$ (Left:discrete mass; Right: discrete energy)

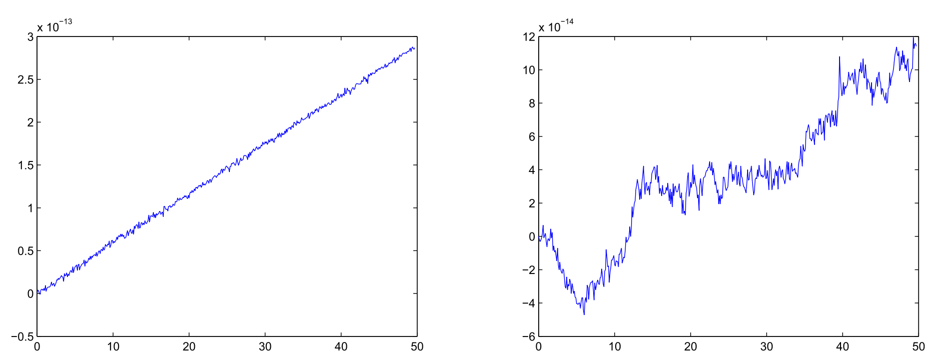

Figure 7 The errors of discrete mass and energy of the fractional quantum Zakharov system with $\alpha = 1.7$, $\beta = 1.8$ (Left:discrete mass; Right: discrete energy)

order 2 in time, and the Fourier method approach also achieves spectral convergence up to machine precision. In addition, we also show the convergence orders and errors by the numerical scheme in [39]. Figure 4 shows the numerical orders of time and space accuracy by the numerical scheme in [39] when $t = 1$. It clearly indicates that the proposed

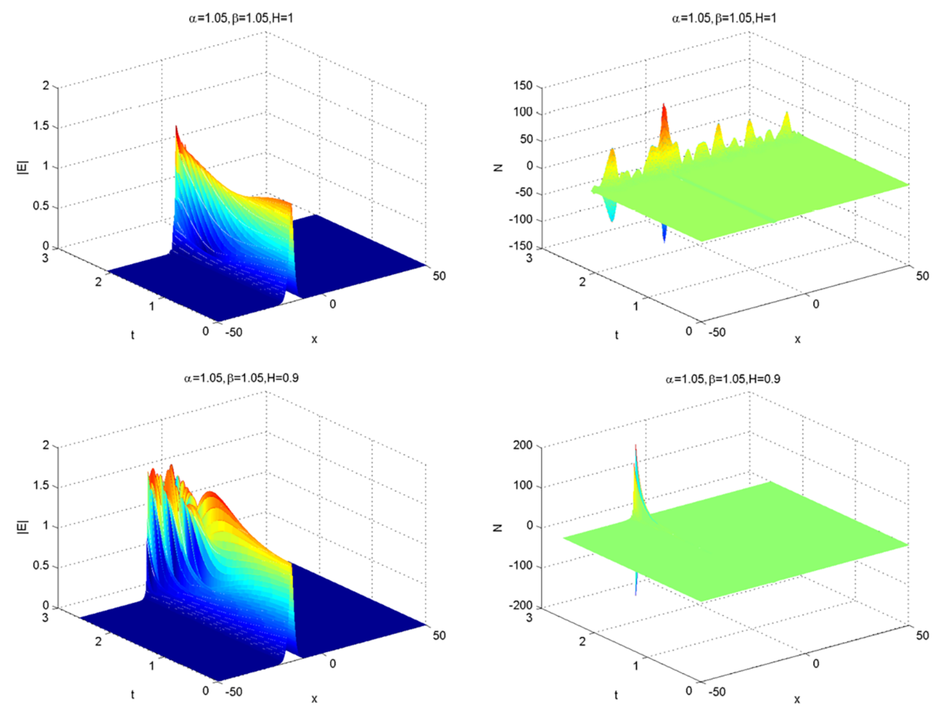

Figure 8 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

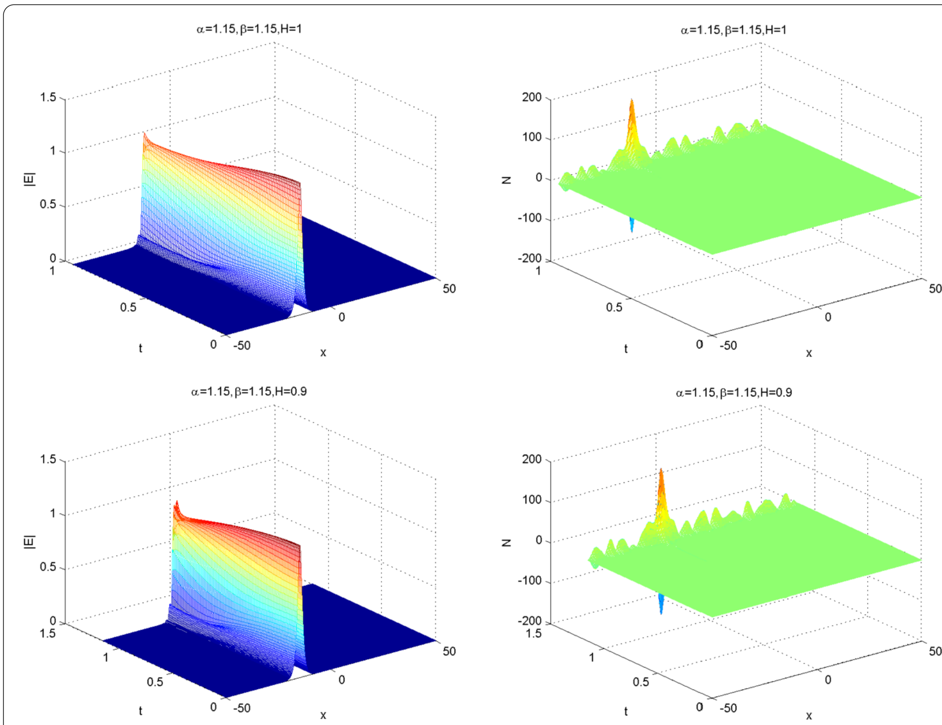

Figure 9 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

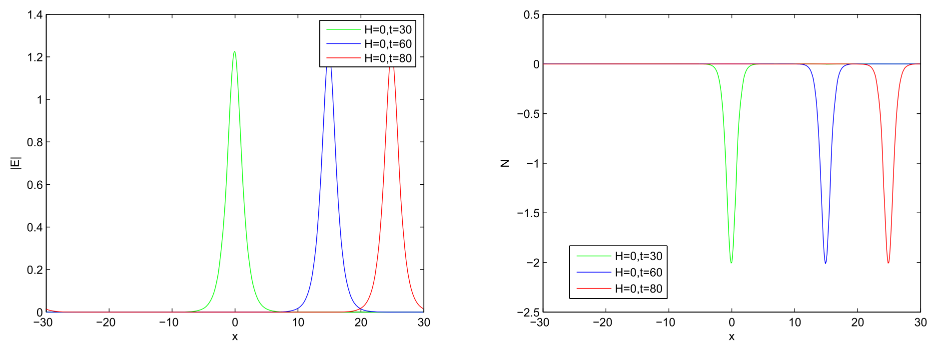

Figure 10 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

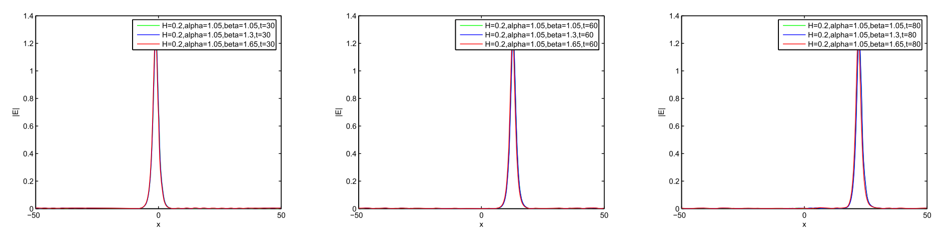

Figure 11 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

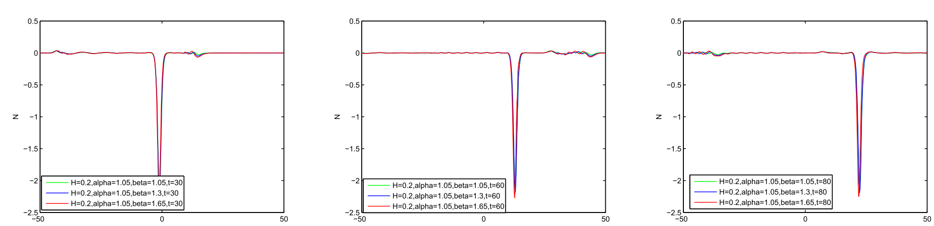

Figure 12 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

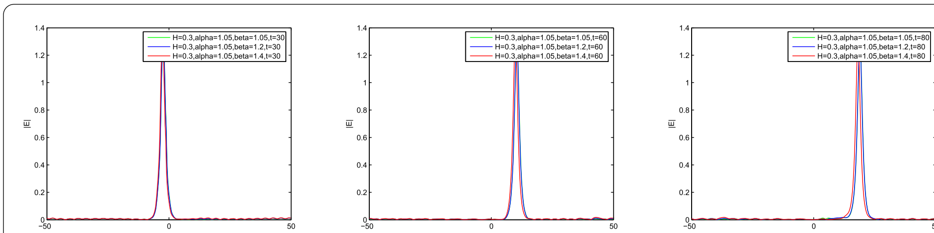

Figure 13 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

scheme is also of order 2 in time and space. Moreover, we show also CPU times with $\alpha = 1.4, 1.6$ by numerical scheme (9)–(13), numerical scheme (15)–(16), and the finite difference scheme in [39]. The results are listed in Table 9. From Table 9, it is found that numerical scheme (15)–(16) takes the least time and has the highest computational efficiency, followed by numerical scheme (9)–(13), and the finite difference scheme takes the

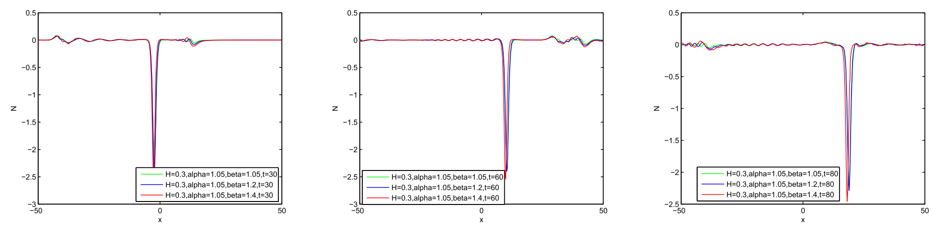

Figure 14 The module of numerical E and numerical N for the fractional quantum Zakharov system

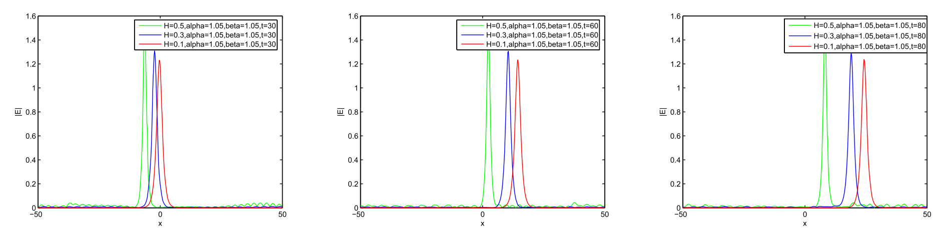

Figure 15 The module of numerical E and numerical N for the fractional quantum Zakharov system

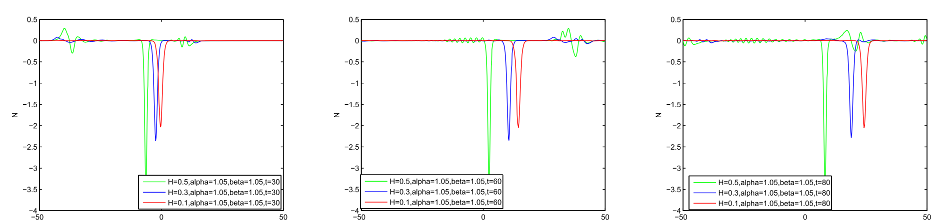

Figure 16 The module of numerical E and numerical N for the fractional quantum Zakharov system

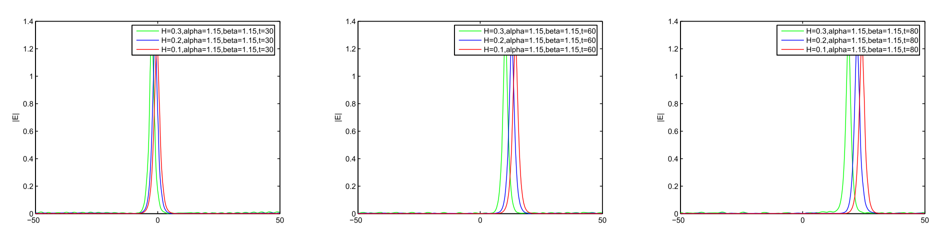

Figure 17 The module of numerical E and numerical N for the fractional quantum Zakharov system

longest time. Moreover, we also find that the calculation time is related to α. In a similar method, we can obtain corresponding conclusions for different fractional order β.

Then, we will test the conservative property of the conservative Fourier spectral method (9)–(13) for the fractional quantum Zakharov system. We choose the parameters $t \in [0, 50]$, $\tau = 0.001$, $M = 512$ different α, β. Figures 5–7 display the errors in the total mass M^n and energy E^n. It clearly indicates that the Fourier spectral scheme (9)–(13) preserves also the mass and energy conservation laws very well simultaneously for the quantum Zakharov system or the fractional quantum Zakharov system.

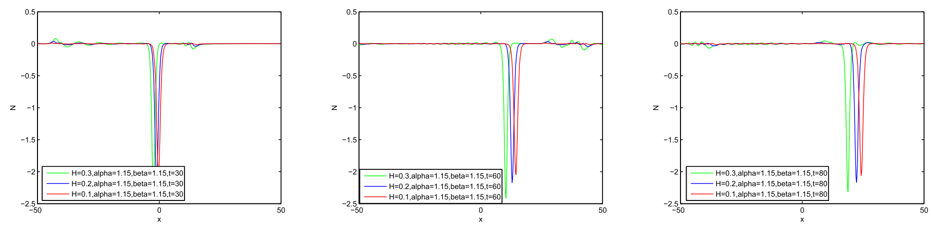

Figure 18 The module of numerical *E* and numerical *N* for the fractional quantum Zakharov system

Finally, the effects of the parameter H and fractional orders α, β on the solitary solution behaviors are investigated. We simulate the solitary wave solution with $t \in [0, 100]$, $\tau = 0.001$, $M = 512$. Figures 8–9 show the wave forms of numerical solutions $|E|$, N for different H, α, β. From Figs. 8–9, it is found that the numerical results indicate that the maximum of solution N increases faster with H, and the solution N will eventually blow up. Figures 10–18 show the wave forms of numerical solutions $|E|$, N for different H, α, β with the same time. From Figs. 10–18, it is found that some small oscillations appear on the two sides of solitary wave $|E|$ and N respectively. From Figs. 10–18, we find also that the values H will affect the propagation velocity of the solitary wave. When H becomes small, the propagation of the soliton will be quick.

7 Conclusion

In the paper, the Fourier spectral method for a class of modified Zakharov systems with high-order space fractional quantum correction is proposed. It is shown that this method preserves the discrete mass and energy conservation laws. The stability and convergence of the scheme are proved. Numerical tests are presented to demonstrate the theoretical results and the method availability, and to investigate the conservation property for different values of orders α and β. Moreover, the effects of the parameter H and fractional orders α, β on the solitary solution behaviors are also investigated numerically.

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Availability of data and materials
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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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