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Stability and dynamics of a stochastic discrete fractional-order chaotic system with short memory

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Abstract

In this paper, a stochastic discrete fractional-order chaotic system with short memory is proposed, which possesses two equilibrium points. With the help of the Lyapunov function theory, some sufficient conditions for the stability in probability of the two equilibrium points are given. Secondly, the effects of fractional order and memory steps on the stability of the system are discussed. Finally, the path dynamical behavior of the system is investigated using numerical methods such as Lyapunov exponents, bifurcation diagram, phase diagram, and 0–1 test. The numerical simulation results validate the findings.

Keywords: Stochastic discrete fractional-order chaotic system; Short-term memory effect; Lyapunov function; Asymptotically mean square stable

1 Introduction

Fractional-order mathematical models are often used as powerful tools to simulate real-world problems with memory effects, see [1–8] and so on. Recently, Wu, Baleanu, and Zeng [9] proposed a discrete fractional-order sine map and a discrete fractional-order standard map, and discussed their chaotic dynamics. Then Wu and Baleanu [10] studied delayed logistic maps that exist chaos. Later, Shukla and Sharma [11] extended a discrete fractional-order chaotic map to a fractional-order generalized hyperchaotic setting, which also has chaotic and hyperchaotic phenomenon. Moreover, Wu, Çankaya, and Banerjee [12] constructed a fractional-order q-deformed chaotic map employing a weight function approach. Furthermore, Khennaoui, Ouannas, Bendoukha, Grassi, Lozi, and Pham investigated chaos, stabilization, and synchronization in some fractional-order discrete-time systems [13], while Ran [14] studied chaos in a two-dimensional fractional-order chaotic map. Other studies on these topics can be seen in [15–20]and their references.

In addition, on the application of discrete fractional-order chaotic systems, Wu et al. [21] gave a review including some examples, for instance, an image encryption technique based on fractional-order chaotic time series. And Liu and Xia [22] proposed a novel two-dimensional fractional-order discrete chaotic map and studied image encryption by using the system. More applications of fractional-order systems have been described in [23, 24].

In fact, in modeling some practical problems, we only need short-term memory rather than full historical information. Therefore, the topic of fractional-order systems with short-term memory has attracted more and more attention. Dzielinski and Sierociuk [25] used a fractional-order operator to establish a short-term memory system and analyzed its local controllability. Later, Mozyrska and Pawłuszewiz [26] studied the stability of a discrete fractional-order state-space system in the Grünwald–Letnikov difference operator sense. In [27], Coll, Herrero, Ginestar, and Sánchez used a discrete fractional-order operator involving short-term memory to build an infectious disease system and obtained stability conditions of its equilibrium points. Very recently, Atici, Chang, and Jonnalagadda systematically summarized theoretical results for some fractional-order systems with long or short memory in [28].

However, systems in the world will inevitably be affected by external interference. In this paper, our purpose is to extend a deterministic discrete fractional-order chaotic system to a stochastic setting, which may find applications in economics, biology, and other fields. To explore the influence of uncertainty and memory, stability, and dynamical behavior are investigated using Lyapunov function theory and numerical methods.

This paper is structured as follows. Section 2 introduces some preliminaries. In Sect. 3, a stochastic fractional-order chaotic system is proposed. In Sect. 4, the stability of equilibrium points of the system is studied, and some numerical examples are given. In Sect. 5, the effects of fractional order and memory steps on the stability of the system are discussed. In Sect. 6, some path dynamical behaviors of the system are illustrated. Conclusions are summarized in Sect. 7.

2 Preliminary

Firstly, the definition of fractional-order Grünwald–Letnikov difference operator is presented below; see [25, 26].

Definition 1 For a function x on $Z = \{0, 1, \ldots\}$, the α-order Grünwald–Letnikov difference operator Δ^α is defined as

$$\Delta^\alpha x_t = \frac{1}{l^\alpha}\sum_{j=0}^{t}(-1)^j\binom{\alpha}{j}x_{t-jl}, \tag{1}$$

where $l \in (0, +\infty)$ is a sampling period, $\alpha \in (0, 1]$ denotes a fractional order, and the binomial coefficient $\binom{\alpha}{j}$ is computed by

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots, t. \end{cases} \tag{2}$$

Now as in [29–31], a stochastic difference system with state variable $x_i \in \mathbf{R}^n$, $i \in Z$ is introduced, which satisfies the equation

$$x_{i+1} = F(i, x_{-h}, \ldots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \ldots, x_j)\xi_j, \tag{3}$$

where h is a given non-negative integral number, the initial condition $x_i = \varphi_i$ for $i \in Z_h = \{-h, \ldots, 0\}$, $F : Z \times S \to \mathbf{R}^n$, $G : Z \times Z \times S \to \mathbf{R}^n$ (S is a space of sequences with elements in $\mathbf{R}^n$), and $\xi_j, j \in Z$ is a sequence of $\mathfrak{F}_j$-adapted random variables on a filtered probability space $(\Omega, \{\mathfrak{F}_j, j = 0, 1, 2, \ldots\}, \mathfrak{F}, \mathbf{P})$. Below are some stability concepts for the trivial solution, if it exists.

Definition 2 The trivial solution of the system (3) is called mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^2 < \varepsilon$, $i \in Z$ for any initial condition $\varphi = (\varphi_{-h}, \ldots, \varphi_0)$ satisfying $\|\varphi\|^2 = \sup_{i\in Z_h} \mathbf{E}|\varphi_i|^2 < \delta$; and it is called asymptotically mean square stable if $\lim_{i\to\infty} \mathbf{E}|x_i|^2 = 0$.

Definition 3 The trivial solution of the system (3) is called stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that

$$\mathbf{P}\left\{\sup_{i\in Z} |x_i| > \varepsilon | \mathfrak{F}_0\right\} < \varepsilon_1, \tag{4}$$

for any initial condition $\varphi = (\varphi_{-h}, \ldots, \varphi_0)$ satisfying

$$\mathbf{P}\left\{\max_{j\in Z_h} |\varphi_j| < \delta\right\} = 1. \tag{5}$$

Clearly, asymptotically mean square stable implies both mean square stable and stable in probability. The following two important theorems can be found in [29–31].

Theorem 1 *If there exists a non-negative functional $V_i = V(i, x_{-h}, \ldots, x_i)$ and two positive numbers c_1, c_2 such that the inequalities*

$$\begin{aligned} &\mathbf{E}V(0, \varphi_{-h}, \ldots, \varphi_0) \le c_1\|\varphi\|^2, \\ &\mathbf{E}\Delta V_i \le -c_2 \mathbf{E}|x_i|^2, \quad i \in Z, \end{aligned} \tag{6}$$

hold, then the trivial solution of (3) is asymptotically mean square stable.

Theorem 2 *If there exists a non-negative functional $V_{1i} = V(i, x_{-h}, \ldots, x_i)$, which satisfies the first condition of (6) and the inequalities*

$$\begin{aligned} &\mathbf{E}\Delta V_{1i} \le a\mathbf{E}|x_i|^2 + \sum_{k=-h}^{i} A_{ik}\mathbf{E}|x_k|^2, \quad i \in Z, A_{ik} \ge 0, \\ &a + b < 0, \quad b = \sup_{i\in Z} \sum_{j=i}^{\infty} A_{ji}, \end{aligned} \tag{7}$$

then the trivial solution of (3) is asymptotically mean square stable.

3 Model

It is well known that a classical one-dimensional discrete logistic chaotic model is defined by

$$x_{i+1} = ux_i(1 - x_i), \tag{8}$$

where $u \in \mathbf{R}$ is a parameter. Now taking the Grünwald–Letnikov difference operator Δ^α, we can obtain

$$\Delta^\alpha x_{i+1} = ux_i(1-x_i) - x_i. \tag{9}$$

For simplicity, let sampling period be $l = 1$, then the above fractional-order equation can be written as

$$x_{i+1} = (u-1+\alpha)x_i - ux_i^2 - \sum_{j=2}^{i+1}(-1)^j \binom{\alpha}{j} x_{i-j+1}. \tag{10}$$

An extension of the above deterministic equation in a stochastic environment can thus be constructed as

$$x_{i+1} = (u-1+\alpha)x_i - ux_i^2 - \sum_{j=2}^{i+1}(-1)^j \binom{\alpha}{j} x_{i-j+1} + \sigma\left(x_i - x^*\right)\xi_{i+1}, \tag{11}$$

where x^* is an equilibrium point of (10) (of course if and only if it is an equilibrium point of (11)), σ is a known constant, and ξ_i, $i \in Z$ is a sequence of mutually independent $\mathfrak{F}_i$-adapted normal random variables with mean $\mathbf{E}\xi_i = 0$ and variance $\mathbf{E}\xi_i^2 = 1$.

In this paper, similar to [29–31], we focus on a stochastic system with short memory or with truncation operator of k steps, that is, for $i \in Z$ and $k = 2,\dots,i+1$,

$$x_{i+1} = (u-1+\alpha)x_i - ux_i^2 - \sum_{j=2}^{k}(-1)^j \binom{\alpha}{j} x_{i-j+1} + \sigma\left(x_i - x^*\right)\xi_{i+1}. \tag{12}$$

The following proposition about equilibrium points for the above system is obvious, the proof is omitted.

Proposition 3 *The model* (12) *has two equilibrium points*:

$$x_1^* = 0, \quad u \in \mathbf{R}; \qquad x_2^* = \frac{u-1-A_k}{u}, \quad u \neq 0,$$

where $A_k = \sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}$.

4 Stability

Now, for $k = 1,2,\dots$, let us define a norm space $H_k = \{x | x = x_i : Z_k \cup Z \to \mathbf{R}, \sup_{i\in Z_k\cup Z} \mathbf{E}|x_i|^2 < \infty\}$, where $Z_k = \{-(k-1),\dots,0\}$. The stability of the two equilibrium points in Proposition 3 will be investigated later.

4.1 Stability of equilibrium points

Let us consider the equilibrium point $x^* = 0$ first. By putting $y_i = x_i - x^*$, the model (12) reduces to

$$y_{i+1} = (u-1+\alpha)y_i - uy_i^2 - \sum_{j=2}^{k}(-1)^j \binom{\alpha}{j} y_{i-j+1} + \sigma y_i \xi_{i+1}. \tag{13}$$

It is shown that the investigation of stability in probability of the above nonlinear stochastic difference equation can be reduced to the investigation of asymptotically mean square stability of its linear part:

$$y_{i+1} = (u - 1 + \alpha)y_i - \sum_{j=2}^{k}(-1)^j \binom{\alpha}{j} y_{i-j+1} + \sigma y_i \xi_{i+1}, \tag{14}$$

and applying the Lyapunov function method to study the stability of the trivial solution will be very useful [29].

Obviously, there is an auxiliary difference equation to the above stochastic system (14):

$$z_{i+1} = \sum_{j=1}^{k} b_j z_{i-j+1}, \tag{15}$$

where $b_1 = u - 1 + \alpha$ and $b_j = (-1)^{j+1}\binom{\alpha}{j}$ for $j \geq 2$. Taking the vector $z(i) = (z_{i-k+1}, \ldots, z_i)'$, we rewrite (15) in a matrix form

$$z(i + 1) = Bz(i), \tag{16}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ b_k & b_{k-1} & b_{k-2} & \cdots & b_2 & b_1 \end{pmatrix}. \tag{17}$$

Then some sufficient conditions to guarantee the stability of equilibrium can be stated below.

Theorem 4 *Assume that a positive semi-definite symmetric matrix $D = (d_{i,j})_{k\times k}$ with $d_{k,k} > 0$ and $d_{k,k}\sigma^2 - 1 < 0$ is a solution to the matrix equation*

$$B'DB - D = -M, \tag{18}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{19}$$

Then the trivial solution of (14) is asymptotically mean square stable, and the equilibrium point $x^* = 0$ of (12) is stable in probability.

Proof By the assumption, a Lyapunov function of (16) can be constructed as

$$v_i = z(i)'Dz(i). \tag{20}$$

In fact, we have $\Delta v_i = z(i+1)'Dz(i+1) - z(i)'Dz(i) = -z_i^2 \leq 0$.

Then a Lyapunov function of (14) is constructed as

$$V_i = V_{1i} = y(i)'Dy(i),$$

where

$$y(i+1) = By(i) + \eta(i) \tag{21}$$

with $y(i) = (y_{i-k+1}, \dots, y_i)'$ and $\eta(i) = (0, \dots, \sigma y_i \xi_{i+1})'$.

Then, we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\big[y(i+1)'Dy(i+1) - y(i)'Dy(i)\big] \\
&= \mathbf{E}\big[\big(By(i) + \eta(i)\big)'D\big(By(i) + \eta(i)\big) - y(i)'Dy(i)\big] \\
&= \mathbf{E}\big[-y_i^2 + \eta(i)'D\eta(i) + 2\eta(i)'DBy(i)\big] \\
&= -\mathbf{E}y_i^2 + \mathbf{E}\big[\eta(i)'D\eta(i)\big] + \mathbf{E}\big[\eta(i)'DAy(i)\big] + \mathbf{E}\big[y(i)'B'D\eta(i)\big].
\end{aligned} \tag{22}$$

Since

$$\mathbf{E}\big[\eta(i)'D\eta(i)\big] = \mathbf{E}\big[d_{kk}\sigma^2 y_i^2 \xi_{i+1}^2\big] \leq d_{kk}\sigma^2 \mathbf{E}y_i^2, \tag{23}$$

$$\mathbf{E}\big[\eta(i)'DBy(i)\big] = \mathbf{E}\big[y(i)'B'D\eta(i)\big] = 0, \tag{24}$$

so, $\mathbf{E}\Delta V_{1i} \leq (d_{kk}\sigma^2 - 1)\mathbf{E}y_i^2$. Therefore, via Theorem 1, we obtain a sufficient condition of asymptotically mean square stability $d_{kk}\sigma^2 - 1 < 0$ for (14). According to Remark 7.9 in [29], this condition is also a sufficient condition for the stability in probability of (12). The proof is completed. □

Repeating the same procedure above, we can obtain the following theorem about the constant equilibrium point $x^* = \frac{u-1-A_k}{u}$, with its proof omitted.

Theorem 5 *Assume that a positive semi-definite symmetric matrix* $D = (d_{i,j})_{k\times k}$ *with* $d_{k,k} > 0$ *and* $d_{k,k}\sigma^2 - 1 < 0$ *is a solution to the matrix equation*

$$C'DC - D = -M, \tag{25}$$

where M is the same matrix as in Theorem 4 and

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_2 & a_1 \end{pmatrix}, \tag{26}$$

with $a_1 = 3u - 3 + \alpha - 2A_k$ *and* $a_j = (-1)^{j+1}\binom{\alpha}{j}$ *for* $j \geq 2$. *Then the equilibrium point* $x^* = \frac{u-1-A_k}{u}$ *of* (12) *is stable in probability.*

4.2 Numerical examples

In this section we present simulations for model (12). Note that in the following three examples we still use x_i to denote the new variable after variable transformation.

Example 1 Consider the equilibrium point $x^* = 0$. Let $k = 2$, then the system (12) can be reduced as

$$x_{i+1} = (u - 1 + \alpha)x_i - ux_i^2 - \frac{\alpha(\alpha - 1)}{2}x_{i-1} + \sigma x_i \xi_{i+1}. \tag{27}$$

Then according to Theorem 4, the following result can be deduced.

Corollary 1 *When*

$$\begin{aligned} &\frac{\sigma^2(2 + \alpha^2 - \alpha)}{(2 - \alpha^2 + \alpha)[(\frac{2+\alpha^2-\alpha}{2})^2 - (u - 1 + \alpha)^2]} < 1, \\ &\left|\frac{\alpha^2 - \alpha}{2}\right| < 1 \quad \textit{and} \\ &|u - 1 + \alpha| < \frac{2 + \alpha^2 - \alpha}{2}, \end{aligned} \tag{28}$$

the zero equilibrium point of (27) *is stable in probability, equivalently, the equilibrium point* $x^* = 0$ *of* (12) *with* $k = 2$ *is stable in probability.*

Let $\alpha = 0.8$, $u = 0.3$, and $\sigma = 0.8$, then it is easy to verify that this special case satisfies the above conditions. The mean path along with the standard deviation (shaded area) [32] of 3×10^3 sample trajectories is presented in Fig. 1 and 500 sample trajectories are presented in Fig. 2.

Example 2 Consider the equilibrium point $x^* = \frac{u-1-A_k}{u}$. Let $k = 2$, then the system (12) turns to be the equation

$$x_{i+1} = (3u - 3 + \alpha - 2A_2)x_i - ux_i^2 - \frac{\alpha(\alpha - 1)}{2}x_{i-1} + \sigma x_i \xi_{i+1}. \tag{29}$$

From Theorem 5 follows the following result.

Corollary 2 *When*

$$\begin{aligned} &\frac{\sigma^2(2 + \alpha^2 - \alpha)}{(2 - \alpha^2 + \alpha)[(\frac{2+\alpha^2-\alpha}{2})^2 - (3u - 3 + \alpha - 2A_2)^2]} < 1, \\ &\left|\frac{\alpha^2 - \alpha}{2}\right| < 1 \quad \textit{and} \\ &|3u - 3 + \alpha - 2A_2| < \frac{2 + \alpha^2 - \alpha}{2}, \end{aligned} \tag{30}$$

the zero equilibrium point of (29) *is stable in probability, equivalently, the equilibrium point* $x^* = \frac{u-1-A_2}{u}$ *of* (12) *with* $k = 2$ *is stable in probability.*

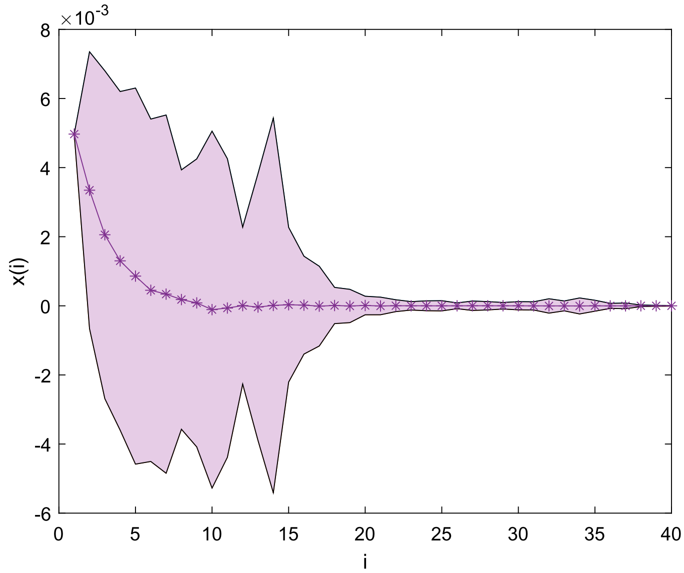

Figure 1 The mean along with standard deviation of 3×10^3 sample trajectories of (27) with $\alpha = 0.8$, $u = 0.3$, $\sigma = 0.8$

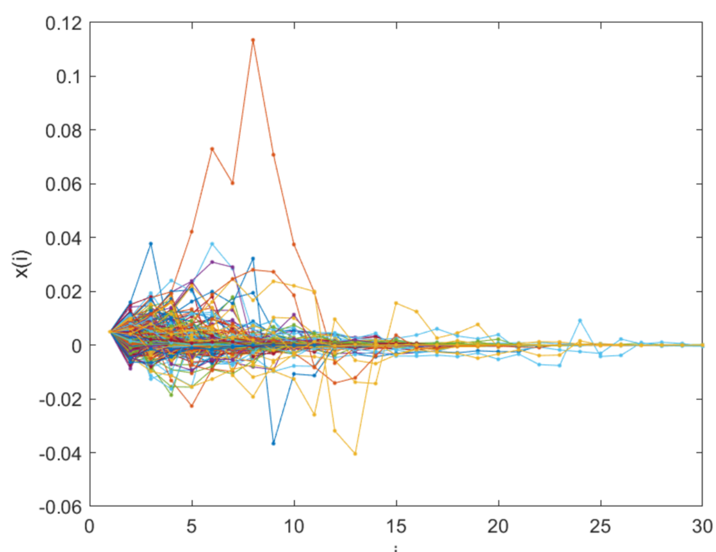

Figure 2 The sample trajectories of (27) with $\alpha = 0.8$, $u = 0.3$, $\sigma = 0.8$

Then the case with $\alpha = 0.8$, $u = 1$, and $\sigma = 0.6$ satisfies the above conditions. The mean path together with the standard deviation (shaded area) of 3×10^3 sample trajectories are plotted in Fig. 3 and 500 sample trajectories are plotted in Fig. 4.

Example 3 Consider the equilibrium point $x^* = 0$. Let $k = 3$, then the system (12) turns to be the equation

$$x_{i+1} = (u - 1 + \alpha)x_i - ux_i^2 - \frac{\alpha(\alpha - 1)}{2}x_{i-1} + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6}x_{i-2} + \sigma x_i \xi_{i+1}. \tag{31}$$

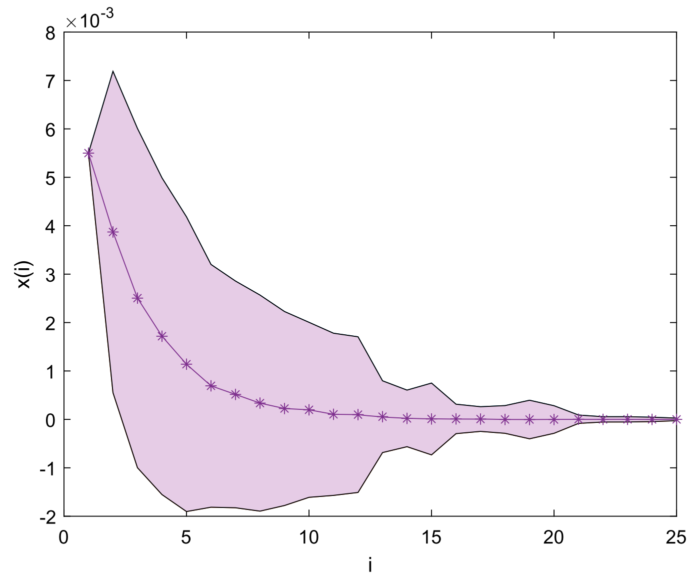

Figure 3 The mean along with standard deviation of 3×10^3 sample trajectories of (29) with $\alpha = 0.8$, $u = 1$, $\sigma = 0.6$

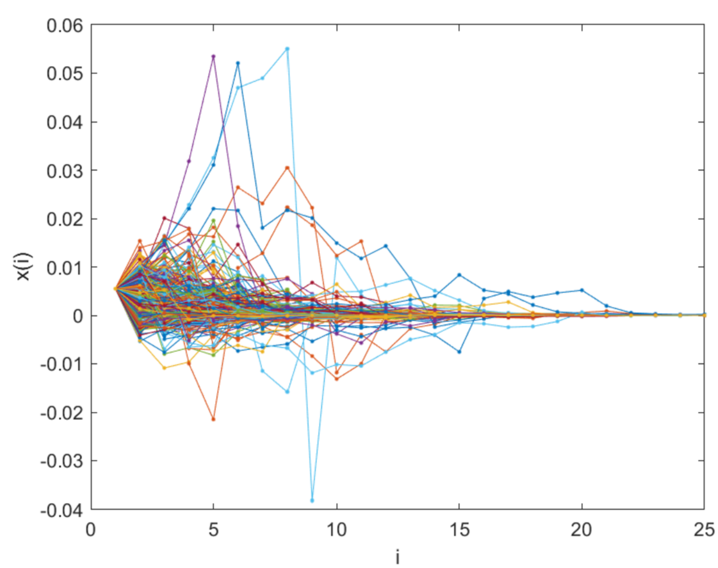

Figure 4 The sample trajectories of (29) with $\alpha = 0.8$, $u = 1$, $\sigma = 0.6$

By taking parameters α and u the same as those in Example 1, respectively, via (17), we have

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.032 & 0.08 & 0.1 \end{pmatrix}. \tag{32}$$

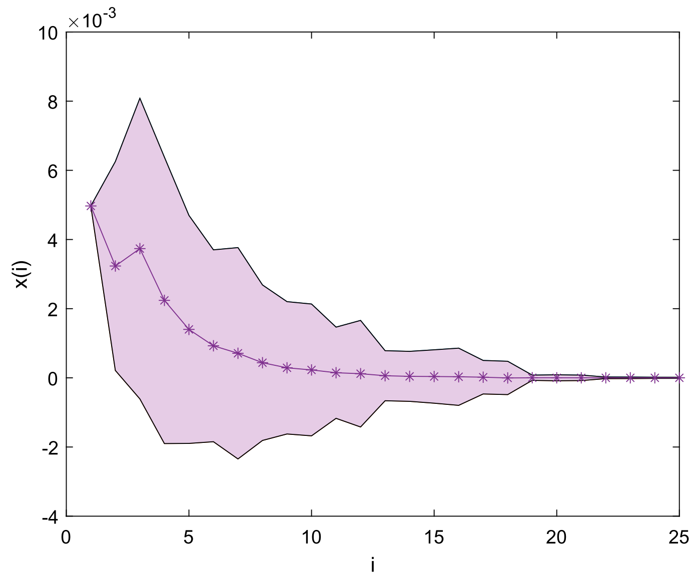

Figure 5 The mean along with standard deviation of 3×10^3 sample trajectories of (31) with $\alpha = 0.8$, $u = 0.3$, $\sigma = 0.6$

Then the matrix

$$D = \begin{pmatrix} 0.1189 & 0.0487 & 0.0594 \\ 0.0487 & 0.1359 & 0.0695 \\ 0.0594 & 0.0695 & 1.1614 \end{pmatrix} \tag{33}$$

solves the equation (18). So it follows from Theorem 4 that the following corollary holds.

Corollary 3 *When $\sigma^2 < \frac{1}{1.1614}$, the zero equilibrium point of (31) is stable in probability, equivalently, the equilibrium point $x^* = 0$ of (12) with $k = 3$ is stable in probability.*

Clearly, $\sigma = 0.6$ satisfies the above condition. The mean path together with the standard deviation (shaded area) of 3×10^3 sample trajectories are displayed in Fig. 5 and 500 sample trajectories are displayed in Fig. 6.

In view of the above discussion, we find that the stability of equilibrium points is determined by the value of parameters σ, α, u, k, and the truncation steps k, and fractional order α play important roles.

5 Effects of fractional order and memory steps on stability

In this section, we will use numerical method to study briefly effects of fractional order α and memory steps k on the region of stability in the system (13).

Assume that $\alpha = 0.8$, $u = 0.3$, and M is the same as that in Theorem 4. The stability conditions are shown in Table 1 for memory steps $k = 2 \sim 5$, respectively.

Assume that $k = 2$, $u = 1$, and M is the same as that in Theorem 4. The stability conditions are shown in Table 2 for fractional order $\alpha = 0.2 \sim 0.9$, respectively.

As illustrated above, the stability zone appears to shrink as the number of memory steps k rises, and the same is true for the fractional order α.

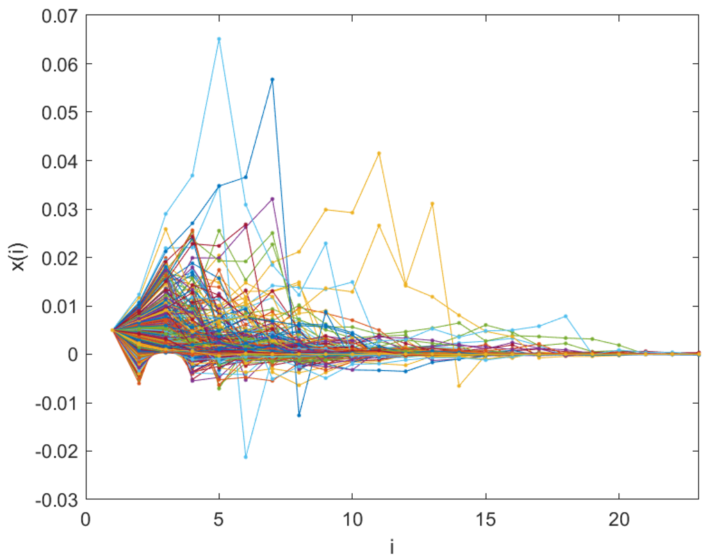

Figure 6 The sample trajectories of (31) with $\alpha = 0.8$, $u = 0.3$, $\sigma = 0.6$

Table 1 Stability condition with different values of k

memory steps k	2	3	4	5
stability condition	$\sigma^2 < \frac{1}{1.0185}$	$\sigma^2 < \frac{1}{1.1614}$	$\sigma^2 < \frac{1}{1.1680}$	$\sigma^2 < \frac{1}{1.1720}$

Table 2 Stability condition with different values of α

fractional order α	0.2 0.6	0.3 0.7	0.4 0.8	0.5 0.9
stability condition	$\sigma^2 < \frac{1}{1.0564}$ $\sigma^2 < \frac{1}{1.8960}$	$\sigma^2 < \frac{1}{1.1391}$ $\sigma^2 < \frac{1}{2.6041}$	$\sigma^2 < \frac{1}{1.2788}$ $\sigma^2 < \frac{1}{4.1272}$	$\sigma^2 < \frac{1}{1.5084}$ $\sigma^2 < \frac{1}{8.9574}$

6 Dynamics

In this section, we will study path dynamics of the system (13) by means of numerical methods, i.e., Lyapunov exponents, bifurcation diagram, and 0–1 test.

Consider a stochastic discrete-time dynamical system

$$X_{i+1} = f(X_i, \xi_i), \tag{34}$$

where $X_i \in R^n$ is state variable, $f: R^n \to R^n$ is a vector function and ξ_i is a process. It follows from linearization of (34) that

$$X_{i+1} = J(X_i, \xi_i), = J(i)X_i, \tag{35}$$

where $J(X_i, \xi_i)$ is the Jacobian matrix of (34), which implies

$$X_{i+1} = J(i) \cdots J(1)J(0)X_0. \tag{36}$$

As a result, we can establish Lyapunov exponents of (34) using products of random matrices. Next, we will study the path dynamics of the system (13) in the sense of probability.

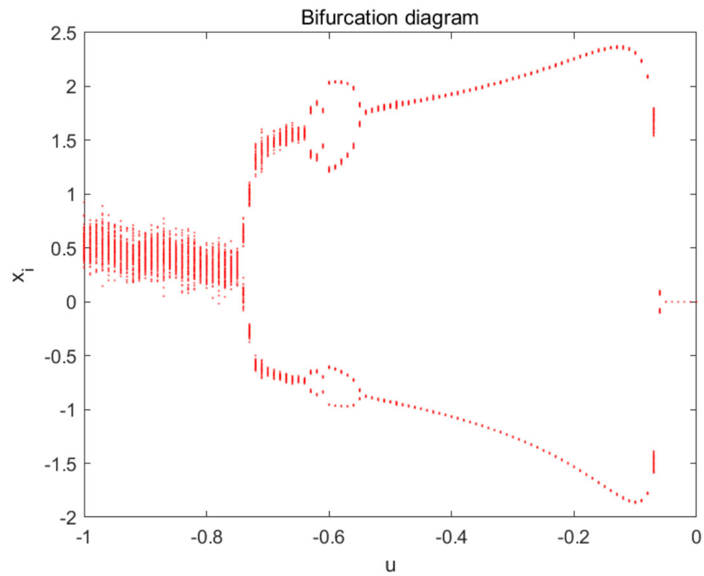

Figure 7 Bifurcation diagram of (13) versus $u \in [-1, 0]$ with $\alpha = 0.1$, $\sigma = 0.01$

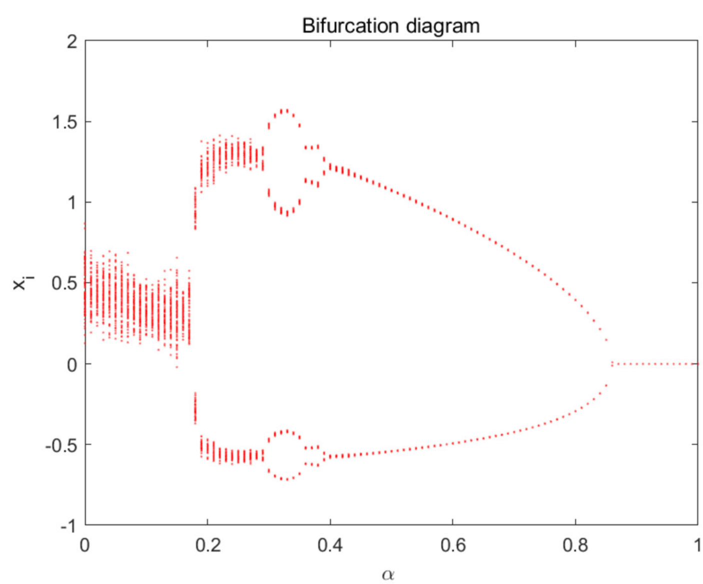

Figure 8 Bifurcation diagram of (13) versus $\alpha \in (0, 1]$ with $u = -0.8$, $\sigma = 0.01$

Firstly, consider u as a bifurcation parameter. Fix $k = 2$, $\alpha = 0.1$, and $\sigma = 0.01$ and take the mean of 10^2 sample trajectories of the system (13), then we can obtain the bifurcation diagram versus $u \in [-1, 1]$ as shown in Fig. 7.

Next, similar to the above procedure, the bifurcation diagram versus $\alpha \in (0, 1]$ is shown in Fig. 8, where $k = 2$, $u = -0.8$, and $\sigma = 0.01$.

Here a concrete example is taken to demonstrate periodic and chaotic dynamics in (13) with $k = 2$. Let $u = -0.8$, $\sigma = 0.01$, then the time history diagrams and 0–1 test results for $\alpha = 0.15$ and $\alpha = 0.6$ are shown in Fig. 9 and Fig. 10, respectively (the algorithm of 0–1

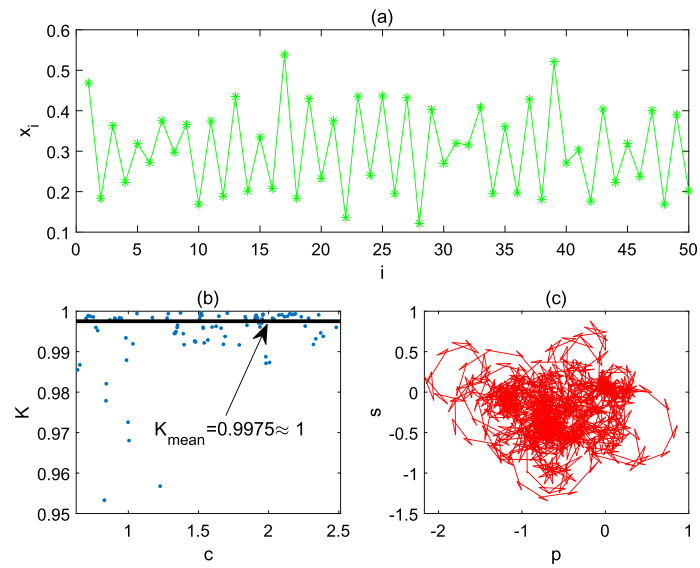

Figure 9 Numerical results for $u = -0.8$, $\alpha = 0.15$, and $\sigma = 0.01$. (a) Time history diagram, (b) K value, (c) p–s plot

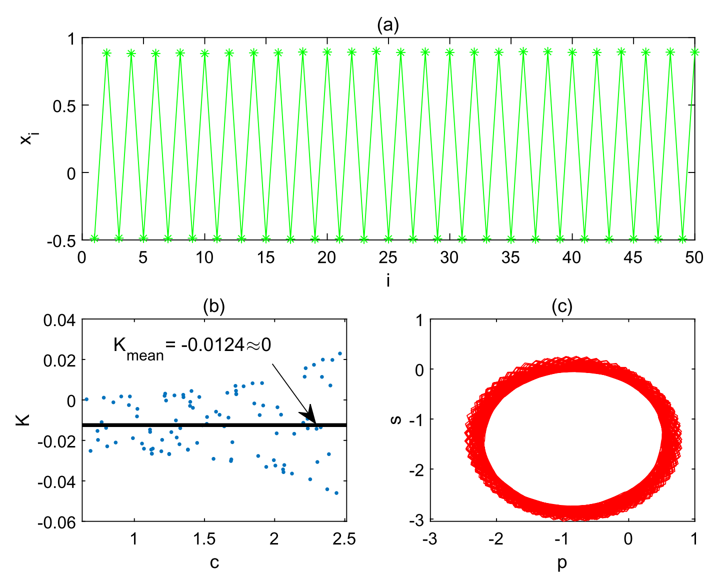

Figure 10 Numerical results for $u = -0.8$, $\alpha = 0.6$, and $\sigma = 0.01$. (a) Time history diagram, (b) K value, (c) p–s plot

test is omitted here; for details, please refer to the [33, 34]). Furthermore, by the use of QR decomposition, the corresponding Lyapunov exponents are obtained, as shown in Fig. 11 and Fig. 12, respectively.

The above numerical study shows that the system exhibits a rich dynamical behavior, and contains chaotic solutions and periodic solutions, which verifies once again that truncation steps k and fractional order α play an important role in the system.

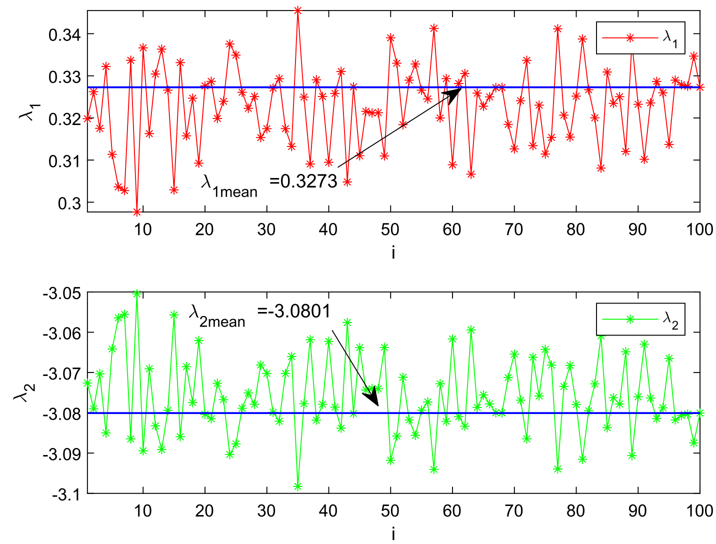

Figure 11 Lyapunov exponents for $u = -0.8$, $\alpha = 0.15$, and $\sigma = 0.01$

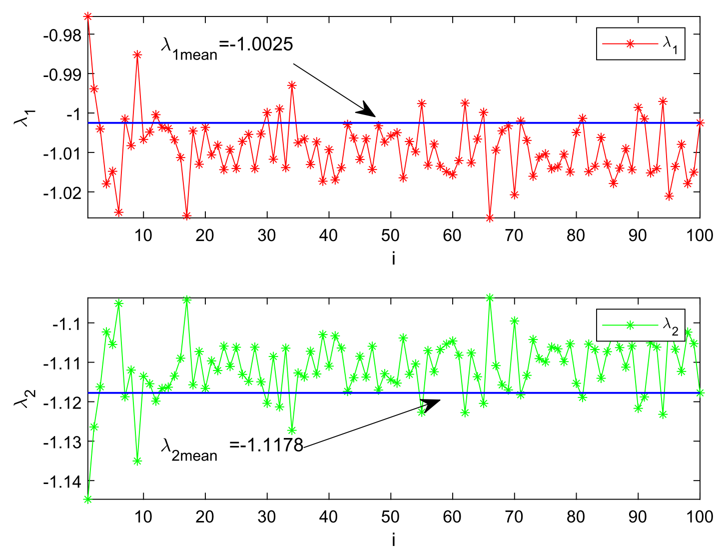

Figure 12 Lyapunov exponents for $u = -0.8$, $\alpha = 0.6$, and $\sigma = 0.01$

7 Discussion and conclusion

In this paper, a stochastic discrete fractional-order chaotic system with short memory is proposed, which possesses two equilibrium points. By the use of the Lyapunov function theory, the stability conditions of equilibrium points in the sense of probability are obtained. It shows that the stability region seems to decrease as memory step k increases or as fractional-order α increases. Unfortunately, however, we have only observed the phenomenon from the point of view of numerical methods, and the theoretical study is still an open question. In addition, the Lyapunov exponents are captured by using the QR decomposition algorithm. The study shows that the system exhibits rich dynamical behavior in the sense of paths, such as chaotic solutions and periodic solutions. From a modeling point of view, these results lead us to believe that the stochastic fractional-order version has better application prospects. We hope that our results will be helpful in exploring the modeling and dynamical behavior of discrete fractional-order systems.

Based on our work, there are some extensions to investigate more information for the discrete fractional-order chaotic system with stochastic perturbations, such as the Caputo fractional-order chaotic system or system under multiple stochastic disturbances, which will be studied in our future work.

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Declarations

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The authors declare no competing interests.

Author contributions
JR: writing, original draft preparation, methodology; JR, JQ, and YZ: review and editing; JQ: visualization; YZ: conceptualization. All authors read and approved the final manuscript.

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