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An analytical approach to the pricing of an exchange option with default risk under a stochastic volatility model

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Abstract

The exchange option, which has two correlated underlying assets, is one of the most popular exotic options in the over-the-counter markets. This paper studies the valuation of exchange options with default risk of option issuer, where default is allowed only at maturity. Moreover, we consider three underlying assets with stochastic volatilities and assume that fast mean-reverting processes determine the stochastic volatilities. Based on the partial differential equation approach, we derive the analytical pricing formula of the exchange option price with default risk using the asymptotic expansion. To verify the accuracy and efficiency of our pricing formula, we compare the results by our pricing formula with those by Monte Carlo simulation, which is considered a benchmark. In addition, we provide several graphs to illustrate the properties of the option for significant parameters.

Keywords: Exchange option; Default risk; Stochastic volatility; Asymptotic expansion

1 Introduction

The exchange option, derived from the difference between the values of two underlying assets, is one of the most popular exotic options in the over-the-counter (OTC) market. Exchange option that offers the option holder the right to exchange one risky asset for another at maturity was first introduced by Margrabe [1]. Thus, it is occasionally called the Margrabe option. Since Margrabe provided a closed-form pricing formula under the Black-Scholes model [2], there have been various extensions of the exchange option pricing model such as stochastic volatility [3–6], jump-diffusion [7–9], fractional Brownian motion [10], and stochastic correlation [11].

Since default risk exists for option issuers in the OTC market, credit risk must be considered when pricing options in the over-the-counter market. The option with default risk is called a vulnerable option. The valuation of a vulnerable option has been studied with two approaches: the reduced-form and structural model. Under the reduced-form model, credit events are determined by the counting process with some intensities. On the other hand, credit events under the structural model are determined by the relation between the firm value and the option issuer's value. Based on these models, various studies have been on the valuation of vulnerable options.

The pricing of vulnerable option under the reduced-form model has been studied by many researchers. Fard [12] provided an analytical pricing solution of vulnerable option using the Esscher–Girsanov transform when the underlying asset follows a generalized jump–diffusion model. Wang [13] dealt with the valuation of vulnerable option under the generalized autoregressive conditional heteroskedasticity (GARCH) model which is one of the discrete models. Koo and Kim [14] derived an explicit pricing formula of the catastrophe option with credit risk based on the reduced-form model of Fard. More recently, Pasricha and Goel [15] studied the valuation of European power exchange option in a reduced-form model. They assume that two underlying assets have the correlated jump-diffusion processes.

Structural model also has been used a lot for pricing of vulnerable option by many researchers. Johnson and Stulz [16] first studied on vulnerable option pricing under the structural model. Klein [17] improved the model of [16] considering the correlated default risk. Klein and Inglis [18, 19] considered stochastic interest rate and a default boundary, which depends on the liabilities of option issuer and option itself when vulnerable option is priced. Zhou et al. [20] extended the pricing model of Klein [17] with a variable default boundary based on the option's potential debt and the option writer's other liabilities. These results were based on the probabilistic approaches similar to vulnerable option valuation under the reduced-form model. However, unlike the reduced-form model, partial differential equation (PDE) approach has been also widely used for valuing vulnerable options under the structural model. The PDEs for vulnerable options have been mainly solved using Mellin transforms, and many researchers have developed the pricing models of vulnerable options (for instance, see [21–27]). In this paper, we also develop the vulnerable option pricing model based on PDE approach. More specifically, we study the valuation of vulnerable exchange option and extend the pricing model for the option with a stochastic volatility model.

Stochastic volatility models have been developed to overcome the limitation of the Black-Scholes model. Stochastic volatility models present the time-varying volatilities and explain implied volatility patterns that arise in real option markets unlike the Black-Scholes model. In fact, there have been many kinds of stochastic volatility models to describe the phenomena in financial option markets, and there exist many studies on vulnerable option pricing based on the stochastic volatility models. Yang et al. [28] first considered a stochastic volatility model for vulnerable option pricing. They chose a stochastic volatility model with fast mean-reversion introduced by Fouque et al. [29] and derived asymptotic expansion formula of European vulnerable option price. Wang et al. [30] investigated the pricing of vulnerable option under a stochastic volatility which has the short-term fluctuation with a mean-reverting process. In addition, Lee and Kim [31] dealt with a multiscale generalized Heston's stochastic volatility model for the pricing of defaultable options. Wang [32] proposed a stochastic volatility model which captures leverage effects and stochastic correlation and obtained an analytic pricing formula of Asian option with counterparty risk under the proposed model. Ma et al. [33] considered the GARCH diffusion model as the stochastic volatility model for pricing of the vulnerable European option, and calculated the price using the fast Fourier transform (FFT) algorithm. We also study vulnerable option pricing with a stochastic volatility model. Specifically, we derive the analytical pricing formula of exchange option with default risk based on the stochastic model of [29] and verify our formula with some numerical results.

The contribution of this work is to find the pricing formula of exchange option with default risk under the stochastic volatility. To present our results, this paper is structured as follows. In Sect. 2, we introduce the model of a stochastic volatility model for vulnerable exchange option and derived the pricing formula of vulnerable exchange option under the proposed model. In Sect. 3, we provide some numerical results to examine the features of the vulnerable exchange option with a stochastic volatility model. In Sect. 4, we provide the concluding remarks.

2 Model and option pricing

In this section, we investigate an asymptotic analysis approach to price the vulnerable exchange option under a stochastic volatility model. Specifically, we introduce the model for the vulnerable exchange option with a stochastic volatility model and derive an analytical pricing formula for the option using an asymptotic approach.

Let S_t^1 and S_t^2 be underlying assets and V_t be an asset value process of the option issuer. We assume that the stochastic differential equations for the processes S_t^1, S_t^2, and V_t are as follows.

$$
\begin{aligned}
dS_t^1 &= rS_t^1\,dt + f_1\big(Y_t^1\big)S_t\,dW_t^1,\\
dY_t^1 &= \left[\frac{k_1}{\epsilon}\big(m_1 - Y_t^1\big) - \frac{\sqrt{2}\nu_1}{\sqrt{\epsilon}}\Lambda_1\big(Y_t^1\big)\right]dt + \frac{\sqrt{2}\nu_1}{\sqrt{\epsilon}}\,dZ_t^1,\\
dS_t^2 &= rS_t^2\,dt + f_2\big(Y_t^2\big)S_t^2\,dW_t^2,\\
dY_t^2 &= \left[\frac{k_2}{\epsilon}\big(m_2 - Y_t^2\big) - \frac{\sqrt{2}\nu_2}{\sqrt{\epsilon}}\Lambda_2\big(Y_t^2\big)\right]dt + \frac{\sqrt{2}\nu_2}{\sqrt{\epsilon}}\,dZ_t^2,\\
dV_t &= rV_t\,dt + f_v\big(Y_t^v\big)V_t\,dW_t^v,\\
dY_t^v &= \left[\frac{k_v}{\epsilon}\big(m_v - Y_t^v\big) - \frac{\sqrt{2}\nu_v}{\sqrt{\epsilon}}\Lambda_v\big(Y_t^v\big)\right]dt + \frac{\sqrt{2}\nu_v}{\sqrt{\epsilon}}\,dZ_t^v,
\end{aligned}
\tag{1}
$$

where r is the risk-free interest rate, Λ_i $(i = 1, 2, v)$ are the market prices of volatility risk and W_t^i and Z_t^i for $i = 1, 2, v$ are the standard Brownian motions under the risk-neutral measure Q. Here, The Ornstein–Uhlenbeck (OU) processes Y_t^1, Y_t^2, and Y_t^v mean the volatility driven process for S_t^1, S_t^2, and V_t, respectively, and are set to reflect the fast mean reverting volatility environment. The correlation structures of the Brownian motions are set up as follows.

$$
\begin{aligned}
&dW_t^1 W_t^2 = \rho_{12}\,dt, \qquad dW_t^1\,dW_t^v = \rho_{1v}\,dt, \qquad dW_t^2\,dW_t^v = \rho_{2v}\,dt,\\
&dW_t^1\,dZ_t^1 = \eta_1\,dt, \qquad dW_t^2\,dZ_t^2 = \eta_2\,dt, \qquad dW_t^v\,dZ_t^v = \eta_v\,dt.
\end{aligned}
\tag{2}
$$

Then, the price of vulnerable exchange option under the measure Q is given by

$$
\begin{aligned}
&P^\epsilon(t, s_1, s_2, v, y_1, y_2, y_v)\\
&\quad = \mathrm{E}^Q\big[e^{-r(T-t)}h\big(S_T^1, S_T^2, V_T\big)|S_t^1 = s_1, S_t^2 = s_2, V_t = v, Y_t^1 = y_1, Y_t^2 = y_2, Y_t^v = y_v\big],
\end{aligned}
$$

where h is the payoff function of the vulnerable exchange option, which is defined by

$$h\left(S_T^1, S_T^2, V_T\right) = \left(S_T^1 - S_T^2\right)^+ \left(1_{\{V_T \geq D^*\}} + 1_{\{V_T < D^*\}} \frac{(1-\alpha)V_T}{D}\right).$$

The default of option issuer occurs if asset value V_T is less than the default level D^* at maturity T and recovery rate is set to $(1-\alpha)V_T/D$, where α is the deadweight cost of bankruptcy and D is the expected value of the option issuer's total liability at maturity. From the Feynman-Kac formula, the price P^ϵ satisfies the following PDE:

$$\begin{aligned} &\mathcal{L}^\epsilon P^\epsilon(t, s_1, s_2, v, y_1, y_2, y_v) = 0, \\ &P^\epsilon(T, s_1, s_2, v, y_1, y_2, y_v) = h(s_1, s_2, v), \end{aligned} \tag{3}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

and

$$\begin{aligned} \mathcal{L}_0 &= k_1(m_1 - y_1)\frac{\partial}{\partial y_1} + k_2(m_2 - y_2)\frac{\partial}{\partial y_2} + k_v(m_v - y_v)\frac{\partial}{\partial y_v} + v_1^2\frac{\partial^2}{\partial y_1^2} + v_2^2\frac{\partial^2}{\partial y_2^2} + v_v^2\frac{\partial^2}{\partial y_v^2}, \\ \mathcal{L}_1 &= -\sqrt{2}v_1\Lambda_1(y_1)\frac{\partial}{\partial y_1} - \sqrt{2}v_2\Lambda_2(y_2)\frac{\partial}{\partial y_2} - \sqrt{2}v_v\Lambda_v(y_v)\frac{\partial}{\partial y_v} \\ &\quad + \sqrt{2}v_1\eta_1 f_1(y_1)s_1\frac{\partial^2}{\partial s_1 \partial y_1} + \sqrt{2}v_2\eta_2 f_2(y_2)s_2\frac{\partial^2}{\partial s_2 \partial y_2} + \sqrt{2}v_v\eta_v f_v(y_v)s_v\frac{\partial^2}{\partial s_v \partial y_v}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + r\left(s_1\frac{\partial}{\partial s_1} + s_2\frac{\partial}{\partial s_2} + v\frac{\partial}{\partial v} - \cdot\right) + \frac{1}{2}f_1^2(y_1)s_1^2\frac{\partial^2}{\partial s_1^2} \\ &\quad + \frac{1}{2}f_2^2(y_2)s_2^2\frac{\partial^2}{\partial s_2^2} + \frac{1}{2}f_v^2(y_v)v^2\frac{\partial^2}{\partial v^2} \\ &\quad + \rho_{12}f_1(y_1)f_2(y_2)s_1s_2\frac{\partial^2}{\partial s_1 \partial s_2} + \rho_{1v}f_1(y_1)f_v(y_v)s_1v\frac{\partial^2}{\partial s_1 \partial v} + \rho_{2v}f_2(y_2)f_v(y_v)s_2v\frac{\partial^2}{\partial s_2 \partial v}. \end{aligned}$$

To obtain a pricing formula for the option by using the asymptotic approach in [29], we first expand P in power of $\sqrt{\epsilon}$, $P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \cdots$. Then we have

$$\begin{aligned} &\frac{1}{\epsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0P_1 + \mathcal{L}_1P_0) + (\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0) \\ &\quad + \sqrt{\epsilon}(\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1) + \cdots = 0. \end{aligned} \tag{4}$$

In order to derive a solution for the equation $\mathcal{L}_0P_0 = 0$ in the $\frac{1}{\epsilon}$-order term in (4), P_0 should be independent of y_1, y_2, and y_v, in other words, $P_0 = P_0(t, s_1, s_2, v)$. Inserting P_0 to the $\frac{1}{\sqrt{\epsilon}}$-order term in (4), P_1 is also independent of y_1, y_2, and y_v similar to P_0. We then have the Poisson equation

$$\mathcal{L}_0P_2 + \mathcal{L}_2P_0 = 0 \tag{5}$$

in zero-order term, which must hold its solvability condition $\langle \mathcal{L}_2 P_0 \rangle = 0$. Here, $\langle \cdot \rangle$ denotes the expectation with respect to the invariant distribution Φ of the three-dimensional processes (Y_t^1, Y_t^2, Y_t^ν). From the independence assumption, the invariant distribution Φ is defined as the product of each invariant distribution of Y_t^1, Y_t^2 and Y_t^ν. That is,

$$\Phi(y_1, y_2, y_\nu) = \prod_{i=1,2,\nu} \frac{1}{\sqrt{2\pi \tilde{\nu}_i^2}} \exp\left\{ -\frac{(y_i - m_i)^2}{2\tilde{\nu}_i^2} \right\},$$

where $\tilde{\nu}_i^2 = \frac{\nu_i^2}{k_i}$. Therefore, P_0 satisfies the following PDE:

$$\begin{aligned} &\langle \mathcal{L}_2 \rangle P_0(t, s_1, s_2, \nu) = 0, \\ &P_0(T, s_1, s_2, \nu) = (s_1 - s_2)^+ \left(1_{\{\nu \geq D^*\}} + 1_{\{\nu < D^*\}} \frac{(1-\alpha)\nu}{D} \right), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \langle \mathcal{L}_2 \rangle = {}& \frac{\partial}{\partial t} + r(\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_\nu - \cdot) + \frac{1}{2}\bar{\sigma}_1^2 \mathcal{D}_1^2 + \frac{1}{2}\bar{\sigma}_2^2 \mathcal{D}_2^2 + \frac{1}{2}\bar{\sigma}_\nu^2 \mathcal{D}_\nu^2 \\ & + \bar{\rho}_{12}\bar{\sigma}_1\bar{\sigma}_2 \mathcal{D}_1\mathcal{D}_2 + \bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu \mathcal{D}_1\mathcal{D}_\nu + \bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu \mathcal{D}_2\mathcal{D}_\nu. \end{aligned}$$

and $\mathcal{D}_i^n = s_i^n \frac{\partial^n}{\partial s_i^n}$ for $i = 1, 2$, $\mathcal{D}_\nu^n = \nu^n \frac{\partial^n}{\partial \nu^n}$,

$$\begin{aligned} \bar{\sigma}_1 &= \sqrt{\langle f_1^2(y_1) \rangle}, \\ \bar{\sigma}_2 &= \sqrt{\langle f_2^2(y_2) \rangle}, \\ \bar{\sigma}_\nu &= \sqrt{\langle f_\nu^2(y_\nu) \rangle}, \\ \bar{\rho}_{12} &= \frac{\rho_{12} \langle f_1(y_1) f_2(y_2) \rangle}{\bar{\sigma}_1 \bar{\sigma}_2}, \\ \bar{\rho}_{1\nu} &= \frac{\rho_{1\nu} \langle f_1(y_1) f_\nu(y_\nu) \rangle}{\bar{\sigma}_1 \bar{\sigma}_\nu}, \\ \bar{\rho}_{2\nu} &= \frac{\rho_{2\nu} \langle f_2(y_2) f_\nu(y_\nu) \rangle}{\bar{\sigma}_2 \bar{\sigma}_\nu}. \end{aligned} \tag{7}$$

Since Eq. (6) represents a three-dimensional Black-Scholes equation, we can derive the Black-Scholes formula for P_0 from the work outlined in [22]. The price P_0 includes the volatilities $\bar{\sigma}_1$, $\bar{\sigma}_2$, and $\bar{\sigma}_\nu$, and the correlation coefficients $\bar{\rho}_{12}$, $\bar{\rho}_{1\nu}$, and $\bar{\rho}_{2\nu}$. This can be summarized in the following theorem:

Theorem 1 *If P_0, P_1, and their y_i-partial derivatives do not grow exponentially as y_i goes to infinity for $i = 1, 2, \nu$, then the leading order term P_0 is independent of y_1, y_2, and y_ν, and is given by*

$$\begin{aligned} P_0(t, s_1, s_2, \nu) = {}& s_1 \mathcal{N}_2(a_1, a_2; \theta_3) - s_2 \mathcal{N}_2(b_1, b_2; \theta_3) \\ & + \frac{(1-\alpha)\nu}{D} \big(s_1 e^{(\theta_1 + \theta_2)(T-t)} \mathcal{N}_2(c_1, c_2; -\theta_3) - s_2 e^{\theta_2 (T-t)} \mathcal{N}_2(d_1, d_2; -\theta_3) \big), \end{aligned}$$

where

$$
\begin{aligned}
a_1 &= \frac{1}{\bar{\sigma}_{12}\sqrt{T-t}}\ln\frac{s_1}{s_2} + \frac{\bar{\sigma}_{12}}{2}\sqrt{T-t},\\
a_2 &= \frac{1}{\bar{\sigma}_{\nu}\sqrt{T-t}}\ln\frac{\nu}{D^*} + \left(\frac{\bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu + r}{\bar{\sigma}_\nu} - \frac{\bar{\sigma}_\nu}{2}\right)\sqrt{T-t},\\
b_1 &= \frac{1}{\bar{\sigma}_{12}\sqrt{T-t}}\ln\frac{s_1}{s_2} - \frac{\bar{\sigma}_{12}}{2}\sqrt{T-t},\\
b_2 &= \frac{1}{\bar{\sigma}_{\nu}\sqrt{T-t}}\ln\frac{\nu}{D^*} + \left(\frac{\bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu + r}{\bar{\sigma}_\nu} - \frac{\bar{\sigma}_\nu}{2}\right)\sqrt{T-t},\\
c_1 &= \frac{1}{\bar{\sigma}_{12}\sqrt{T-t}}\ln\frac{s_1}{s_2} + \left(\frac{\bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu - \bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu}{\bar{\sigma}_{12}} + \frac{\bar{\sigma}_{12}}{2}\right)\sqrt{T-t},\\
c_2 &= -\frac{1}{\bar{\sigma}_{\nu}\sqrt{T-t}}\ln\frac{\nu}{D^*} - \left(\frac{\bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu + r}{\bar{\sigma}_\nu} + \frac{\bar{\sigma}_\nu}{2}\right)\sqrt{T-t},\\
d_1 &= \frac{1}{\bar{\sigma}_{12}\sqrt{T-t}}\ln\frac{s_1}{s_2} + \left(\frac{\bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu - \bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu}{\bar{\sigma}_{12}} - \frac{\bar{\sigma}_{12}}{2}\right)\sqrt{T-t},\\
d_2 &= -\frac{1}{\bar{\sigma}_{\nu}\sqrt{T-t}}\ln\frac{\nu}{D^*} - \left(\frac{\bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu + r}{\bar{\sigma}_\nu} + \frac{\bar{\sigma}_\nu}{2}\right)\sqrt{T-t},\\
\theta_1 &= \bar{\rho}_{1\nu}\bar{\sigma}_1\bar{\sigma}_\nu - \bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu,\\
\theta_2 &= \bar{\rho}_{2\nu}\bar{\sigma}_2\bar{\sigma}_\nu + r,\\
\theta_3 &= \frac{\bar{\rho}_{1\nu}\bar{\sigma}_1 - \bar{\rho}_{2\nu}\bar{\sigma}_2}{\bar{\sigma}_{12}},\\
\bar{\sigma}_{12} &= \sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2 - 2\bar{\rho}_{12}\bar{\sigma}_1\bar{\sigma}_2},
\end{aligned}
$$

and $\mathcal{N}_2$ is the cumulative density function for two-dimensional normal distribution defined by

$$
\mathcal{N}_2(x,y;\rho) = \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right)du\,dv.
$$

Next, we consider Eq. (6) to find the correction term P_1. Equation (6) yields

$$
\begin{aligned}
\mathcal{L}_2P_0 &= \mathcal{L}_2P_0 - \langle\mathcal{L}_2\rangle P_0\\
&= \frac{1}{2}\left(f_1^2(y_1) - \bar{\sigma}_1^2\right)s_1^2\frac{\partial^2}{\partial s_1^2}P_0 + \frac{1}{2}\left(f_2^2(y_2) - \bar{\sigma}_2^2\right)s_2^2\frac{\partial^2}{\partial s_2^2}P_0 + \frac{1}{2}\left(f_\nu^2(y_\nu) - \bar{\sigma}_\nu^2\right)\nu^2\frac{\partial^2}{\partial \nu^2}P_0\\
&\quad + \rho_{12}\left(f_1(y_1)f_2(y_2) - \langle f_1(y_1)f_2(y_2)\rangle\right)s_1s_2\frac{\partial^2}{\partial s_1\partial s_2}P_0\\
&\quad + \rho_{1\nu}\left(f_1(y_1)f_\nu(y_\nu) - \langle f_1(y_1)f_\nu(y_\nu)\rangle\right)s_1\nu\frac{\partial^2}{\partial s_1\partial \nu}P_0\\
&\quad + \rho_{2\nu}\left(f_2(y_2)f_\nu(y_\nu) - \langle f_2(y_2)f_\nu(y_\nu)\rangle\right)s_2\nu\frac{\partial^2}{\partial s_2\partial \nu}P_0,
\end{aligned}
$$

and Eq. (5) leads

$$
\begin{aligned}
P_2 = & -\mathcal{L}_0^{-1}(\mathcal{L}_2 P_0) \\
= & -\mathcal{L}_0^{-1}\big(\mathcal{L}_2 - \langle \mathcal{L}_2\rangle\big)P_0 \\
= & -\frac{1}{2}\big(\phi_1(y_1) + c_1(t,s_1,s_2,\nu,z)\big)s_1^2 \frac{\partial^2}{\partial s_1^2}P_0 \\
& -\frac{1}{2}\big(\phi_2(y_2) + c_2(t,s_1,s_2,\nu,z)\big)s_2^2 \frac{\partial^2}{\partial s_2^2}P_0 \\
& -\frac{1}{2}\big(\phi_\nu(y_\nu) + c_\nu(t,s_1,s_2,\nu,z)\big)\nu^2 \frac{\partial^2}{\partial \nu^2}P_0 \\
& -\rho_{12}\big(\phi_{12}(y_1,y_2) + c_{12}(t,s_1,s_2,\nu,z)\big)s_1 s_2 \frac{\partial^2}{\partial s_1 \partial s_2}P_0 \\
& -\rho_{1\nu}\big(\phi_{1\nu}(y_1,y_\nu) + c_{1\nu}(t,s_1,s_2,\nu,z)\big)s_1 \nu \frac{\partial^2}{\partial s_1 \partial \nu}P_0 \\
& -\rho_{2\nu}\big(\phi_{2\nu}(y_2,y_\nu) + c_{2\nu}(t,s_1,s_2,\nu,z)\big)s_2 \nu \frac{\partial^2}{\partial s_2 \partial \nu}P_0.
\end{aligned}
$$

Here, the functions ϕ_i $(i = 1,2,\nu)$, ϕ_{12}, $\phi_{1\nu}$, and $\phi_{2\nu}$ are the solutions of the following Poisson equations.

$$
\begin{aligned}
&\mathcal{L}_0\phi_1(y_1) = f_1^2(y_1) - \bar{\sigma}_1^2, \\
&\mathcal{L}_0\phi_2(y_2) = f_2^2(y_2) - \bar{\sigma}_2^2, \\
&\mathcal{L}_0\phi_\nu(y_3) = f_\nu^2(y_\nu) - \bar{\sigma}_\nu^2, \\
&\mathcal{L}_0\phi_{12}(y_1,y_2) = f_1(y_1)f_2(y_2) - \big\langle f_1(y_1)f_2(y_2)\big\rangle, \\
&\mathcal{L}_0\phi_{1\nu}(y_1,y_\nu) = f_1(y_1)f_\nu(y_\nu) - \big\langle f_1(y_1)f_\nu(y_\nu)\big\rangle, \\
&\mathcal{L}_0\phi_{2\nu}(y_2,y_\nu) = f_2(y_2)f_\nu(y_\nu) - \big\langle f_2(y_2)f_\nu(y_\nu)\big\rangle,
\end{aligned}
\tag{8}
$$

where c_i $(i = 1,2,\nu)$, c_{12}, $c_{1\nu}$, and $c_{2\nu}$ are the integral constant function. Since the operator $\mathcal{L}_0$ depends on the variables y_1, y_2, and y_ν, ϕ_i $(i = 1,2,\nu)$, ϕ_{12}, $\phi_{1\nu}$, and $\phi_{2\nu}$ are the functions with three variables. However, as a result of the spectral theory, the functions can be defined as above. We leave the details of this in the Appendix. In $\sqrt{\epsilon}$-order term in (4), we have the PDE for P_3, $\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1 = 0$, and $\langle\mathcal{L}_1P_2\rangle + \langle\mathcal{L}_2\rangle P_1 = 0$ holds from the solvability condition for P_3. Furthermore, since the terminal condition in Eq. (3) does not depend on the small parameter ϵ, it leads us to the conclusion that $P_1(T) = 0$ from $P^\epsilon(T) = P_0(T) + \sqrt{\epsilon}P_1(T) + \cdots$. As a result of this, we can further derive the following PDE for $P_1^\epsilon (:= \sqrt{\epsilon}P_1)$.

$$
\begin{aligned}
&\langle\mathcal{L}_2\rangle P_1^\epsilon = \mathcal{H}^\epsilon P_0, \\
&P_1^\epsilon(T,s_1,s_2,\nu) = 0,
\end{aligned}
\tag{9}
$$

where

$$
\mathcal{H}^\epsilon = \sqrt{\epsilon}\big\langle\mathcal{L}_1\mathcal{L}_0^{-1}\big(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\big)\big\rangle
$$

$$
\begin{aligned}
&= \sqrt{\epsilon}\Bigg\langle \bigg(-\sqrt{2}\nu_1\Lambda_1(y_1)\frac{\partial}{\partial y_1} - \sqrt{2}\nu_2\Lambda_2(y_2)\frac{\partial}{\partial y_2} - \sqrt{2}\nu_\nu\Lambda_\nu(y_\nu)\frac{\partial}{\partial y_\nu} \\
&\quad + \sqrt{2}\nu_1\eta_1 f_1(y_1)s_1\frac{\partial^2}{\partial s_1\partial y_1} + \sqrt{2}\nu_2\eta_2 f_2(y_2)s_2\frac{\partial^2}{\partial s_2\partial y_2} + \sqrt{2}\nu_\nu\eta_\nu f_\nu(y_\nu)\nu\frac{\partial^2}{\partial \nu\partial y_\nu} \bigg) \\
&\quad \times \bigg[\frac{1}{2}\big(\phi_1(y_1) + c_1(t,s_1,s_2,\nu,z)\big)s_1^2\frac{\partial^2}{\partial s_1^2} + \frac{1}{2}\big(\phi_2(y_2) + c_2(t,s_1,s_2,\nu,z)\big)s_2^2\frac{\partial^2}{\partial s_2^2} \\
&\quad + \frac{1}{2}\big(\phi_\nu(y_\nu) + c_\nu(t,s_1,s_2,\nu,z)\big)\nu^2\frac{\partial^2}{\partial \nu^2} + \rho_{12}\big(\phi_{12}(y_1,y_2) \\
&\quad + c_{12}(t,s_1,s_2,\nu,z)\big)s_1 s_2\frac{\partial^2}{\partial s_1\partial s_2} \\
&\quad + \rho_{1\nu}\big(\phi_{1\nu}(y_1,y_\nu) + c_{1\nu}(t,s_1,s_2,\nu,z)\big)s_1\nu\frac{\partial^2}{\partial s_1\partial \nu} \\
&\quad + \rho_{2\nu}\big(\phi_{2\nu}(y_2,y_\nu) + c_{2\nu}(t,s_1,s_2,\nu,z)\big)s_2\nu\frac{\partial^2}{\partial s_2\partial \nu} \bigg] \Bigg\rangle.
\end{aligned}
$$

Consequently, the following theorem is derived.

Theorem 2 *Under the same conditions as in Theorem 1, the fast correction term P_1^ϵ, which is the solution of the PDE (9), is given by*

$$
P_1^\epsilon = -(T-t)\mathcal{H}^\epsilon P_0,
$$

where

$$
\begin{aligned}
\mathcal{H}^\epsilon = &-\big[\mathcal{D}_1\big(V_1^\epsilon\mathcal{D}_1^2 + V_{1,2}^\epsilon\mathcal{D}_1\mathcal{D}_2 + V_{1,\nu}^\epsilon\mathcal{D}_1\mathcal{D}_\nu\big) + \mathcal{D}_2\big(V_2^\epsilon\mathcal{D}_2^2 + V_{2,1}^\epsilon\mathcal{D}_1\mathcal{D}_2 + V_{2,\nu}^\epsilon\mathcal{D}_2\mathcal{D}_\nu\big) \\
&+ \mathcal{D}_\nu\big(V_\nu^\epsilon\mathcal{D}_\nu^2 + V_{\nu,1}^\epsilon\mathcal{D}_1\mathcal{D}_\nu + V_{\nu,2}^\epsilon\mathcal{D}_2\mathcal{D}_\nu\big) + W_1^\epsilon\mathcal{D}_1^2 + W_{1,2}^\epsilon\mathcal{D}_1\mathcal{D}_2 + W_{1,\nu}^\epsilon\mathcal{D}_1\mathcal{D}_\nu \\
&+ W_2^\epsilon\mathcal{D}_2^2 + W_{2,1}^\epsilon\mathcal{D}_1\mathcal{D}_2 + W_{2,\nu}^\epsilon\mathcal{D}_2\mathcal{D}_\nu + W_\nu^\epsilon\mathcal{D}_\nu^2 + W_{\nu,1}^\epsilon\mathcal{D}_1\mathcal{D}_\nu + W_{\nu,2}^\epsilon\mathcal{D}_2\mathcal{D}_\nu\big],
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
V_i^\epsilon &= -\frac{\sqrt{\epsilon}\eta_i\nu_i}{\sqrt{2}}\bigg\langle f_i(y_i)\frac{\partial}{\partial y_i}\phi_i(y_i)\bigg\rangle, \\
V_{i,j}^\epsilon &= -\sqrt{2\epsilon}\eta_i\rho_{ij}\nu_i\bigg\langle f_i(y_i)\frac{\partial}{\partial y_i}\phi_{ij}(y_i,y_j)\bigg\rangle, \\
W_i^\epsilon &= \frac{\sqrt{\epsilon}\nu_i}{\sqrt{2}}\bigg\langle \Lambda_i(y_i)\frac{\partial}{\partial y_i}\phi_i(y_i)\bigg\rangle, \\
W_{i,j}^\epsilon &= \sqrt{2\epsilon}\rho_{ij}\nu_i\bigg\langle \Lambda_i(y_i)\frac{\partial}{\partial y_i}\phi_{ij}(y_i,y_j)\bigg\rangle,
\end{aligned}
\tag{11}
$$

for $i,j = 1,2,\nu$. Here, $\rho_{ji} = \rho_{ij}$ and $\phi_{ji} = \phi_{ij}$ are considered for the case where ϕ_{ji} is not defined (for example, $\phi_{\nu 1}$ is replaced by $\phi_{1\nu}$).

Proof Since the differential operators $\frac{\partial}{\partial t}$, $\mathcal{D}_i^n$ $(i = 1,2,\nu)$ are commutative for any $n \in \mathbb{N}$, the operator $\langle\mathcal{L}_2\rangle$ and $\mathcal{H}^\epsilon$ are also commutative, and

$$
\langle\mathcal{L}_2\rangle P_1^\epsilon = \langle\mathcal{L}_2\rangle(t-T)\mathcal{H}^\epsilon P_0
$$

$$
\begin{aligned}
&= \mathcal{H}^\epsilon P_0 + (t-T)\big(\langle\mathcal{L}_2\rangle\mathcal{H}^\epsilon P_0\big) \\
&= \mathcal{H}^\epsilon P_0 + (t-T)\big(\mathcal{H}^\epsilon\langle\mathcal{L}_2\rangle P_0\big) \\
&= \mathcal{H}^\epsilon P_0. \qquad \square
\end{aligned}
$$

From Theorem 1 and Theorem 2, we can obtain the analytical pricing formula of vulnerable exchange option as the approximate solution. Moreover, we verify the accuracy of the solution in the following theorem.

Theorem 3 *Let $\tilde{P}^\epsilon = P_0 + P_1^\epsilon$, then $\tilde{P}^\epsilon$ approximates to P^ϵ in (3) with order $\epsilon^{2/3}\log\epsilon$. In other words, there exist a positive constant C such that*

$$\left|P^\epsilon - \tilde{P}^\epsilon\right| < C\epsilon^{2/3}\log|\epsilon|.$$

Proof Since the option in this study has non-smooth and discontinuous payoff, the proof will sequentially cover three steps. First, we analyze the error of accuracy for smooth payoffs. We then consider the case of continuous but non-smooth payoffs. Finally, we address the error of accuracy for discontinuous payoffs.

Firstly, if the payoff function for P^ϵ is smooth, it is known that $\tilde{P}^\epsilon$ approximates to P^ϵ with order ϵ (refer to [34, 35]). Reviewing the whole process, first let $R^\epsilon = P^\epsilon - \hat{P}^\epsilon$, where $\hat{P}^\epsilon = \tilde{P}^\epsilon + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3$. We also suppose that the payoff function for P^ϵ and its derivatives are smooth and bounded. We then obtain the following from $\mathcal{L}^\epsilon P^\epsilon = 0$ in Eq. (3):

$$
\begin{aligned}
&\mathcal{L}^\epsilon R^\epsilon + \frac{1}{\epsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0P_1 + \mathcal{L}_1P_0) + (\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0) \\
&\quad + \sqrt{\epsilon}(\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1) + \epsilon(\mathcal{L}_1P_3 + \mathcal{L}_2P_2 + \sqrt{\epsilon}\mathcal{L}_2P_3) \qquad (12)\\
&= 0.
\end{aligned}
$$

From $\mathcal{L}_0P_0 = 0$, $\mathcal{L}_0P_1 + \mathcal{L}_1P_0 = 0$, $\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0 = 0$, and $\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1 = 0$ in Eq. (4), The Eq. (12) is rewritten as

$$\mathcal{L}^\epsilon R^\epsilon + \epsilon R_1^\epsilon = 0,$$

where $R_1^\epsilon = \mathcal{L}_1P_3 + \mathcal{L}_2P_2 + \sqrt{\epsilon}\mathcal{L}_2P_3$. Considering the terminal condition for P_0 and P_1 in Eq. (6) and Eq. (9), respectively, we obtain

$$R^\epsilon(T,s_1,s_2,v,y_1,y_2,y_v) = -\epsilon P_2(T,s_1,s_2,v,y_1,y_2,y_v) - \epsilon\sqrt{\epsilon}P_3(T,s_1,s_2,v,y_1,y_2,y_v).$$

Hence, one can use the Feynman–Kac probabilistic representation formula to lead

$$
\begin{aligned}
R^\epsilon = \epsilon\mathrm{E}^{\mathcal{Q}}\Bigg[& e^{-r(T-t)}\big(-P_2\big(T,S_T^1,S_T^2,V_T,Y_T^1,Y_T^2,Y_T^v\big) \\
& - \sqrt{\epsilon}P_3\big(T,S_T^1,S_T^2,V_T,Y_T^1,Y_T^2,Y_T^v\big)\big) \\
& + \int_t^T e^{-r(u-t)}R_1^\epsilon\big(u,S_u^1,S_u^2,V_u,Y_u^1,Y_u^2,Y_u^v\big)\,du\,\Big|\,S_t^1,S_t^2,V_t,Y_t^1,Y_t^2,Y_t^v\Bigg],
\end{aligned}
$$

and then

$$\begin{aligned}\left|P^\epsilon - \tilde{P}^\epsilon\right| &\le \left|R^\epsilon\right| + \left|\hat{P}^\epsilon - \tilde{P}^\epsilon\right| \\ &\le C_1\epsilon,\end{aligned}$$

for some constant $C_1 > 0$.

We then apply the regularization method described in [36] for handling non-smooth payoff functions. As a brief sketch of the second step of the proof, we start by regularizing an option with a continuous yet non-smooth payoff (such as a call option). Since our option has three underlying assets, for simplicity, let $C^{\epsilon,\eta}$ be an one-dimensional call option where the payoff is replaced by the Black-Scholes call option price at $t = T - \eta$ for small parameter η. Further, let $\tilde{C}^{\epsilon,\eta}$ represent the first-order approximation of $C^{\epsilon,\eta}$, such that

$$C^{\epsilon,\eta} \approx \tilde{C}^{\epsilon,\eta} = C_0^\eta + \sqrt{\epsilon}C_1^\eta,$$

where

$$\begin{aligned}&C_0^\eta(t,s_1) = C_{BS}(t-\eta, s_1; \bar{\sigma}_1), \\ &\sqrt{\epsilon}C_1^\eta = -(T-t)\mathcal{H}_1^\epsilon C_0^\eta.\end{aligned} \tag{13}$$

Here, the operator $\mathcal{H}_1^\epsilon$ is the one-dimensional reduced form of the operator $\mathcal{H}^\epsilon$ in Eq. (10). From this, we can deduce that

$$\left|C^\epsilon - \tilde{C}^\epsilon\right| \le \left|C^\epsilon - C^{\epsilon,\eta}\right| + \left|C^{\epsilon,\eta} - \tilde{C}^{\epsilon,\eta}\right| + \left|\tilde{C}^{\epsilon,\eta} - \tilde{C}^\epsilon\right| \tag{14}$$

and the right hand side of Eq. (14) is bounded by three terms $(\eta, \epsilon \ln|\eta|, \frac{\epsilon^{3/2}}{\sqrt{\eta}})$. Refer to [36] for details.

Lastly, in the case of a discontinuous payoff (such as digital options), an additional factor $\eta^{-1/2}$ is required due to the successive derivatives of C_0^η in Eq. (13). This modification bounds the error term by the three terms $(\eta, \frac{\epsilon \ln|\eta|}{\sqrt{\eta}}, \frac{\epsilon^{3/2}}{\eta})$. For further details on discontinuous payoffs, please refer to [37]. By substituting $\eta = \epsilon^q$, we derive the max-min problem:

$$\max\min\left\{q, 1 - \frac{q}{2}, \frac{3}{2} - q\right\}.$$

This problem has the solution $q = \frac{2}{3}$, leading to an order of accuracy for the error:

$$\left|C^\epsilon - \tilde{C}^\epsilon\right| = \mathcal{O}\left(\epsilon^{2/3}\log|\epsilon|\right).$$

Given that the exchange option in this study carries a default risk, it has a discontinuous payoff for V_t. Although the exchange option has three underlying asset processes, it is not difficult to extend the methodology from the one-dimensional digital option case mentioned earlier. As a results, we establish

$$\left|P^\epsilon - \tilde{P}^\epsilon\right| \le C\epsilon^{2/3}\log|\epsilon|$$

for some constant $C > 0$. □

3 Numerical results

In this section, we carry out several numerical experiments to illustrate the value of vulnerable option and to verify our results. Specifically, we provide some graphs to show the movements for the sensitivity analysis with respect to the significant parameters and implement the Monte-Carlo (MC) simulation, which is generally used for pricing of multi-asset options, for the verification of our result.

For feasible conduction of numerical experiment, the volatilities f_i in (1) are specified as follows.

$$dS_t^1 = rS_t^1\,dt + e^{Y_t^1} S_t^1\,dW_t^1, \tag{15}$$

$$dY_t^1 = \frac{k_1}{\epsilon}\left(m_1 - Y_t^1\right)dt + \frac{\sqrt{2}\nu_1}{\sqrt{\epsilon}}\,dZ_t^1, \tag{16}$$

$$dS_t^2 = rS_t^2\,dt + e^{Y_t^2} S_t^2\,dW_t^2, \tag{17}$$

$$dY_t^2 = \frac{k_2}{\epsilon}\left(m_2 - Y_t^2\right)dt + \frac{\sqrt{2}\nu_2}{\sqrt{\epsilon}}\,dZ_t^2, \tag{18}$$

$$dV_t = rV_t\,dt + e^{Y_t^\nu} V_t\,dW_t^\nu, \tag{19}$$

$$dY_t^\nu = \frac{k_\nu}{\epsilon}\left(m_\nu - Y_t^\nu\right)dt + \frac{\sqrt{2}\nu_\nu}{\sqrt{\epsilon}}\,dZ_t^\nu. \tag{20}$$

From the calculation in the appendix of [35], the group parameters in (7) and (11) can be obtained by

$$\bar{\sigma}_i = e^{m_i + \tilde{\nu}_i^2}, \tag{21}$$

$$\bar{\rho}_{ij} = \rho_{ij}\exp\left\{-\frac{1}{2}\left(\tilde{\nu}_i^2 + \tilde{\nu}_j^2\right)\right\}, \tag{22}$$

$$U_i^\epsilon = \frac{\sqrt{\epsilon}\eta_i}{\sqrt{2}\nu_i}\exp\left\{3m_i + \frac{5}{2}\tilde{\nu}_i^2\right\}\left(e^{2\tilde{\nu}_i^2} - 1\right), \tag{23}$$

$$U_{i,j}^\epsilon = \frac{\sqrt{2\epsilon}\eta_i\rho_{ij}}{\nu_i}\exp\left\{2m_i + m_j + \frac{1}{2}\left(2\tilde{\nu}_i^2 + \tilde{\nu}_j^2\right)\right\}\left(e^{\tilde{\nu}_i^2} - 1\right), \tag{24}$$

for $i,j = 1,2,\nu$, where $\tilde{\nu}_i^2 = \nu_i^2/k_i$. For the experiment, the baseline parameters are needed. We use the following baseline parameters $r = 0.05$, $\alpha = 0.25$, $D = D^* = 70$, $S_0^1 = 100$, $S_0^2 = 100$, $V_0 = 100$, $k_1 = k_2 = k_\nu = 1$, $m_1 = m_2 = m_3 = -1.45$, $\rho_{12} = \rho_{1\nu} = \rho_{2\nu} = 0.2$, $\eta_1 = \eta_2 = \eta_\nu = -0.5$, $\epsilon = 0.01$ and $T = 3$ unless otherwise stated. From these parameters and (21), (22), (23) and (24), we can determine the group parameters. The parameters are as follows.

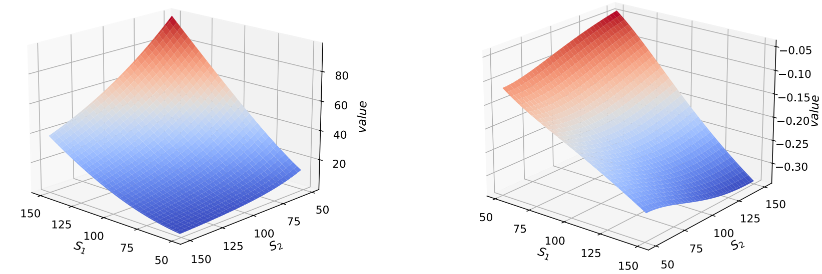
$$\bar{\sigma}_1 = 0.3012, \qquad \bar{\sigma}_2 = 0.3012, \qquad \bar{\sigma}_\nu = 0.2019,$$

$$\bar{\rho}_{12} = 0.1557, \qquad \bar{\rho}_{1\nu} = 0.1557, \qquad \bar{\rho}_{2\nu} = 0.1557,$$

$$U_1^\epsilon = -1.11 \times 10^{-3}, \qquad U_2^\epsilon = -1.11 \times 10^{-3}, \qquad U_\nu^\epsilon = -3.33 \times 10^{-4},$$

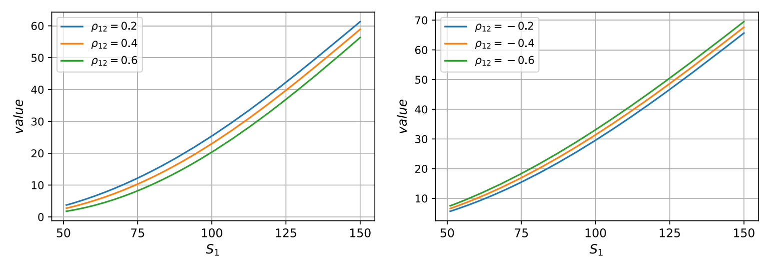
$$U_{1,2}^\epsilon = -1.51 \times 10^{-4}, \qquad U_{1,\nu}^\epsilon = -1.01 \times 10^{-4}, \qquad U_{2,1}^\epsilon = -1.51 \times 10^{-4},$$

$$U_{2,\nu}^\epsilon = -1.01 \times 10^{-4}, \qquad U_{\nu,1}^\epsilon = -6.78 \times 10^{-5}, \qquad U_{\nu,2}^\epsilon = -6.78 \times 10^{-5}.$$



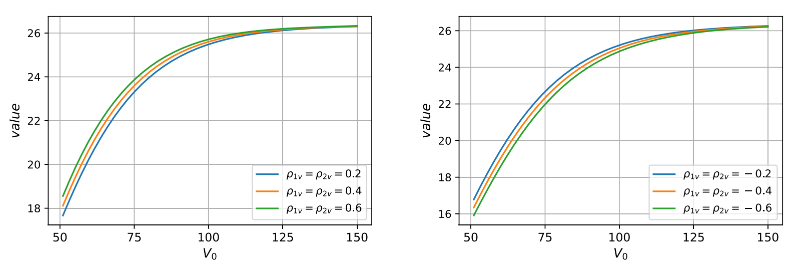
(a) $\tilde{P}_1^\epsilon$ (b) P_1^ϵ

Figure 1 Vulnerable exchange option value $\tilde{P}_1^\epsilon$ and the correction term P_1^ϵ



(a) $\tilde{P}_1^\epsilon$ varying with positive ρ_{12}. (b) $\tilde{P}_1^\epsilon$ varying with negative ρ_{12}.

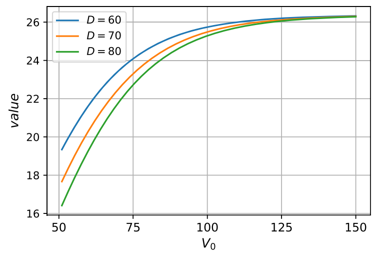
Figure 2 Option values with respect to S_0^1 for different ρ_{12}



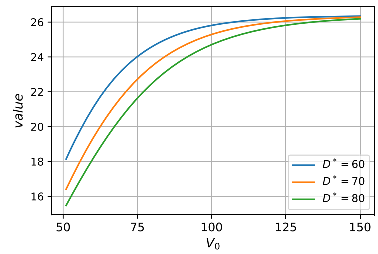
(a) $\tilde{P}_1^\epsilon$ varying with positive ρ_{1v} and ρ_{2v}. (b) $\tilde{P}_1^\epsilon$ varying with negative ρ_{1v} and ρ_{2v}.

Figure 3 Option values with respect to V_0 for different ρ_{1v} and ρ_{2v}

Figure 1(a) and Fig. 1(b) illustrate the sensitivities of the option value $\tilde{P}_1^\epsilon$ and the correction term P_1^ϵ with respect to initial values of two underlying assets. In Fig. 1, we can observe that the effect of stochastic volatility correction is nonlinear with respect to the underlying assets. Note that the values $\tilde{P}_1^\epsilon$ and P_1^ϵ are more sensitive with respect to asset S_0^1 than asset S_0^2. Figure 2(a), Fig. 2(b), Fig. 3(a), and Fig. 3(b) present how the option value with the stochastic volatility changes for different correlations when the underlying assets S^1 and V increase. In Fig. 2, it can be seen that high value of correlation ρ_{12} lowers the option value. We also find the results in higher value difference for positive values of



(a) $\tilde{P}_1^{\epsilon}$ varying with different D.



(b) $\tilde{P}_1^{\epsilon}$ varying with different D^*.

Figure 4 Option values with respect to V_0 for different D and D^*

correlation ρ_{12}. In contrast to Fig. 2, Fig. 3 shows that the option value increases as the correlation between underlying asset and the option issuer's asset increases. However, we can see that the option values converge to the same value for all correlations if V is sufficiently large. Figure 4(a) and Fig. 4(b) present how the option value changes for different D and D^* as the option issuer's asset V varies. As expected, higher values of D and D^* correspond to lower values of vulnerable exchange option, and the option values converge to the same value for all values of D and D^* if the value of V is sufficiently large. We also find that the option values are more sensitive with respect to values of D than values of D^*.

In what follows, we show the accuracy of our pricing formula by comparing the values of $\tilde{P}^{\epsilon}$ and the values by MC simulation. The MC simulation is based on the processes (15), (16), (17), (18), (19), (20), and the baseline parameters. It is well known that the MC method is very costly in terms of time. However, the MC method is widely used to verify the accuracy in pricing of financial derivatives since the value by the MC method converges to exact value for sufficiently large sample paths. For the experiment of MC simulation with the stability, we adopt the Euler scheme with N = 1,000,000 sample paths and $M = 1000 \times T$ time steps for the discretization of the stochastic processes. The numerical results are presented in Table 1. Values of 'R-err' in Table 1 denote the relative error defined by

$$\text{R-err} \triangleq \left| \frac{\text{'Pricing formula'} - \text{'Monte Carlo'}}{\text{'Pricing formula'}} \right|.$$

The results of Table 1 show that our analytical pricing formula for vulnerable exchange option provides the exact value. Moreover, we can see that it takes less than 0.01 seconds to calculate each option value. In other words, we consider the value obtained by the MC simulation as the benchmark and verify the accuracy and efficiency of our formula compared to benchmark value.

4 Concluding remarks

In this paper, we study the valuation of exchange option with default risk, which is called 'vulnerable exchange option', under a stochastic volatility model. The stochastic volatility model is assumed as a fast mean-reverting model. Since vulnerable exchange option consist of three underlying assets, we assume that the processes of three assets follow the stochastic volatility model. We derive the PDE for vulnerable exchange option based on asymptotic expansion approach and provide an explicit analytical pricing formula of the

Table 1 Values obtained from our approach and values from Monte-Carlo simulation. The baseline parameters are $r = 0.02$, $\alpha = 0.5$, $D = 100$, $S_0^1 = 100$, $S_0^2 = 100$, $V_0 = 100$, $k_1 = k_2 = k_v = 1$, $m_1 = m_2 = m_v = -2$, $\rho_{12} = \rho_{1v} = \rho_{2v} = 0.5$, and $\eta_1 = \eta_2 = \eta_v = -0.7$. Pricing formula* in P_0 denotes the value obtained from Theorem 1. 'Av. run time' denotes average CPU execution time for the computation when $T = 3$

		P_0			$\tilde{P}^\epsilon (= P_0 + P_1^\epsilon)$		
T	D^*	Pricing formula*	Monte Carlo	R-err	Pricing formula	Monte Carlo	R-err
Panel A. $\epsilon = 0.01$							
1	70	6.1275	6.1324	8.0×10^{-4}	6.0996	6.1141	2.4×10^{-3}
	80	5.9549	5.9610	1.0×10^{-3}	5.9069	5.9208	2.3×10^{-3}
	90	5.4297	5.4353	1.0×10^{-3}	5.3947	5.4077	2.4×10^{-3}
3	70	10.2437	10.2326	1.1×10^{-3}	10.1801	10.2058	2.5×10^{-3}
	80	9.6883	9.6794	9.0×10^{-4}	9.6293	9.6498	2.1×10^{-3}
	90	8.9200	8.9118	9.0×10^{-4}	8.8798	8.9014	2.4×10^{-3}
Panel B. $\epsilon = 0.02$							
1	70	6.1275	6.6.1324	8.0×10^{-4}	6.0881	6.0596	4.7×10^{-3}
	80	5.9549	5.9610	1.0×10^{-3}	5.8870	5.8639	3.9×10^{-3}
	90	5.4297	5.4353	1.0×10^{-3}	5.3802	5.3629	3.2×10^{-3}
3	70	10.2437	10.2326	1.1×10^{-3}	10.1536	10.1374	1.6×10^{-3}
	80	9.6883	9.6794	9.0×10^{-4}	9.6049	9.5900	1.6×10^{-3}
	90	8.9200	8.9118	9.0×10^{-4}	8.8631	8.8481	1.7×10^{-3}
Av. run time	($T = 3$)				0.0097 s	26 min 48 s	

option with the correction term. We also carry out the numerical experiments to show some features of vulnerable exchange option and the accuracy of our formula. We provide some graphs to illustrate the features of the option. From the graphs, we observe the stochastic volatility correction effect on the vulnerable exchange option and the behaviors of the option values with respect to some significant parameters. In addition, MC simulation is applied to obtain the value that is considered as a benchmark. From the numerical results with MC simulation, we can find that our formula is accurate and efficient in pricing of the vulnerable exchange option.

Appendix: Spectral theory

In this section, we investigate the dependence of the functions defined in (8) on certain variables. Without loss of generality, we consider only ϕ_1 and ϕ_{12} here, which should be defined as

$$\begin{aligned}
\mathcal{L}_0\phi_1(y_1, y_2, y_v) &= f_1^2(y_1) - \bar{\sigma}_1^2, \\
\mathcal{L}_0\phi_{12}(y_1, y_2, y_v) &= f_1(y_1)f_2(y_2) - \langle f_1(y_1)f_2(y_2)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_0 = {} & k_1(m_1 - y_1)\frac{\partial}{\partial y_1} + k_2(m_2 - y_2)\frac{\partial}{\partial y_2} + k_v(m_v - y_v)\frac{\partial}{\partial y_v} \\
& + k_1\tilde{\nu}_1^2\frac{\partial^2}{\partial y_1^2} + k_2\tilde{\nu}_2^2\frac{\partial^2}{\partial y_2^2} + k_v\tilde{\nu}_v^2\frac{\partial^2}{\partial y_v^2}
\end{aligned}$$

and $\tilde{\nu}_i^2 = \frac{\nu_i^2}{k_i}$ for $i = 1, 2, \nu$. Using the change of variables

$$y_i = m_i + \tilde{\nu}_i u_i \quad (i = 1, 2, \nu),$$

the operator $\mathcal{L}_0$ can be converted to $\tilde{\mathcal{L}}_0$, which is given by

$$\tilde{\mathcal{L}}_0 = k_1\left(\frac{\partial^2}{\partial u_1^2} - u_1\frac{\partial}{\partial u_1}\right) + k_2\left(\frac{\partial^2}{\partial u_2^2} - u_2\frac{\partial}{\partial u_2}\right) + k_\nu\left(\frac{\partial^2}{\partial u_\nu^2} - u_\nu\frac{\partial}{\partial u_\nu}\right).$$

The operator $\tilde{\mathcal{L}}_0$ is the infinitesimal generator of a multivariate OU process with zero mean-reverting level. Also the invariant distribution γ of the OU process is given by the product of the univariate OU processes because of their independence, as follows.

$$\gamma(du_1, du_2, du_\nu) = \prod_{i=1,2,\nu} \gamma_i(du_i),$$

where

$$\gamma_i(du_i) = \frac{1}{\sqrt{2\pi}} e^{-u_i^2/2}\, du_i,$$

for $i = 1, 2, \nu$. According to [38, 39], the operator $\tilde{\mathcal{L}}_0$ has the eigenfunction H_α, which is known as the Hermite polynomial and given by

$$H_\alpha(u_1, u_2, u_\nu) = \prod_{i=1,2,\nu} H_{\alpha_i}(u_i),$$

for index $\alpha = (\alpha_1, \alpha_2, \alpha_\nu) \in \mathbb{N}^3$. Here, $\mathbb{N}^3$ denotes the set of all three-dimensional positive integers. Also, its corresponding eigenvalue is $-k \cdot \alpha$, where $k = (k_1, k_2, k_\nu)$, and it means that the following holds.

$$\tilde{\mathcal{L}}_0 H_\alpha(u_1, u_2, u_\nu) = -(k_1\alpha_1 + k_2\alpha_2 + k_\nu\alpha_\nu) H_\alpha(u_1, u_2, u_\nu).$$

By a result of spectral theory, the set of the polynomial H_α forms a complete orthogonal basis for the Hilbert space $L^2(\gamma)$ in which the inner product is given by

$$\langle f, g\rangle = \iiint_{\mathbb{R}^3} f(u_1, u_2, u_\nu) g(u_1, u_2, u_\nu)\gamma(du_1, du_2, du_\nu),$$

for $f, g \in L^2(\gamma)$. Therefore, the three-dimensional Poisson equation,

$$\tilde{\mathcal{L}}_0 \tilde{\phi}_1(u_1, u_2, u_\nu) = \tilde{f}_1^2(u_1) - \bar{\sigma}_1^2,$$

has the following solution

$$\tilde{\phi}_1(u_1, u_2, u_\nu) = -\sum_{\alpha\in\mathbb{N}^3} \frac{c_\alpha}{k\cdot\alpha} \frac{1}{\sqrt{\alpha!}} H_\alpha(u_1, u_2, u_\nu),$$

where

$$c_\alpha = \left\langle \frac{1}{\sqrt{\alpha!}} H_\alpha(u_1, u_2, u_\nu), \tilde{f}_1^2(u_1) - \bar{\sigma}_1^2 \right\rangle,$$

and $\alpha! = \alpha_1!\alpha_2!\alpha_\nu!$. By simple calculation, one can obtain

$$\begin{aligned} c_\alpha &= \left\langle \frac{1}{\sqrt{\alpha!}} H_\alpha(u_1,u_2,u_\nu), \tilde{f}_1^2(u_1) - \bar{\sigma}_1^2 \right\rangle \\ &= \frac{1}{\sqrt{\alpha!}} \int_{\mathbb{R}} H_{\alpha_1}(u_1)\big(\tilde{f}_1^2(u_1) - \bar{\sigma}_1^2\big)\gamma(du_1) \int_{\mathbb{R}} H_{\alpha_2}(u_2)\gamma(du_2) \int_{\mathbb{R}} H_{\alpha_\nu}(u_\nu)\gamma(du_\nu) \end{aligned}$$

and

$$\int_{\mathbb{R}} H_{\alpha_i}(u_i)\gamma(du_i) = \begin{cases} 1 & \text{if } \alpha_i = 0, \\ 0 & \text{if } \alpha_i > 0, \end{cases}$$

for $i = 1, 2, \nu$. Hence, we conclude that

$$c_\alpha = \begin{cases} \frac{1}{\sqrt{\alpha_1!}} \int_{\mathbb{R}} H_{\alpha_1}(u_1)(\tilde{f}_1^2(u_1) - \bar{\sigma}_1^2)\gamma(du_1) & \text{if } \alpha_2 = 0 \text{ and } \alpha_\nu = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\tilde{\phi}_1$ is the function of u_1, which means ϕ_1 is the function of y_1.

Similarly, the equation,

$$\tilde{\mathcal{L}}_0 \tilde{\phi}_{12}(u_1,u_2,u_\nu) = \tilde{f}_1(u_1)\tilde{f}_2(u_2) - \big\langle f_1(y_1)f_2(y_2)\big\rangle,$$

has the solution of the form

$$\tilde{\phi}_{12}(u_1,u_2,u_\nu) = -\sum_{\alpha\in\mathbb{N}^3} \frac{d_\alpha}{k\cdot\alpha} \frac{1}{\sqrt{\alpha!}} H_\alpha(u_1,u_2,u_\nu),$$

where

$$\begin{aligned} d_\alpha &= \left\langle \frac{1}{\sqrt{\alpha!}} H_\alpha(u_1,u_2,u_\nu), \tilde{f}_1(u_1)\tilde{f}_2(u_2) - \big\langle f_1(y_1)f_2(y_2)\big\rangle \right\rangle \\ &= \begin{cases} \frac{1}{\sqrt{\alpha_1!\alpha_2!}} \iint_{\mathbb{R}^2} H_{\alpha_1}(u_1)H_{\alpha_2}(u_1)(\tilde{f}_1(u_1)\tilde{f}_2(u_2) & \\ \quad - \langle f_1(y_1)f_2(y_2)\rangle)\gamma(du_1)\gamma(du_2) & \text{if } \alpha_\nu = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, ϕ_{12} is also the function of y_1 and y_2.

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Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contributions

JJ and GK designed the model; JJ and JH contributed analysis of the mathematical model; JJ proved the Theorems in the paper; JH carried out the numerical experiments; JJ, JH and GK wrote the paper. All authors read and approved the final manuscript.

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