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Existence results for fractional neutral functional differential equations with infinite delay and nonlocal boundary conditions

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Abstract

In this paper, we establish sufficient criteria for ensuring the existence of solutions and uniqueness for a class of nonlinear neutral Caputo fractional differential equations supplemented with infinite delay and nonlocal boundary conditions involving fractional derivatives. The theory of infinite delay and standard fixed point theorems are employed to obtain the existence results for the given problem. Examples will be constructed to illustrate the obtained results.

Mathematics Subject Classification: 26A33; 34K05

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1 Introduction

Delay functional differential equations have emerged as a great tool for describing and modeling a wide range of real-world processes and changes involving long-term delays. There are many applications for this type of equations in the literature, for instance, population dynamics [1], immunology [2], disease models [3], ecological models [4], physiology and epidemiology [5], and neural networks [6–8]. The differential equation system with time delay is more complicated to treat and analyze than the classical one as its solution not only depends on the current situation but also takes the past state into consideration. The concept of the phase space $\mathfrak{F}$ plays a significant role in the study of equations with unbounded delay, which is specified by fundamental axioms that were presented by Hale and Kato [9]; also to find more discussion on these axioms, see [10, 11]. For further details on the theoretical developments of delayed differential equations, we refer to the works [9, 12–16].

In recent years, there have been interesting results in the study of the neutral fractional differential equation with infinite delay. Benchohra et al. [16] established some existence results relying on the Leray–Schauder type nonlinear alternative theorem and the Banach fixed point theorem for a class of initial value problems affected by infinite delay. Nouri et al. [17] investigated the existence of solutions by applying Krasnoselskii's fixed point theorem and contraction mapping principle to integro-fractional delayed differential equations. Ahmad et al. [18] discussed some existence results for a class of impulsive multi-order fractional differential equations with unbounded delay. Very recently, Chen and Dong in [19] studied the existence and uniqueness of a class of two-term boundary value problems with infinite delay by employing the standard fixed point theorems. Also, by using the Hyers–Ulam stability theorem, they discussed the stability of solutions for the given problem. However, the work on delayed fractional differential equations is still interesting, and new contributions in this field are needed.

Motivated by the previous studies, in this paper we are devoted to studying the existence and uniqueness of solutions for a new class of nonlinear nonlocal boundary value problems involving Caputo fractional derivatives with infinite delay and nonlocal fractional derivative conditions. In precise terms, we investigate the following problem:

$$\begin{cases} {}^{C}D^{\delta}_{0^+}[u(t) - \int_0^t h(s,u_s)\,ds] = f(t,u_t), & t \in \Omega := [0,a], \\ u(t) = \theta(t), & t \in (-\infty,0], \\ u(a) = \sum_{i=1}^{m} \lambda_i {}^{C}D^{\gamma}_{0^+} u(\mu_i) + \zeta, & \mu_i \in (0,a), \end{cases} \tag{1.1}$$

where ${}^{C}D^{\delta}_{0^+}$, ${}^{C}D^{\gamma}_{0^+}$ are the Caputo fractional derivatives of order $1 < \delta \leq 2$, $0 < \gamma < 1$, respectively. $f : \Omega \times \mathfrak{F} \to \mathbb{R}$, $h : \Omega \times \mathfrak{F} \to \mathbb{R}$, and $\theta \in \mathfrak{F}$ such that $\theta(0) = 0$, where $\mathfrak{F}$ is a phase space that will be explained in detail in Sect. 2. We define, for any $u : (-\infty,a] \to \mathbb{R}$ and any $t \in \Omega$, the function $u_t : (-\infty,0] \to \mathbb{R}$ to be an element of the phase space $\mathfrak{F}$ such that $u_t(s) = u(t+s)$, $s \leq 0$.

We arrange our work as follows: We recall some spaces, definitions, and lemmas needed in this work, and the equivalent integral equation to the linear variant of problem (1.1) is deduced in Sect. 2. Next, in Sect. 3, we obtain our main results with the aid of Krasnoselskii's fixed point theorem, the Leray–Schauder type nonlinear alternative theorem, and the Banach fixed point theorem. Finally, illustrative examples are provided.

2 Preliminaries

For the present work, the space $(\mathfrak{F}, \|.\|_{\mathfrak{F}})$ is defined as a seminormed linear space of functions that map $(-\infty,0]$ into $\mathbb{R}$ and satisfy the following axioms that were established by Hale and Kato in [9]:

(B_1) For every $t \in [0,a]$, if $u : (-\infty,a] \to \mathbb{R}$ and $u_0 \in \mathfrak{F}$, then the following conditions hold:

(1) u_t is in $\mathfrak{F}$,

(2) $|u(t)| \leq A\|u_t\|_{\mathfrak{F}}$,

(3) $\|u_t\|_{\mathfrak{F}} \leq \rho(t)\|u_0\|_{\mathfrak{F}} + \eta(t)\sup\{|u(\tau)| : 0 \leq \tau \leq t\}$,

where $A \geq 0$ is a constant, $\eta : [0,a] \to [0,\infty)$ is continuous, $\rho : [0,\infty) \to [0,\infty)$ is locally bounded, and A, η, ρ are independent of $u(.)$ and

$$\eta_a = \sup_{t\in[0,a]} \eta(t), \qquad \rho_a = \sup_{t\in[0,a]} \rho(t); \tag{2.1}$$

(B_2) For the function $u(.)$ in (B_1), u_t is a $\mathfrak{F}$-valued continuous function on $[0,a]$;

(B_3) The space $\mathfrak{F}$ is complete.

Let the space $\mathfrak{F}_a = \{u : (-\infty,a] \to \mathbb{R} : u|_{(-\infty,0]} \in \mathfrak{F} \text{ and } u|_{[0,a]} \in C(\Omega,\mathbb{R})\}$, and let $\|.\|_{\mathfrak{F}_a}$ be a seminorm in $\mathfrak{F}_a$ defined by $\|u\|_{\mathfrak{F}_a} = \|\theta\|_{\mathfrak{F}} + \sup_{s\in\Omega}|u(s)|$, $u \in \mathfrak{F}_a$.

Definition 2.1 [20] For $\delta > 0$ and a function $h : [0,\infty) \to \mathbb{R}$, the Riemann–Liouville fractional integral of order δ is defined by

$$I^{\delta}_{0+}h(x) = \int_0^x \frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)} h(\tau)\, d\tau, \quad x > 0.$$

Definition 2.2 [20] For $n-1 < \delta \le n$, $n \in \mathbb{N}$, the Caputo derivative of order δ for a function $h : [0,\infty] \to \mathbb{R}$ with $h(x) \in AC^n[0,\infty)$ is defined by

$${}^{C}D^{\delta}_{0+}h(x) = \frac{1}{\Gamma(n-\delta)} \int_0^x \frac{h^{(n)}(\tau)}{(x-\tau)^{\delta-n+1}}\, d\tau, \quad x > 0.$$

Lemma 2.1 [20] *Let $\delta > 0$ and $h(x) \in AC^n[0,\infty)$ or $C^n[0,\infty)$. Then*

$$\big(I^{\delta}_{0+}\, {}^{C}D^{\delta}_{0+}h\big)(x) = h(x) - \sum_{j=0}^{n-1} \frac{h^{(j)}(0)}{j!} x^j, \quad x > 0, n-1 < \delta < n.$$

The following lemma is related to the solution of the linear variant of problem (1.1).

Lemma 2.2 *Let $K \in C(0,a)$, $S \in AC(0,a)$ $u \in AC^2(\Omega,\mathbb{R}) \cap \mathfrak{F}_a$, and*

$$\Lambda_1 = a - \sum_{i=1}^{m} \lambda_i \frac{\mu_i^{1-\gamma}}{\Gamma(2-\gamma)} \neq 0. \tag{2.2}$$

Then the solution of the following problem

$$\begin{cases} {}^{C}D^{\delta}_{0+}[u(t) - \int_0^t S(s)\, ds] = K(t), & t \in \Omega := [0,a], \\ u(t) = \theta(t), & t \in (-\infty, 0], \\ u(a) = \sum_{i=1}^m \lambda_i {}^{C}D^{\gamma}_{0+}u(\mu_i) + \zeta, & \mu_i \in (0,a), \end{cases} \tag{2.3}$$

is given by

$$u(t) = \begin{cases} \theta(t), \quad t \in (-\infty,0], \\ \int_0^t S(s)\, ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} K(s)\, ds \\ \quad + \frac{t}{\Lambda_1}(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} K(s)\, ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)} S(s)\, ds \\ \quad - \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} K(s)\, ds - \int_0^a S(s)\, ds + \zeta), \quad t \in [0,a]. \end{cases} \tag{2.4}$$

Proof At first, we apply the fractional integral I^{δ}_{0+} to both sides of the fractional differential equation in (2.3), and with the aid of Lemma 2.1, the general solution of (2.3) for $t \in [0,a]$ can be written as

$$u(t) = \int_0^t S(s)\, ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} K(s)\, ds + c_1 + c_2 t, \tag{2.5}$$

where c_1, c_2 are arbitrary constants. Then, by using the condition $u(0) = \theta(0) = 0$ in (2.5), we get $c_1 = 0$. In consequence, (2.5) takes the form

$$u(t) = \int_0^t S(s)\, ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} K(s)\, ds + c_2 t. \tag{2.6}$$

For $t \in (0,a)$, we find

$$
\begin{aligned}
{}^C D^{\gamma}_{0^+} u(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} S(s)\,ds \\
&\quad + \frac{1}{\Gamma(\delta-\gamma)} \int_0^t (t-s)^{\delta-\gamma-1} K(s)\,ds + c_2 \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}.
\end{aligned}
$$

The condition $u(a) = \sum_{i=1}^m \lambda_i {}^C D^{\gamma}_{0^+} u(\mu_i) + \zeta$ together with (2.6) implies that

$$
\begin{aligned}
c_2 = {} & \frac{1}{(a - \sum_{i=1}^m \lambda_i \frac{\mu_i^{1-\gamma}}{\Gamma(2-\gamma)})} \Biggl(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} K(s)\,ds \\
& + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} S(s)\,ds \\
& - \frac{1}{\Gamma(\delta)} \int_0^a (a-s)^{\delta-1} K(s)\,ds - \int_0^a S(s)\,ds + \zeta \Biggr),
\end{aligned}
$$

which, on inserting in (2.6), gives the solution (2.4). By direct computation, we can easily obtain the converse of the lemma. This finishes the proof. □

3 Main results

Using Lemma 2.2, we convert problem (1.1) into a fixed point problem by introducing an operator $\mathcal{F} : \mathfrak{F}_a \to \mathfrak{F}_a$ as follows:

$$
(\mathcal{F}u)(t) = \begin{cases} \theta(t), \quad t \in (-\infty, 0], \\ \int_0^t h(s,u_s)\,ds + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f(s,u_s)\,ds \\ \quad + \frac{t}{\Lambda_1} (\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} f(s,u_s)\,ds + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} h(s,u_s)\,ds \\ \quad - \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} f(s,u_s)\,ds - \int_0^a h(s,u_s)\,ds + \zeta), \quad t \in [0,a]. \end{cases}
$$

Then we assume that the solution $u(.)$ that satisfies (3.1) is a decomposition of two functions $v, \bar{w} : (-\infty, a] \to \mathbb{R}$ such that $u(t) = v(t) + \bar{w}(t)$, which implies $u_t = v_t + \bar{w}_t$ for $t \in \Omega$.

These two functions, v and $\bar{w}$, have the following definitions:

$$
v(t) = \begin{cases} \theta(t), \quad t \in (-\infty, 0], \\ 0, \quad t \in [0,a], \end{cases} \tag{3.1}
$$

and

$$
\bar{w}(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ w(t), \quad t \in [0,a], \end{cases} \tag{3.2}
$$

where $w \in C([0,a], \mathbb{R})$ with $w(0) = 0$ and satisfies

$$
w(t) = \int_0^t h(s, v_s + \bar{w}_s)\,ds + \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f(s, v_s + \bar{w}_s)\,ds
$$

$$+\frac{t}{\Lambda_1}\left(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}f(s,v_s+\bar{w}_s)\,ds\right.$$
$$+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}h(s,v_s+\bar{w}_s)\,ds$$
$$\left.-\int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}f(s,v_s+\bar{w}_s)\,ds-\int_0^{a}h(s,v_s+\bar{w}_s)\,ds+\zeta\right). \tag{3.3}$$

Then we have $u_0=\theta$.

Now, consider the space $\mathfrak{F}'_a=\{w\in\mathfrak{F}_a:w_0=0\}$ and define a seminorm $\|.\|_{\mathfrak{F}'_a}$ on $\mathfrak{F}'_a$ by

$$\|w\|_{\mathfrak{F}'_a}=\sup_{t\in[0,a]}\left|w(t)\right|+\|w_0\|_{\mathfrak{F}}=\sup_{t\in[0,a]}\left|w(t)\right|,\quad w\in\mathfrak{F}'_a.$$

This implies that $\|.\|_{\mathfrak{F}'_a}$ defines a norm on $\mathfrak{F}'_a$, and as a consequence, $(\mathfrak{F}'_a,\|.\|_{\mathfrak{F}'_a})$ is a Banach space. Then we define the operator $\mathcal{P}:\mathfrak{F}'_a\to\mathfrak{F}'_a$ by

$$\mathcal{P}w(t)=\int_0^{t}h(s,v_s+\bar{w}_s)\,ds+\int_0^{t}\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}f(s,v_s+\bar{w}_s)\,ds$$
$$+\frac{t}{\Lambda_1}\left(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}f(s,v_s+\bar{w}_s)\,ds\right.$$
$$+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}h(s,v_s+\bar{w}_s)\,ds$$
$$\left.-\int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}f(s,v_s+\bar{w}_s)\,ds-\int_0^{a}h(s,v_s+\bar{w}_s)\,ds+\zeta\right),\quad t\in[0,a]. \tag{3.4}$$

Obviously, we note that the operator $\mathcal{F}$ has a fixed point if and only if $\mathcal{P}$ has a fixed point. In the following, for convenience, we define the notations:

$$\Lambda_2=\frac{a^\delta}{\Gamma(\delta+1)}+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)}+\frac{a^\delta}{\Gamma(\delta+1)}\right), \tag{3.5}$$
$$\Lambda_3=a+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)}+a\right). \tag{3.6}$$

In the first result, we prove the existence of solutions to problem (1.1) by applying Krasnoselskii's fixed point theorem [21].

Lemma 3.1 (*Krasnoselskii's fixed point theorem*). *Let $\mathcal{B}$ be a nonempty convex and closed subset of a Banach space E. Assume that Ψ_1, Ψ_2 are two operators from B to E such that* (1) *$\Psi_1 v+\Psi_2 u\in\mathcal{B}$ whenever $v,u\in\mathcal{B}$*; (2) *Ψ_1 is continuous and compact*; *and* (3) *Ψ_2 is a contraction mapping. Then there exists a fixed point $j\in\mathcal{B}$ such that $j=\Psi_1 j+\Psi_2 j$.*

Theorem 3.1 *Assume that $f,h:\Omega\times\mathfrak{F}\to\mathbb{R}$ are continuous functions such that the following conditions hold*:

(H_1) *There exists a constant $L_1 > 0$ such that*

$$\left|h(t,u) - h(t,v)\right| \le L_1\|u-v\|_{\mathfrak{F}} \quad \textit{for all } t \in \Omega \textit{ and every } u,v \in \mathfrak{F}.$$

(H_2) *There are nonnegative continuous functions $\kappa_1, \kappa_2 : \Omega \to (0,\infty)$ such that $|f(t,u)| \le \kappa_1(t)$, $|h(t,u)| \le \kappa_2(t)$ for all $t \in \Omega$ and every $u \in \mathfrak{F}$.*

Then problem (1.1) has at least one solution on $(-\infty, a]$ if

$$L_1\eta_a\Lambda_3 < 1, \tag{3.7}$$

where η_a and Λ_3 are respectively given by (2.1) and (3.6).

Proof Consider $B_r = \{w \in \mathfrak{F}'_a : \|w\|_{\mathfrak{F}'_a} \le r\}$ with $r > \kappa_1^*\Lambda_2 + \kappa_2^*\Lambda_3 + \frac{a}{|\Lambda_1|}|\zeta|$, where $\kappa_i^* = \sup_{t\in[0,a]}\kappa_i(t)$, $i = 1,2$, and Λ_2 is defined by (3.5). Then let us define the operators $\mathcal{R} : \mathfrak{F}'_a \to \mathfrak{F}'_a$ and $\mathcal{Q} : \mathfrak{F}'_a \to \mathfrak{F}'_a$ on B_r as follows:

$$\begin{aligned}(\mathcal{R}w)(t) &= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1} f(s, v_s + \bar{w}_s)\,ds \\ &\quad + \frac{t}{\Lambda_1}\left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} f(s, v_s + \bar{w}_s)\,ds \right.\\ &\quad \left. - \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} f(s, v_s + \bar{w}_s)\,ds\right)\end{aligned}$$

and

$$\begin{aligned}(\mathcal{Q}w)(t) &= \int_0^t h(s, v_s + \bar{w}_s)\,ds + \frac{t}{\Lambda_1}\left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} h(s, v_s + \bar{w}_s)\,ds \right.\\ &\quad \left. - \int_0^a h(s, v_s + \bar{w}_s)\,ds + \zeta\right).\end{aligned}$$

It is clear that the operator $\mathcal{P} : \mathfrak{F}'_a \to \mathfrak{F}'_a$ defined by (3.4) can be split as $\mathcal{R} + \mathcal{Q} = \mathcal{P}$. For $w, w^* \in B_r$ and $t \in \Omega$, we find

$$\begin{aligned}&\left|\mathcal{R}w(t) + \mathcal{Q}w^*(t)\right| \\ &\quad \le \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\left|f(s, v_s + \bar{w}_s)\right|\,ds \\ &\qquad + \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\left|f(s, v_s + \bar{w}_s)\right|\,ds \right.\\ &\qquad \left. + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\left|f(s, v_s + \bar{w}_s)\right|\,ds\right) + \int_0^t \left|h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds \\ &\qquad + \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}\left|h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds + \int_0^a \left|h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds + |\zeta|\right) \\ &\quad \le \kappa_1^*\left(\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\,ds\right.\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\,ds + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\,ds\right)\Bigg) \\
&+ \kappa_2^*\left(t + \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}\,ds + a\right)\right) + \frac{t}{|\Lambda_1|}|\zeta| \\
\le{}& \kappa_1^*\left(\frac{a^\delta}{\Gamma(\delta+1)} + \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)} + \frac{a^\delta}{\Gamma(\delta+1)}\right)\right) \\
&+ \kappa_2^*\left(a + \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)} + a\right)\right) + \frac{a}{|\Lambda_1|}|\zeta| \\
={}& \kappa_1^*\Lambda_2 + \kappa_2^*\Lambda_3 + \frac{a}{|\Lambda_1|}|\zeta| < r.
\end{aligned}
$$

Thus, for $w, w^* \in B_r$ and $t \in [0,a]$, we have

$$
\left\|\mathcal{R}w + \mathcal{Q}w^*\right\|_{\mathfrak{F}'_a} = \sup_{t\in[0,a]}\left|\mathcal{R}w(t) + \mathcal{Q}w^*(t)\right| \le \kappa_1^*\Lambda_2 + \kappa_2^*\Lambda_3 + \frac{a}{|\Lambda_1|}|\zeta| < r,
$$

which implies that $\mathcal{R}w + \mathcal{Q}w^* \in B_r$. Now, in view of condition (H_1), we show that $\mathcal{Q}$ is a contraction. Let $w, w^* \in B_r$ and $t \in [0,a]$. Then

$$
\begin{aligned}
&\sup_{t\in[0,a]}\left|\mathcal{Q}w(t) - \mathcal{Q}w^*(t)\right| \\
&\quad\le \sup_{t\in[0,a]}\Bigg\{\int_0^t \left|h(s, v_s + \bar{w}_s) - h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds \\
&\qquad+ \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}\left|h(s, v_s + \bar{w}_s) - h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds\right. \\
&\qquad\left.+ \int_0^a \left|h(s, v_s + \bar{w}_s) - h\left(s, v_s + \bar{w}_s^*\right)\right|\,ds\right)\Bigg\} \\
&\quad\le L_1 a\left\|w_t - w_t^*\right\|_{\mathfrak{F}} \\
&\qquad+ \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}L_1\left\|w_s - w_s^*\right\|_{\mathfrak{F}}\,ds + L_1 a\left\|w_a - w_a^*\right\|_{\mathfrak{F}}\right) \\
&\quad\le L_1 a\eta_a \sup_{t\in[0,a]}\left|w(t) - w^*(t)\right| \\
&\qquad+ \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}L_1\eta_a \sup_{s\in[0,a]}\left|w(s) - w^*(s)\right|\,ds\right. \\
&\qquad\left.+ L_1 a\eta_a \sup_{t\in[0,a]}\left|w(t) - w^*(t)\right|\right) \\
&\quad\le L_1\eta_a\left(a + \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)} + a\right)\right)\sup_{t\in[0,a]}\left|w(t) - w^*(t)\right|.
\end{aligned}
$$

Consequently, for $w, w^* \in B_r$ and $t \in [0,a]$, we have

$$\left\| \mathcal{Q}w - \mathcal{Q}w^* \right\|_{\mathfrak{F}'_a} = \sup_{t\in[0,a]} \left| \mathcal{Q}w(t) - \mathcal{Q}w^*(t) \right| \le L_1 \eta_a \Lambda_3 \left\| w - w^* \right\|_{\mathfrak{F}'_a}.$$

The continuity of the operator $\mathcal{R}$ can be directly deduced from the continuity of the functions f and h. Furthermore, $\mathcal{R}$ is uniformly bounded on B_r as

$$\|\mathcal{R}w\|_{\mathfrak{F}'_a} \le \kappa_1^* \Lambda_2.$$

Finally, for the compactness of the operator $\mathcal{R}$, we let $w \in B_r$ and, in view of hypothesis (H_2), for $t_1, t_2 \in [0,a]$, with $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathcal{R}w)(t_2) - (\mathcal{R}w)(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\delta)} \int_0^{t_1} \left|(t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1}\right| \left|f(s, v_s + \bar{w}_s)\right| ds \\
&\qquad + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_2} \left|(t_2 - s)^{\delta-1}\right| \left|f(s, v_s + \bar{w}_s)\right| ds \\
&\qquad + \frac{t_2 - t_1}{|\Lambda_1|} \left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} \left|f(s, v_s + \bar{w}_s)\right| ds \right. \\
&\qquad \left. + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} \left|f(s, v_s + \bar{w}_s)\right| ds \right) \\
&\quad \le \kappa_1^* \left(\int_0^{t_1} \left|(t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1}\right| ds + \int_{t_1}^{t_2} \left|(t_2 - s)^{\delta-1}\right| ds \right. \\
&\qquad \left. + \frac{t_2 - t_1}{|\Lambda_1|} \left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} ds + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} ds \right) \right) \\
&\quad \le \kappa_1^* \left(\frac{2(t_2 - t_1)^\delta}{\Gamma(\delta+1)} + \frac{t_2^\delta - t_1^\delta}{\Gamma(\delta+1)} + \frac{t_2 - t_1}{|\Lambda_1|} \left(\sum_{i=1}^m \frac{\lambda_i \mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)} + \frac{a^\delta}{\Gamma(\delta+1)} \right) \right).
\end{aligned}$$

From the above inequalities, it follows that $|(\mathcal{R}w)(t_2) - (\mathcal{R}w)(t_1)| \to 0$ as $t_2 - t_1 \to 0, \forall t_1, t_2 \in \Omega$ independently of $w \in B_r$. Therefore, $\mathcal{R}$ is equicontinuous, which implies that $\mathcal{R}$ is relatively compact on B_r. Thus, by the conclusion of the Arzelá–Ascoli theorem, $\mathcal{R}$ is compact on B_r. In consequence, as all the assumptions of Lemma 3.1 hold true, we conclude that problem (1.1) has at least one solution on $(-\infty, a]$. □

Next, we apply the following nonlinear Leray–Schauder alternative theorem [22] for our second existence result.

Lemma 3.2 (*Leray–Schauder nonlinear alternative*). *For a closed, convex, nonempty subset $\mathcal{G}$ of a Banach space E and for an open subset B of $\mathcal{G}$ with $0 \in B$, assume that $\mathcal{N} : \overline{B} \to \mathcal{G}$ is a continuous, compact (in other words, $\mathcal{N}(\overline{B})$ is a relatively compact subset of $\mathcal{G}$) map. Then either*

(1) *$\mathcal{N}$ has a fixed point in $\overline{B}$, or*
(2) *there exist $v \in \partial B$ (the boundary of B in $\mathcal{G}$) and $\mu \in (0,1)$ with $v = \mu\mathcal{N}(v)$.*

Theorem 3.2 *Let the following hypotheses hold*:

(A_1) *There exist constants* $0 \le \eta_a C_1 < 1/\Lambda_3$ *and* $C_2 \ge 0$ *such that* $|h(t,u)| \le C_1\|u\|_{\mathfrak{F}} + C_2$, $\forall (t,u) \in [0,a] \times \mathfrak{F}$.

(A_2) *There exist a nonnegative function* $\alpha \in C([0,a],\mathbb{R}^+)$ *and a continuous nondecreasing function* $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,u)| \le \alpha(t)\vartheta(\|u\|_{\mathfrak{F}})$, $\forall (t,u) \in [0,a] \times \mathfrak{F}$.

(A_3) *A constant* $\mathcal{W} > 0$ *exists such that*

$$\frac{(1-\eta_a C_1\Lambda_3)\mathcal{W}}{(C_2 + C_1\rho_a\|\theta\|_{\mathfrak{F}})\Lambda_3 + \vartheta(\eta_a\mathcal{W} + \rho_a\|\theta\|_{\mathfrak{F}})\alpha^*\Lambda_2 + \frac{a}{|\Lambda_1|}|\zeta|} > 1,$$

where $\alpha^* = \sup_{t\in[0,a]}\alpha(t)$, η_a, ρ_a, Λ_2, Λ_3 *are respectively given by* (2.1), (3.5), *and* (3.6).

Then problem (1.1) *has at least one solution on* $(-\infty, a]$.

Proof Firstly, we prove that the operator $\mathcal{P} : \mathfrak{F}'_a \to \mathfrak{F}'_a$ defined by (3.1) is continuous and completely continuous. This will be done in three steps.

(1) *$\mathcal{P}$ is continuous.*

Let us take the sequence $\{w_n\}$ such that $w_n \to w$ in $\mathfrak{F}'_a$. Then we have

$$\begin{aligned}
\big|\mathcal{P}(w_n)(t) - \mathcal{P}(w)(t)\big| \le{}& \int_0^t \big|h(s,\nu_s+\bar{w}_{n_s}) - h(s,\nu_s+\bar{w}_s)\big|\,ds \\
&+ \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\big|f(s,\nu_s+\bar{w}_{n_s}) - f(s,\nu_s+\bar{w}_s)\big|\,ds \\
&+ \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\big|f(s,\nu_s+\bar{w}_{n_s}) - f(s,\nu_s+\bar{w}_s)\big|\,ds \\
&+ \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}\big|h(s,\nu_s+\bar{w}_{n_s}) - h(s,\nu_s+\bar{w}_s)\big|\,ds \\
&+ \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\big|f(s,\nu_s+\bar{w}_{n_s}) - f(s,\nu_s+\bar{w}_s)\big|\,ds \\
&+ \int_0^a \big|h(s,\nu_s+\bar{w}_{n_s}) - h(s,\nu_s+\bar{w}_s)\big|\,ds) \\
\le{}& \Lambda_2\big\|f(\cdot,\nu_{(.)}+\bar{w}_{n_{(.)}}) - f(\cdot,\nu_{(.)}+\bar{w}_{(.)})\big\| \\
&+ \Lambda_3\big\|h(\cdot,\nu_{(.)}+\bar{w}_{n_{(.)}}) - h(\cdot,\nu_{(.)}+\bar{w}_{(.)})\big\|,
\end{aligned}$$

which, in view of the continuity of h and f, leads to

$$\begin{aligned}
&\big\|\mathcal{P}(w_n) - \mathcal{P}(w)\big\| \\
&\quad\le \Lambda_2\big\|f(\cdot,\nu_{(.)}+\bar{w}_{n_{(.)}}) - f(\cdot,\nu_{(.)}+\bar{w}_{(.)})\big\| + \Lambda_3\big\|h(\cdot,\nu_{(.)}+\bar{w}_{n_{(.)}}) - h(\cdot,\nu_{(.)}+\bar{w}_{(.)})\big\| \to 0,
\end{aligned}$$

as $n \to \infty$.

(2) *$\mathcal{P}$ maps bounded sets into bounded sets in $\mathfrak{F}'_a$*

For any $\ell > 0$, we show that there exists a positive constant ξ such that for $w \in B_\ell = \{w \in \mathfrak{F}'_a : \|w\|_{\mathfrak{F}'_a} \le \ell\}$ we have $\|\mathcal{P}(w)\|_{\mathfrak{F}'_a} \le \xi$. Let $w \in B_\ell$, for each $t \in [0,a]$, we have

$$\big|\mathcal{P}(w)(t)\big| \le \int_0^t \big|h(s,\nu_s+\bar{w}_s)\big|\,ds + \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\big|f(s,\nu_s+\bar{w}_s)\big|\,ds$$

$$
\begin{aligned}
&+ \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\left|f(s, v_s + \bar{w}_s)\right| ds\right.\\
&+ \sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}\left|h(s, v_s + \bar{w}_s)\right| ds\\
&+ \int_0^{a} \frac{(a - s)^{\delta-1}}{\Gamma(\delta)}\left|f(s, v_s + \bar{w}_s)\right| ds\\
&\left.+ \int_0^{a} \left|h(s, v_s + \bar{w}_s)\right| ds + |\zeta|\right)\\
\le{}& \int_0^{t}\left[C_1\|v_s + \bar{w}_s\|_{\mathfrak{F}} + C_2\right] ds + \frac{1}{\Gamma(\delta)}\int_0^{t}(t-s)^{\delta-1}\left[\alpha(s)\vartheta\left(\|v_s + \bar{w}_s\|_{\mathfrak{F}}\right)\right] ds\\
&+ \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\left[\alpha(s)\vartheta\left(\|v_s + \bar{w}_s\|_{\mathfrak{F}}\right)\right] ds\right.\\
&+ \sum_{i=1}^{m}\lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)}\left[C_1\|v_s + \bar{w}_s\|_{\mathfrak{F}} + C_2\right] ds\\
&+ \int_0^{a} \frac{(a - s)^{\delta-1}}{\Gamma(\delta)}\left[\alpha(s)\vartheta\left(\|v_s + \bar{w}_s\|_{\mathfrak{F}}\right)\right] ds\\
&\left.+ \int_0^{a}\left[C_1\|v_s + \bar{w}_s\|_{\mathfrak{F}} + C_2\right] ds + |\zeta|\right)\\
\le{}& \left[C_1\left(\eta_a\ell + \rho_a\|\theta\|_{\mathfrak{F}}\right) + C_2\right]\left(a + \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} ds + a\right)\right)\\
&+ \alpha^*\vartheta\left(\eta_a\ell + \rho_a\|\theta\|_{\mathfrak{F}}\right)\left(\frac{1}{\Gamma(\delta)}\int_0^{t}(t-s)^{\delta-1} ds\right.\\
&+ \frac{t}{|\Lambda_1|}\left(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} ds\right.\\
&\left.\left.+ \int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)} ds\right)\right) + \frac{a}{|\Lambda_1|}|\zeta|.
\end{aligned}
$$

So, by taking the norm on the space $\mathfrak{F}'_a$, we have

$$
\begin{aligned}
\left\|\mathcal{P}(w)\right\|_{\mathfrak{F}'_a} \le{}& [C_1L + C_2]\left(a + \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)} + a\right)\right)\\
&+ \vartheta(L)\alpha^*\left(\frac{a^\delta}{\Gamma(\delta+1)} + \frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)} + \frac{a^\delta}{\Gamma(\delta+1)}\right)\right)\\
&+ \frac{a}{|\Lambda_1|}|\zeta|\\
={}& [C_1L + C_2]\Lambda_3 + \vartheta(L)\alpha^*\Lambda_2 + \frac{a}{|\Lambda_1|}|\zeta| := \xi,
\end{aligned}
$$

where

$$
\|v_s + \bar{w}_s\|_{\mathfrak{F}} \le \|v_s\|_{\mathfrak{F}} + \|\bar{w}_s\|_{\mathfrak{F}} \le \eta_a\ell + \rho_a\|\theta\|_{\mathfrak{F}} := L.
$$

(3) $\mathcal{P}$ *maps bounded sets into equicontinuous sets of* $\mathfrak{F}'_a$.

For a bounded set B_ℓ of $\mathfrak{F}'_a$ defined as in Step 2, let $w \in B_\ell$ and $0 < t_1 < t_2 < a$. Then we have

$$
\begin{aligned}
&\left|\mathcal{P}(w)(t_2) - \mathcal{P}(w)(t_1)\right| \\
&\quad\leq \left| \int_0^{t_2} h(s, v_s + \bar{w}_s)\,ds - \int_0^{t_1} h(s, v_s, \bar{w}_s)\,ds \right. \\
&\qquad + \frac{1}{\Gamma(\delta)} \int_0^{t_1} \left((t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1}\right) f(s, v_s + \bar{w}_s)\,ds \\
&\qquad + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_2} (t_2 - s)^{\delta-1} f(s, v_s + \bar{w}_s)\,ds \\
&\qquad + \frac{t_2 - t_1}{\Lambda_1} \left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} f(s, v_s + \bar{w}_s)\,ds \right. \\
&\qquad + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} h(s, v_s + \bar{w}_s)\,ds + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} f(s, v_s + \bar{w}_s)\,ds \\
&\qquad \left.\left. + \int_0^a h(s, v_s + \bar{w}_s)\,ds + \zeta \right) \right| \\
&\quad\leq \int_{t_1}^{t_2} [C_1 L + C_2]\,ds + \frac{1}{\Gamma(\delta)} \int_0^{t_1} \left((t_2 - s)^{\delta-1} - (t_1 - s)^{\delta-1}\right) \alpha^* \vartheta(L)\,ds \\
&\qquad + \frac{1}{\Gamma(\delta)} \int_{t_1}^{t_2} (t_2 - s)^{\delta-1} \alpha^* \vartheta(L)\,ds + \frac{t_2 - t_1}{|\Lambda_1|} \left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} \alpha^* \vartheta(L)\,ds \right. \\
&\qquad + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} [C_1 L + C_2]\,ds + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} \alpha^* \vartheta(L)\,ds \\
&\qquad \left. + \int_0^a [C_1 L + C_2]\,ds + |\zeta| \right),
\end{aligned}
$$

which implies that $|\mathcal{P}(w)(t_2) - \mathcal{P}(w)(t_1)| \longrightarrow 0$ as $t_1 \longrightarrow t_2$. In view of the Arzelá–Ascoli theorem, we deduce from the foregoing three steps that $\mathcal{P} : \mathfrak{F}'_a \to \mathfrak{F}'_a$ is completely continuous.

Finally, we show that for $0 < \sigma < 1$ there exists an open set $\Theta \subseteq \mathfrak{F}'_a$ and $w \in \partial\Theta$ such that $w \neq \sigma \mathcal{P}(w)$.

Let $w \in \mathfrak{F}'_a$ with $w - \sigma \mathcal{P}(w) = 0$ for $\sigma \in (0,1)$. Then, for $t \in [0,a]$, we have

$$
\begin{aligned}
\left|w(t)\right| &= \left|\sigma(\mathcal{P}w)(t)\right| \\
&\leq \int_0^t \left|h(s, v_s + \bar{w}_s)\right|\,ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \left|f(s, v_s + \bar{w}_s)\right|\,ds \\
&\quad + \frac{t}{|\Lambda_1|} \left(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} \left|f(s, v_s + \bar{w}_s)\right|\,ds \right. \\
&\quad + \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} \left|h(s, v_s + \bar{w}_s)\right|\,ds + \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} \left|f(s, v_s + \bar{w}_s)\right|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^a \left|h(s,v_s+\bar{w}_s)\right| ds+|\zeta|\Bigg) \\
&\leq \int_0^t \left[C_1\|v_s+\bar{w}_s\|_{\mathfrak{F}}+C_2\right] ds+\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\left[\alpha(s)\vartheta\left(\|v_s+\bar{w}_s\|_{\mathfrak{F}}\right)\right] ds \\
&+\frac{t}{|\Lambda_1|}\Bigg(\sum_{i=1}^m \lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\left[\alpha(s)\vartheta\left(\|v_s+\bar{w}_s\|_{\mathfrak{F}}\right)\right] ds \\
&+\sum_{i=1}^m \lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}\left[C_1\|v_s+\bar{w}_s\|_{\mathfrak{F}}+C_2\right] ds \\
&+\int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\left[\alpha(s)\vartheta\left(\|v_s+\bar{w}_s\|_{\mathfrak{F}}\right)\right] ds \\
&+\int_0^a \left[C_1\|v_s+\bar{w}_s\|_{\mathfrak{F}}+C_2\right] ds+|\zeta|\Bigg) \\
&\leq \left[C_1\left(\eta_a\|w\|_{\mathfrak{F}'_a}+\rho_a\|\theta\|_{\mathfrak{F}}\right)+C_2\right]\left(a+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^m\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)}+a\right)\right) \\
&+\vartheta\left(\eta_a\|w\|_{\mathfrak{F}'_a}+\rho_a\|\theta\|_{\mathfrak{F}}\right)\alpha^* \\
&\times\left(\frac{a^\delta}{\Gamma(\delta+1)}+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^m\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)}+\frac{a^\delta}{\Gamma(\delta+1)}\right)\right)+\frac{a}{|\Lambda_1|}|\zeta| \\
&\leq \left[C_1\left(\eta_a\|w\|_{\mathfrak{F}'_a}+\rho_a\|\theta\|_{\mathfrak{F}}\right)+C_2\right]\Lambda_3+\vartheta\left(\eta_a\|w\|_{\mathfrak{F}'_a}+\rho_a\|\theta\|_{\mathfrak{F}}\right)\alpha^*\Lambda_2+\frac{a}{|\Lambda_1|}|\zeta|,
\end{aligned}
$$

which, on taking the norm for $t\in[0,a]$, implies that

$$
\frac{(1-\eta_a C_1\Lambda_3)\|w\|_{\mathfrak{F}'_a}}{(C_2+C_1\rho_a\|\theta\|_{\mathfrak{F}})\Lambda_3+\vartheta(\eta_a\|w\|_{\mathfrak{F}'_a}+\rho_a\|\theta\|_{\mathfrak{F}})\alpha^*\Lambda_2+\frac{a}{|\Lambda_1|}|\zeta|}\leq 1.
$$

In view of hypothesis (A_3), there exists a constant $\mathcal{W}>0$ such that $\|w\|_{\mathfrak{F}'_a}\neq\mathcal{W}$. Let us set

$$
\Theta=\left\{w\in\mathfrak{F}'_a:\|w\|_{\mathfrak{F}'_a}<\mathcal{W}\right\}.
$$

Note that the operator $\mathcal{P}:\overline{\Theta}\to\mathfrak{F}'_a$ is continuous and completely continuous.

By this choice of Θ, there is no $w\in\Theta$ such that $w=\sigma\mathcal{P}(w)$ for some $\sigma\in(0,1)$. Consequently, by the conclusion nonlinear alternative of the Leray–Schauder theorem (Lemma 3.2), we deduce that $\mathcal{P}$ has a fixed point $w\in\overline{\Theta}$, which is a solution to problem (1.1). This finishes the proof. □

In our last result, we prove the uniqueness of solutions to (1.1) with the aid of the Banach contraction mapping principle.

Theorem 3.3 *Let $f,h\in C(\Omega\times\mathfrak{F},\mathbb{R})$, and condition (H_1) and the following condition satisfy:*

(H_3) *There exists a positive constant L_2 such that*

$$
\left|f(t,u)-f(t,v)\right|\leq L_2\|u-v\|_{\mathfrak{F}}\quad\textit{for all } t\in\Omega \textit{ and every } u,v\in\mathfrak{F}.
$$

Then problem (1.1) *has a unique solution on* $(-\infty, a]$ *if*

$$\eta_a(L_1\Lambda_3 + L_2\Lambda_2) < 1, \tag{3.8}$$

where η_a, Λ_2, *and* Λ_3 *are respectively defined by* (2.1), (3.5), *and* (3.6).

Proof Putting $\sup_{t\in[0,a]} |f(t,0)| = \hat{f}$, also $\sup_{t\in[0,a]} |h(t,0)| = \hat{h}$, we consider the set

$$B_{\bar{r}} = \{ w \in \mathfrak{F}'_a : \|w\|_{\mathfrak{F}'_a} \le \bar{r} \}$$

with

$$\bar{r} > \frac{(L_1\rho_a\|\theta\|_{\mathfrak{F}} + \hat{h})\Lambda_3 + (L_2\rho_a\|\theta\|_{\mathfrak{F}} + \hat{f})\Lambda_2 + \frac{a}{|\Lambda_1|}|\zeta|}{1 - L_1\eta_a\Lambda_3 - L_2\eta_a\Lambda_2}$$

and show that $\mathcal{P}B_{\bar{r}} \subset B_{\bar{r}}$. For $w \in B_{\bar{r}}$ and $t \in [0,a]$, we have

$$\begin{aligned}
\big|(\mathcal{P}w)(t)\big| \le{}& \Bigg| \int_0^t h(s, \nu_s + \bar{w}_s)\,ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s, \nu_s + \bar{w}_s)\,ds \\
&+ \frac{t}{|\Lambda_1|} \Bigg(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} f(s, \nu_s + \bar{w}_s)\,ds \\
&+ \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} h(s, \nu_s + \bar{w}_s)\,ds \\
&- \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} f(s, \nu_s + \bar{w}_s)\,ds - \int_0^a h(s, \nu_s + \bar{w}_s)\,ds + \zeta \Bigg) \Bigg| \\
\le{}& \int_0^t \big|h(s, \nu_s + \bar{w}_s) - h(s,0)\big| + \big|h(s,0)\big|\,ds \\
&+ \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \big|f(s, \nu_s + \bar{w}_s) - f(s,0)\big| + \big|f(s,0)\big|\,ds \\
&+ \frac{t}{|\Lambda_1|} \Bigg(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} \big|f(s, \nu_s + \bar{w}_s) - f(s,0)\big| + \big|f(s,0)\big|\,ds \\
&+ \sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{-\gamma}}{\Gamma(1-\gamma)} \big|h(s, \nu_s + \bar{w}_s) - h(s,0)\big| + \big|h(s,0)\big|\,ds \\
&+ \int_0^a \frac{(a-s)^{\delta-1}}{\Gamma(\delta)} \big|f(s, \nu_s + \bar{w}_s) - f(s,0)\big| + \big|f(s,0)\big|\,ds \\
&+ \int_0^a \big|h(s, \nu_s + \bar{w}_s) - h(s,0)\big| + \big|h(s,0)\big|\,ds + |\zeta| \Bigg) \\
\le{}& \int_0^t \big(L_1\|\nu_s + \bar{w}_s\|_{\mathfrak{F}} + \hat{h}\big)\,ds + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \big(L_2\|\nu_s + \bar{w}_s\|_{\mathfrak{F}} + \hat{f}\big)\,ds \\
&+ \frac{t}{|\Lambda_1|} \Bigg(\sum_{i=1}^m \lambda_i \int_0^{\mu_i} \frac{(\mu_i - s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)} \big(L_2\|\nu_s + \bar{w}_s\|_{\mathfrak{F}} + \hat{f}\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}\big(L_1\|v_s+\bar{w}_s\|_{\mathfrak{F}}+\hat{h}\big)\,ds\\
&+\int_0^a\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\big(L_2\|v_s+\bar{w}_s\|_{\mathfrak{F}}+\hat{f}\big)\,ds+\int_0^a\big(L_1\|v_s+\bar{w}_s\|_{\mathfrak{F}}+\hat{h}\big)\,ds+|\zeta|\Bigg)\\
\le{}&\big(L_1\big(\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\bar{r}\big)+\hat{h}\big)\left(a+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)}+a\right)\right)\\
&+\big(L_2\big(\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\bar{r}\big)+\hat{f}\big)\\
&\times\left(\frac{a^\delta}{\Gamma(\delta+1)}+\frac{a}{|\Lambda_1|}\left(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)}+\frac{a^\delta}{\Gamma(\delta+1)}\right)\right)+\frac{a}{|\Lambda_1|}|\zeta|\\
\le{}&\big(L_1\big(\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\bar{r}\big)+\hat{h}\big)\Lambda_3+\big(L_2\big(\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\bar{r}\big)+\hat{f}\big)\Lambda_2+\frac{a}{|\Lambda_1|}|\zeta|<\bar{r},
\end{aligned}
$$

which, on taking the norm for $t\in[0,a]$, implies that $\|\mathcal{P}w\|_{\mathfrak{F}'_a}<\bar{r}$, where for $t\in[0,a]$ we have

$$
\begin{aligned}
\|v_t+\bar{w}_t\|_{\mathfrak{F}}&\le\|v_t\|_{\mathfrak{F}}+\|\bar{w}_t\|_{\mathfrak{F}}\\
&\le\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\sup\big\{\big|w(s)\big|:s\in[0,t]\big\}\\
&\le\rho_a\|\theta\|_{\mathfrak{F}}+\eta_a\bar{r}.
\end{aligned}
$$

Thus, $PB_{\bar{r}}\subset B_{\bar{r}}$.

Now, we shall show that the operator $\mathcal{P}:\mathfrak{F}'_a\to\mathfrak{F}'_a$ is a contraction map. For that, let us consider $w,w^*\in\mathfrak{F}'_a$. Then we have for each $t\in[0,a]$

$$
\begin{aligned}
&\big|\mathcal{P}w(t)-\mathcal{P}w^*(t)\big|\\
&\quad\le\int_0^t\big|h(s,v_s+\bar{w}_s)+h\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\\
&\qquad+\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}\big|f(s,v_s+\bar{w}_s)-f\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\\
&\qquad+\frac{t}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}\big|f(s,v_s+\bar{w}_s)-f\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\\
&\qquad+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}\big|h(s,v_s+\bar{w}_s)-h\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\\
&\qquad+\int_0^a\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}\big|f(s,v_s+\bar{w}_s)-f\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\\
&\qquad+\int_0^a\big|h(s,v_s+\bar{w}_s)-h\big(s,v_s+\bar{w}_s^*\big)\big|\,ds\Bigg)\\
&\quad\le\int_0^t L_1\big\|w_s-w_s^*\big\|_{\mathfrak{F}}\,ds+\frac{1}{\Gamma(\delta)}\int_0^t(t-s)^{\delta-1}L_2\big\|w_s-w_s^*\big\|_{\mathfrak{F}}\,ds\\
&\qquad+\frac{t}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}L_2\big\|w_s-w_s^*\big\|_{\mathfrak{F}}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}L_1\|w_s-w_s^*\|_{\mathfrak{F}}\,ds \\
&+\int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}L_2\|w_s-w_s^*\|_{\mathfrak{F}}\,ds+\int_0^{t}L_1\|w_s-w_s^*\|_{\mathfrak{F}}\,ds\Bigg) \\
\le{}&\int_0^{t}L_1\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds+\frac{1}{\Gamma(\delta)}\int_0^{t}(t-s)^{\delta-1}L_2\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds \\
&+\frac{t}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}L_2\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds \\
&+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}L_1\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds \\
&+\int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}L_2\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds+\int_0^{a}L_1\eta_a\sup_{s\in[0,a]}\big|w(s)-w^*(s)\big|\,ds\Bigg) \\
\le{}&\int_0^{t}L_1\eta_a\|w-w^*\|_{\mathfrak{F}'_a}\,ds+\frac{\eta_a}{\Gamma(\delta)}\int_0^{t}(t-s)^{\delta-1}L_2\|w-w^*\|_{\mathfrak{F}'_a}\,ds \\
&+\frac{a\eta_a}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{\delta-\gamma-1}}{\Gamma(\delta-\gamma)}L_2\|w-w^*\|_{\mathfrak{F}'_a}\,ds \\
&+\sum_{i=1}^{m}\lambda_i\int_0^{\mu_i}\frac{(\mu_i-s)^{-\gamma}}{\Gamma(1-\gamma)}L_1\|w-w^*\|_{\mathfrak{F}'_a}\,ds \\
&+\int_0^{a}\frac{(a-s)^{\delta-1}}{\Gamma(\delta)}L_2\|w-w^*\|_{\mathfrak{F}'_a}\,ds+\int_0^{t}L_1\|w-w^*\|_{\mathfrak{F}'_a}\,ds\Bigg) \\
\le{}&\eta_a\Bigg[L_2\Bigg(\frac{a^\delta}{\Gamma(\delta+1)}+\frac{a}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{\delta-\gamma}}{\Gamma(\delta-\gamma+1)}+\frac{a^\delta}{\Gamma(\delta+1)}\Bigg)\Bigg) \\
&+L_1\Bigg(a+\frac{a}{|\Lambda_1|}\Bigg(\sum_{i=1}^{m}\frac{\lambda_i\mu_i^{1-\gamma}}{\Gamma(2-\gamma)}+a\Bigg)\Bigg)\Bigg]\|w-w^*\|_{\mathfrak{F}'_a}.
\end{aligned}
$$

Therefore,

$$\|\mathcal{P}w-\mathcal{P}w^*\|_{\mathfrak{F}'_a}=\sup_{t\in[0,a]}\big|\mathcal{P}w(t)-\mathcal{P}w^*(t)\big|\le\eta_a(L_1\Lambda_3+L_2\Lambda_2)\|w-w^*\|_{\mathfrak{F}'_a},$$

and hence $\mathcal{P}$ is a contraction. Consequently, by the contraction mapping principle, $\mathcal{P}$ has a unique fixed point, which is indeed the unique solution to problem (1.1) on $(-\infty,a]$. □

Remark 3.1 It should be noted that we have needed to assume stronger criteria for the uniqueness result, Theorem 3.3, than the conditions for the existence result, Theorem 3.1. So, in addition to imposing f to satisfy the Lipschitz condition, condition (3.8) has an extra term on the left-hand side of the inequality compared with condition (3.7), and it still has to be less than 1.

3.1 Examples

Let us consider the following problem:

$$\begin{cases} {}^{C}D_{0^+}^{3/2}[u(t) - \int_0^t h(s,u_s)\,ds] = f(t,u_t), & t \in \Omega := [0,2], \\ u(t) = \theta(t), \quad t \in (-\infty, 0], \\ u(2) = \frac{1}{2}\,{}^{C}D_{0^+}^{1/2}u(5/4) + {}^{C}D_{0^+}^{1/2}u(5/3) + 2, \end{cases} \tag{3.9}$$

where $\delta = 3/2$, $\gamma = 1/2$, $m = 2$, $t \in [0,2]$, $\mu_1 = 5/4$, $\mu_2 = 5/3$, $\lambda_1 = 1/2$, $\lambda_2 = 1$, $\zeta = 2$, and $h(t,u_t), f(t,u_t), \theta(t)$ will be fixed later.

Using the given data, we find that $\Lambda_1 = -0.087514371$, $\Lambda_2 = 103.1250204$, and $\Lambda_3 = 95.41355769$, where Λ_1, Λ_2, and Λ_3 are respectively given by (2.2), (3.5), and (3.6).

For a continuous function $g : (-\infty, 0] \to [0,\infty)$ satisfying $l = \int_{-\infty}^0 g(s)\,ds < \infty$, define the space $\mathfrak{F}_g = \{u \in C((-\infty,0],\mathbb{R}) : \int_{-\infty}^0 g(s)\|u\|_{[s,0]}\,ds < \infty\}$, where $\|u\|_{[s,0]} = \sup_{t\in[s,0]} |u(t)|$. Choose $g(s) = e^{3s}$ such that $\int_{-\infty}^0 e^{3s}\,ds = \frac{1}{3}$, and supplement this space with the norm $\|u\|_{\mathfrak{F}_g} = \int_{-\infty}^0 g(s)\|u\|_{[s,0]}\,ds$. Then the space $(\mathfrak{F}_g, \|.\|_{\mathfrak{F}_g})$ satisfies the phase space's axioms with $\eta(t) = \frac{1}{3}$, $\rho(t) = 1$, $A = 3$ as the following:

Let $u : (-\infty, a] \to \mathbb{R}$ be such that $u_0 \in \mathfrak{F}_g$. Then

$$\begin{aligned} \int_{-\infty}^0 e^{3s}\|u_t\|_{[s,0]}\,ds &= \int_{-\infty}^0 e^{3s} \sup_{w\in[s,0]} \bigl|u(t+w)\bigr|\,ds \\ &= \int_{-\infty}^0 e^{3s} \sup_{v\in[s+t,t]} \bigl|u(v)\bigr|\,ds \\ &= \int_{-\infty}^{-t} e^{3(s-t)} \sup_{v\in[s,0]} \bigl|u(v)\bigr|\,ds \\ &\le \int_{-\infty}^0 e^{3(s-t)} \sup_{v\in[s,0]} \bigl|u(v)\bigr|\,ds \\ &= e^{-3t}\int_{-\infty}^0 e^{3s}\|u_0\|_{[s,0]}\,ds < \infty, \quad \text{which implies } u_t \in \mathfrak{F}_g. \end{aligned}$$

Next, to show that

$$\|u_t\|_{\mathfrak{F}_g} \le \frac{1}{3}\sup\bigl\{\bigl|u(\tau)\bigr| : 0 \le \tau \le t\bigr\} + \|u_0\|_{\mathfrak{F}_g},$$

we have, for $-\infty < s \le 0$, the following cases:

For $s \le t + w \le 0$, we find

$$\bigl|u_t(w)\bigr| = \bigl|u(t+w)\bigr| \le \sup_{\tau\in[s,0]} \bigl|u(\tau)\bigr|.$$

If $t + w \ge 0$, $w \le 0$, then we have

$$\bigl|u_t(w)\bigr| = \bigl|u(t+w)\bigr| \le \sup_{\tau\in[0,t]} \bigl|u(\tau)\bigr|.$$

Thus, for $t \in [0,a]$, we have $|u_t(w)| \le \sup_{\tau\in[s,0]} |u(\tau)| + \sup_{\tau\in[0,t]} |u(\tau)|$.

Consequently, for $t \in [0,a]$, we have

$$\begin{aligned}
\|u_t\|_{\mathfrak{F}_g} &= \int_{-\infty}^{0} e^{3s} \sup_{w\in[s,0]} \left|u_t(w)\right| ds \\
&\leq \int_{-\infty}^{0} e^{3s} \sup_{\tau\in[0,t]} \left|u(\tau)\right| ds + \int_{-\infty}^{0} e^{3s} \sup_{\tau\in[s,0]} \left|u(\tau)\right| ds \\
&= \int_{-\infty}^{0} e^{3s}\, ds \sup_{\tau\in[0,t]} \left|u(\tau)\right| + \int_{-\infty}^{0} e^{3s} \|u_0\|_{[s,0]}\, ds \\
&= \frac{1}{3} \sup\left\{\left|u(\tau)\right| : 0 \leq \tau \leq t\right\} + \|u_0\|_{\mathfrak{F}_g}.
\end{aligned}$$

Finally, we find

$$\left|u(t)\right| \leq \sup_{w\in[s,0]} \left|u(t+w)\right| \leq 3 \int_{-\infty}^{0} e^{3s} \|u_t\|_{[s,0]}\, ds = 3\|u_t\|_{\mathfrak{F}_g}.$$

Now, we choose $\theta(t)$ to be $\theta(t) = e^t - e^{2t}$, which is a continuous function and satisfies $\theta(0) = 0$. Also, it is easy to show that $\theta \in \mathfrak{F}_g$, that is, $\int_{-\infty}^{0} e^{3s} \|\theta\|_{[s,0]}\, ds < \infty$.

To illustrate Theorem 3.1, we choose

$$f(t,u_t) = \frac{(1+t)}{120} \left(\int_{-\infty}^{0} e^{3s} \tan^{-1} u_t\, ds + \frac{e^t}{8} \right) \tag{3.10}$$

and

$$h(t,u_t) = \frac{1}{4\sqrt{400+t}} \left(t \int_{-\infty}^{0} e^{3s} \frac{|u_t|}{|u_t|+1}\, ds + \sin t \right). \tag{3.11}$$

Obviously, f and h are continuous functions, and conditions (H_1) and (H_2) are satisfied with $L_1 = 1/40$, $\kappa_1(t) = \frac{(1+t)}{120}(\frac{1}{3} + \frac{e^t}{8})$ and $\kappa_2(t) = \frac{(\frac{t}{3}+\sin t)}{4\sqrt{400+t}}$. Moreover,

$$L_1 \eta_a \Lambda_3 \approx 0.7951129808 < 1.$$

Thus, all the hypotheses of Theorem 3.1 are satisfied, and consequently, problem (3.9) has at least one solution on $(-\infty, 2]$, with $f(t,u_t)$ and $h(t,u_t)$ given by (3.10) and (3.11), respectively.

Next, to demonstrate the application of Theorem 3.2, we take

$$f(t,u_t) = \frac{e^t}{(255+t)^2} \left(\int_{-\infty}^{0} e^{3s} \sin u_t\, ds + \cos t \right) \tag{3.12}$$

and

$$h(t,u_t) = \frac{1}{90(2+t^2)} \sin t \int_{-\infty}^{0} e^{3s} u_t\, ds + \frac{t}{255}. \tag{3.13}$$

Clearly, conditions (A_1) and (A_2) hold true with $C_1 = \frac{1}{180}$, $C_2 = \frac{2}{255}$, $\alpha(t) = \frac{e^t}{(255+t)^2}$, and $\vartheta(\|u\|_{\mathfrak{F}_g}) = \|u\|_{\mathfrak{F}_g} + 1$. Also, by condition (A_3), we have $\mathcal{W} > 56.758894$ such that

$$\frac{(1-\eta_a C_1\Lambda_3)\mathcal{W}}{(C_2 + C_1\rho_a\|\theta\|_{\mathfrak{F}})\Lambda_3 + \vartheta(\eta_a\mathcal{W} + \rho_a\|\theta\|_{\mathfrak{F}})\alpha^*\Lambda_2 + \frac{a}{|\Lambda_1|}|\zeta|} > 1.$$

As all the assumptions of Theorem 3.2 hold true, its conclusion applies to problem (3.9) on $(-\infty, 2]$ with $f(t,u_t)$ and $h(t,u_t)$ given by (3.12) and (3.13), respectively.

Finally, Theorem 3.3 can be illustrated by taking

$$f(t,u_t) = \frac{e^t}{(t+25)^2}\left(\int_{-\infty}^{0} e^{3s}\tan^{-1} u_t\, ds + 1/16\right) \tag{3.14}$$

and

$$h(t,u_t) = \frac{t}{6\sqrt{900+t}}\left(\int_{-\infty}^{0} e^{3s}\frac{|u_t|}{|u_t|+1}\, ds + \tan^{-1} t\right). \tag{3.15}$$

Notice that conditions (H_1) and (H_3) are satisfied with $L_1 = 1/90$ and $L_2 = e^2/625$. In addition, $\eta_a(L_1\Lambda_3 + L_2\Lambda_2) \approx 0.7597817128 < 1$. So, all the conditions of Theorem 3.3 hold true, and as a result, problem (3.9) with $f(t,u_t)$ given by (3.14) and (3.15) has a unique solution on $(-\infty, 2]$.

4 Conclusions

In this article, we have investigated the existence of solutions to a new class of neutral boundary value problems with infinite delay. By imposing an arbitrary phase space that satisfies the fundamental axioms given by Hale and Kato [9] and applying Krasnoselskii's fixed point theorem, the Leray–Schauder type nonlinear alternative theorem, and the Banach fixed point theorem, we have presented three results related to our problem. Also, we have illustrated our results by giving three examples defined on a specific state space. Our results are a new contribution that enriches the literature on delayed fractional order boundary value problems, whereas most of the previous studies on this topic were devoted to differential equations of fractional order between 0 and 1, and to the best of our knowledge, no work has been done on boundary value problems with infinite delay and boundary conditions that involve Caputo fractional derivative; see, for example, [16–19, 23–28].

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