https://doi.org/10.1186/s13662-023-03780-6

Advances in Continuous and Discrete Models

RESEARCH Open Access

On positively invariant polyhedrons for discrete-time positive linear systems

ChengDan Wang[1] and HongLi Yang[1*]

*Correspondence: yhlmath@126.com
[1] Mathematics and Systems Sciences College, Shandong University of Science and Technology, Qingdao, 266590, China

Abstract

In this paper, necessary and sufficient conditions for the polyhedron set to be a positively invariant polyhedron of a discrete-time positive linear system subject to external disturbances are established. By solving a set of inequalities, which is also a linear programming, necessary and sufficient conditions for the existence of positive invariant polyhedra for discrete-time positive linear systems are proposed, and the relationship between Lyapunov stability and positively invariant polyhedron is also investigated, numerical examples illustrate our results.

Keywords: Positive invariant set; Positive discrete-time systems; Positive invariance; Stability; Linear programming

1 Introduction

Positive invariance in control theory of dynamical systems has received extensive attention over the past few decades [1]. Any state trajectory emanating from a set in the state space still remains within the set, such a set is called a positively invariant set. Invariant sets, especially positively invariant sets, play an important role in the theory and application of dynamical systems. Problems related to disturbance rejection can be analyzed and solved with the help of positively invariant sets [2]. Similarly, many constrained control problems of dynamical systems can also be represented and solved by positively invariant sets [3].

For discrete-time linear systems, [4] and [5, 6] give descriptions of necessary and sufficient algebraic conditions for the positive invariance of convex polyhedra under both unperturbed and bounded perturbations, respectively. In the form of linear relationship, a set of inequalities is derived, and the invariant set of related systems is defined by the method of linear programming [7]. There are also many studies on the computational methods of invariant sets [8], and a different linear programming algorithm is proposed in [9] to give sufficient and necessary conditions for any set of polyhedrons to be positively invariant sets for discrete-time linear systems. However, since the algorithm is limited, some algorithms are not suitable for computing all polyhedron positively invariant sets. Daniel Rubin et al. proposed a special supplementary algorithm [10], and they also proposed a new algorithm to compute the polyhedron positively invariant set [11]. Disturbance is also a common problem in the research and analysis of dynamical systems. Reference [12] generalizes not only the concept of self-bounded (A, B) invariant subspaces to sets of convex polyhedra for general discrete-time systems, but also their results to systems subject to control constraints and bounded additive disturbances. A solution to the problem of computing a robustly positively invariant outer approximation of the minimal robustly positively invariant set for a discrete-time linear time-invariant system is proposed in [13]. An algorithm for computing the maximal robustly positively invariant set is described, and sufficient conditions for finite termination of this algorithm are given [14]. [15] presents an algorithm for the computation of full-complexity polytopic robust control invariant sets, which can be extended to linear discrete-time systems subject to additive disturbances and structured norm-bounded or polytopic uncertainties.

For positive systems, their stability has been extensively studied [16]. [17] studied the stability and control problems of positive delayed systems. [18] studied the synthesis problem of interval positive linear systems. About the problems investigated in this paper for positive continuous-time linear systems, reference [19] gives excellent research results, related results can also be found in the references therein. The main contribution of this paper is to give necessary and sufficient conditions for the existence of positively invariant polyhedra for discrete-time positive linear systems by solving a set of linear programming. The same method is applied to discrete-time positive linear systems with external inputs to obtain the conditions for the existence of robustly positively invariant polyhedra. In this paper, some properties of regular invariant polyhedra are elucidated, and their relations with Lyapunov stability are investigated.

The rest of the paper is organized as follows. Section 2 presents a preliminary case of discrete-time positive linear systems. Section 3 defines the positively invariant polyhedron and reveals the close connection between the Lyapunov stability and the positively invariant polyhedron. Section 4 establishes the necessary and sufficient condition for the existence of robustly positively invariant polyhedra under two external input conditions.

Throughout the paper, the following notations are used.

N, N_+	set of integers, set of positive integers
N_0	$\{0\} \cup N$
R, R^n	set of real numbers, set of n-dimensional real vectors
$R^{m\times n}$	set of $m \times n$ real matrices
$\bar{R}^n_+, R^n_+$	nonnegative and positive orthants of R^n
$\mathbf{1}, I$	vector $[1, 1, \dots, 1]^T$, identity matrix
$[\mathbf{1}]$	matrix with all entries assigned to 1
A^T	transpose of matrix A
$\\|x(k)\\|_1$	$\sum_{i=1}^n \|x(k)\|$
$\\|x(k)\\|_\infty$	$\max_{i=1}^n \|x_i(t)\|$
$\\|\omega(k)\\|_{\infty,1}$	$\max_{i=1}^n \\|\omega(k)\\|_1$
$\\|\omega(k)\\|_{\infty,\infty}$	$\max_{i=1}^n \\|\omega(k)\\|_\infty$

In this paper, capital letters denote real matrices and lower case letters denote column vectors of scalars. If $A = (a_{ij})$ is a real matrix, then $|A| = (|a_{ij}|)$, $A^+ = (a^+_{ij})$ with $a^+_{ij} = \max(a_{ij}, 0)$ and $A^- = (a^-_{ij})$ with $a^-_{ij} = \min(a_{ij}, 0)$. $x \geq 0$ denotes that every component of x is nonnegative, $A \geq 0$ denotes that every component of A is nonnegative. It is always assumed that all vectors and matrices have compatible dimensions without specification.

2 Preliminaries

In this section, some definitions and lemmas related to invariant sets of discrete-time linear systems are introduced. Consider a discrete-time linear dynamical system described by a difference equations in the following form:

$$S_0 : x(k+1) = Ax(k), \quad k \in N_0, \tag{1}$$

where $x(k) \in R^n$ is system state, $k \in N_0$ $N_0 = \{0\} \cup N$, and $A \in R^{n\times n}$ is a constant system state matrix.

Definition 1 Any nonempty convex polyhedron in R^n can be characterized by a matrix $G \in R^{r\times n}$ and a vector $\gamma \in R^r$, $r \in N_+$, $n \in N_+$, which is defined by

$$P[G,\gamma] = \{x \in R^n : Gx \le \gamma, G \in R^{r\times n}, \gamma \in R^r\}.$$

In particular, in this paper we mainly study the polyhedron described by a matrix $G \in R^{r\times n}$ and a vector $\gamma \in R^r_+$ ($\gamma_i > 0$) defined as

$$R[G,\gamma] = \{x \in R^n : -\gamma \le Gx \le \gamma, G \in R^{r\times n}, \gamma \in R^r_+\}.$$

And the polyhedron described by a matrix $G \in R^{r\times n}$ and two vectors $\gamma_1, \gamma_2 \in R^r_+$ ($\gamma_1 > 0, \gamma_2 > 0$) is defined as

$$Q[G,\gamma_1,\gamma_2] = \{x \in R^n : -\gamma_1 \le Gx \le \gamma_2, G \in R^{r\times n}, \gamma_1,\gamma_2 \in R^r_+\}.$$

Definition 2 A nonempty subset $M \in R^n$ is said to be a positively invariant set of system S_0 if for each initial state $x_0 \in M$ the motion emanating from x_0 remains in M.

From Definitions 1 and 2, one can derive that a nonempty polyhedron $R[G,\gamma]$ is positively invariant polyhedron for system S_0 if and only if

$$\begin{bmatrix} G \\ -G \end{bmatrix} A^k x_0 \le \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}$$

for any

$$\begin{bmatrix} G \\ -G \end{bmatrix} x_0 \le \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}, \quad k \in N_0.$$

Likewise, the polyhedron $Q[G,\gamma_1,\gamma_2]$ is a positively invariant polyhedron for system S_0 if and only if

$$\begin{bmatrix} G \\ -G \end{bmatrix} A^k x_0 \le \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}$$

for any

$$\begin{bmatrix} G \\ -G \end{bmatrix} x_0 \le \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}, \quad k \in N_0.$$

The following lemma proposed in [9] provides a sufficient and necessary algebraic condition for the positive invariance of $R[G,\gamma]$ and $Q[G,\gamma_1,\gamma_2]$.

Lemma 1 [9] *The polyhedron $P[G,\gamma]$ is a positively invariant polyhedron of system S_0 in* (1) *if and only if there exists a nonnegative matrix $H \in \overline{R}_+^{r\times r}$ such that*

$$GA - HG = 0,$$
$$(H - I)\gamma \leq 0.$$

3 Positive invariance and its relationship with stability

A linear system becomes a positive linear system when matrix A is nonnegative, that is,

$$S_1 : x(k+1) = Ax(k), \quad k \in N_0, \tag{2}$$

where $x(k) \in \overline{R}_+^n$ is system state and $A \in \overline{R}_+^{n\times n}$ is nonnegative, $k \in N_0$, $x_0 \geq 0$ is the initial state. A polyhedron with respect to system S_1 is characterized by

$$R^+[G,\gamma] = \{x \in \overline{R}_+^n : -\gamma \leq Gx \leq \gamma, G \in \overline{R}_+^{r\times n}, \gamma \in R_+^r\} \tag{3}$$

or

$$Q^+[G,\gamma_1,\gamma_2] = \{x \in \overline{R}_+^n : -\gamma_1 \leq Gx \leq \gamma_2, G \in \overline{R}_+^{r\times n}, \gamma_1,\gamma_2 \in R_+^r\}. \tag{4}$$

3.1 Conditions of a positively invariant set

Since A is nonnegative, $x(k) = A^k x_0 \in \overline{R}_+^n$ [20] and a necessary and sufficient condition for the existence of a positively invariant polyhedron $R^+[G,\gamma]$ and $Q^+[G,\gamma_1,\gamma_2]$ with respect to system S_1 can be derived from Lemma 1, as stated in the following theorem.

Theorem 1 *The nonempty set $R^+[G,\gamma]$ is a positively invariant polyhedron of system S_1 in* (2) *if and only if there exists a matrix $H \in R^{r\times r}$ such that*

$$GA - HG \leq 0,$$
$$\bigl(|H| - I\bigr)\gamma \leq 0.$$

Proof Only the necessary condition is proven. According to the description of $R^+[G,\gamma]$ in (3), it can be rewritten as

$$R^+[G,\gamma] = R[G,\gamma] \cap P[-I,0].$$

Observe that the polyhedral set $R[G,\gamma]$ can be written in the form

$$R[G,\gamma] = \left\{x \in R^n : \begin{bmatrix} G \\ -G \end{bmatrix} x \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}\right\},$$

then the polyhedral set $R^+[G,\gamma]$ can be written in the form

$$R^+[G,\gamma] = R[G,\gamma] \cap P[-I,0] = \left\{x \in R^n : \begin{bmatrix} G \\ -G \\ -I \end{bmatrix} x \leq \begin{bmatrix} \gamma \\ \gamma \\ 0 \end{bmatrix}\right\}.$$

From Lemma 1, $R^+[G,\gamma]$ is a positively invariant polyhedron of system S_1 in (2) if and only if there exists a nonnegative matrix $\overline{H} \in \overline{R}_+^{(2r+n)\times(2r+n)}$,

$$\overline{H} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

with $H_{11}, H_{12}, H_{21}, H_{22} \in \overline{R}_+^{r\times r}$,$H_{13}, H_{23} \in \overline{R}_+^{r\times n}$,$H_{31}, H_{32} \in \overline{R}_+^{n\times r}$,$H_{33} \in \overline{R}_+^{n\times n}$ such that

$$\begin{bmatrix} G \\ -G \\ -I \end{bmatrix} A - \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} G \\ -G \\ -I \end{bmatrix} = 0,$$

$$\left(\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} - I \right) \begin{bmatrix} \gamma \\ \gamma \\ 0 \end{bmatrix} \leq 0.$$

In particular,

$$GA - (H_{11} - H_{12})G + H_{13} = 0,$$
$$-A - (H_{31} - H_{32})G + H_{33} = 0,$$
$$(H_{11} + H_{12} - I)\gamma \leq 0,$$
$$(H_{31} + H_{32})\gamma \leq 0.$$

One can set $H_{31} = H_{32} = 0$ and $H_{33} = A$, without losing generality, which is equivalent to

$$GA - (H_{11} - H_{12})G + H_{13} = 0,$$
$$(H_{11} + H_{12} - I)\gamma \leq 0.$$

Now set $H = H_{11} - H_{12}$, then $|H| \leq H_{11} + H_{12}$,

$$GA - HG + H_{13} = 0,$$
$$\big(|H| - I\big)\gamma \leq 0.$$

Note that H_{13} is nonnegative, it concludes that

$$GA - HG \leq 0,$$
$$\big(|H| - I\big)\gamma \leq 0.$$

□

Remark 1 For any matrix H that satisfies the algebraic inequalities condition in Theorem 1, Theorem 1 guarantees the positive invariance of $R^+[G,\gamma]$ for system S_1 and does not have any requirements for the matrix G.

Note that the positively invariant polyhedron $R^+[G,\gamma]$ is symmetric in Theorem 1. Next we consider the more general case where the polyhedral sets $Q^+[G,\gamma_1,\gamma_2]$ are not symmetric. In the following theorem, we establish conditions for the positive invariance of polyhedral sets $Q^+[G,\gamma_1,\gamma_2]$ of system S_1.

Theorem 2 *The polyhedral set $Q^+[G,\gamma_1,\gamma_2]$ is a positively invariant polyhedron of system S_1 in (2) if and only if there exists a matrix $H \in R^{r\times r}$ such that*

$$GA - HG \leq 0,$$
$$\left(\begin{bmatrix} H^+ & -H^- \\ -(-H)^- & (-H)^+ \end{bmatrix} - I\right)\begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \leq 0.$$

Proof Only the necessary condition is proven. The polyhedral sets $Q^+[G,\gamma_1,\gamma_2]$ on the basis of observation and description of $Q^+[G,\gamma_1,\gamma_2]$ in (4) can be rewritten in the form

$$Q^+[G,\gamma_1,\gamma_2] = Q[G,\gamma_1,\gamma_2] \cap P[-I,0] = \left\{ x \in R^n : \begin{bmatrix} G \\ -G \\ -I \end{bmatrix} x \leq \begin{bmatrix} \gamma_2 \\ \gamma_1 \\ 0 \end{bmatrix} \right\}.$$

By virtue of Lemma 1, the positive invariance of $Q^+[G,\gamma_1,\gamma_2]$ implies the existence of a nonnegative matrix $\underline{H} \in \overline{R}_+^{(2r+n)\times(2r+n)}$,

$$\underline{H} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

with $H_{11},H_{12},H_{21},H_{22} \in \overline{R}_+^{r\times r}$,$H_{13},H_{23} \in \overline{R}_+^{r\times n}$,$H_{31},H_{32} \in \overline{R}_+^{n\times r}$,$H_{33} \in \overline{R}_+^{n\times n}$, such that

$$\begin{bmatrix} G \\ -G \\ -I \end{bmatrix} A - \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}\begin{bmatrix} G \\ -G \\ -I \end{bmatrix} = 0,$$
$$\left(\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} - I\right)\begin{bmatrix} \gamma_2 \\ \gamma_1 \\ 0 \end{bmatrix} \leq 0,$$

which can be rewritten as

$$GA - (H_{11} - H_{12})G + H_{13} = 0,$$
$$-GA - (H_{21} - H_{22})G + H_{23} = 0,$$
$$-A - (H_{31} - H_{32})G + H_{33} = 0,$$
$$(H_{11} - I)\gamma_2 + H_{12}\gamma_1 \leq 0,$$
$$H_{21}\gamma_2 + (H_{22} - I)\gamma_1 \leq 0,$$
$$H_{31}\gamma_2 + H_{32}\gamma_1 \leq 0.$$

One can set $H_{31} = H_{32} = 0$ and $H_{33} = A$, without losing generality, which is equivalent to

$$GA - (H_{11} - H_{12})G + H_{13} = 0, \tag{5}$$

$$-GA - (H_{21} - H_{22})G + H_{23} = 0, \tag{6}$$

$$(H_{11} - I)\gamma_2 + H_{12}\gamma_1 \leq 0, \tag{7}$$

$$H_{21}\gamma_2 + (H_{22} - I)\gamma_1 \leq 0. \tag{8}$$

Note that $H_{13} \geq 0$ and $H_{23} \geq 0$, from (5) and (6) it can be obtained that

$$GA - (H_{11} - H_{12})G \leq 0,$$

$$-GA - (H_{21} - H_{22})G \leq 0.$$

Now, setting $H = H_{11} - H_{12} = H_{21} - H_{22}$, we conclude that

$$GA - HG \leq 0.$$

From (7) and (8), which can be written as

$$H_{11}\gamma_2 + H_{12}\gamma_1 \leq \gamma_2,$$

$$H_{21}\gamma_2 + H_{22}\gamma_1 \leq \gamma_1,$$

then they can be rewritten as

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \leq \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}.$$

So,

$$\begin{aligned} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} &\geq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \\ &\geq \begin{bmatrix} (H_{11} - H_{12})^+ & -(H_{11} - H_{12})^- \\ -(H_{22} - H_{21})^- & (H_{22} - H_{21})^+ \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \\ &= \begin{bmatrix} H^+ & -H^- \\ -(-H)^- & (-H)^+ \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}, \end{aligned}$$

which are further equivalent to

$$\left(\begin{bmatrix} H^+ & -H^- \\ -(-H)^- & (-H)^+ \end{bmatrix} - I \right) \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \leq 0. \qquad \square$$

Vectors γ, γ_1 and γ_2 of $R^+[G,\gamma]$ and $Q^+[G,\gamma_1,\gamma_2]$ are positive respectively. A necessary and sufficient condition for the existence of a positively invariant polyhedron $Q^+[G,\gamma_1,\gamma_2]$ when $\gamma_1 = 0$ is given in the following corollary. The proof is omitted since it is similar to the proof of Theorem 2.

Corollary 1 *The set $Q^+[G,0,\gamma_2]$ is a positively invariant polyhedron of system S_1 in (2) if and only if there exists a matrix $H \in R^{r\times r}$ such that*

$$GA - HG \leq 0,$$
$$(H - I)\gamma \leq 0.$$

The positively invariant polyhedron $R^+[G,\gamma]$ of system S_1 can also be constructed by an invariant polyhedron of similar systems

$$S_1^* : y(k+1) = T^{-1}ATy(k) \tag{9}$$

with nonsingular matrix $T \in R^{n\times n}$.

Theorem 3 *Let A be nonnegative, $G \in \overline{R}_+^{r\times n}$, $\gamma \in R_+^r$. $R^+[G,\gamma]$ is a positively invariant polyhedron of positive system S_1 in (2) if and only if $R[GT,\gamma] \cap P[-T,0]$ is a positively invariant polyhedron of system S_1^* in (9).*

Proof Necessity. Since $R^+[G,\gamma]$ is a positively invariant polyhedron of system S_1, it follows that

$$\begin{bmatrix} G \\ -G \end{bmatrix} A^k x_0 \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}$$

for any $x_0 \in \overline{R}_+^n$ satisfying

$$\begin{bmatrix} G \\ -G \end{bmatrix} x_0 \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}, \quad k \in N_0.$$

By the transformation of state $y(k) = T^{-1}x(k) \in R^n$, it follows that

$$\begin{aligned} \begin{bmatrix} G \\ -G \end{bmatrix} Ty(k) &= \begin{bmatrix} G \\ -G \end{bmatrix} T\left(T^{-1}AT\right)^k y_0 \\ &= \begin{bmatrix} G \\ -G \end{bmatrix} TT^{-1}A^k Ty_0 \\ &= \begin{bmatrix} G \\ -G \end{bmatrix} A^k Ty_0 = \begin{bmatrix} G \\ -G \end{bmatrix} A^k x_0 \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} \end{aligned}$$

for any $y_0 \in R^n$ satisfying

$$\begin{bmatrix} G \\ -G \end{bmatrix} Ty_0 = \begin{bmatrix} G \\ -G \end{bmatrix} x_0 \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}.$$

Since $Ty(k) = x(k) \geq 0$ implies $-Ty(k) \leq 0$. Therefore, $R[GT,\gamma] \cap P[-T,0]$ is a positively invariant polyhedron of system S_1^*.

Sufficiency. By a similarity transformation of state $x(k) = Ty(k)$, $x(k)$ satisfies the equation $x(k+1) = Ax(k)$ with a nonnegative matrix A. Consequently, $x(k) \in \overline{R}_+^n$. Since

$$\begin{bmatrix} G \\ -G \end{bmatrix} x_0 = \begin{bmatrix} G \\ -G \end{bmatrix} Ty_0 \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}$$

and

$$\begin{bmatrix} G \\ -G \end{bmatrix} x_k = \begin{bmatrix} G \\ -G \end{bmatrix} A^k x_0 = \begin{bmatrix} G \\ -G \end{bmatrix} T\left(T^{-1}AT\right)^k y_0 = \begin{bmatrix} G \\ -G \end{bmatrix} Ty(k) \leq \begin{bmatrix} \gamma \\ \gamma \end{bmatrix},$$

$R[GT,\gamma] \cap P[-T,0]$ is a positively invariant polyhedron of system S_1^* implies $R^+[G,\gamma]$ is a positively invariant polyhedron of positive system S_1. □

Remark 2 Since $T^{-1}AT$ is not necessarily a nonnegative matrix, system S_1^* may no longer be a positive system. Theorem 3 clarifies the connection between the invariant polyhedron construction of a positive system and a general system.

Meanwhile, the above conclusion is also satisfied for $Q^+[G,\gamma_1,\gamma_2]$ as shown in the following corollary. The proof is omitted.

Corollary 2 *$Q^+[G,\gamma_1,\gamma_2]$ is a positively invariant polyhedron of positive system S_1 in (2) if and only if $Q[GT,\gamma_1,\gamma_2] \cap P[-T,0]$ is a positively invariant polyhedron of system S_1^* in (9).*

Remark 3 In the case of $\gamma_1 = 0$, the conclusion that $Q^+[G,0,\gamma_2]$ is a positively invariant polyhedron of positive system S_1 if and only if $Q[GT,0,\gamma_2] \cap P[-T,0]$ is a positively invariant polyhedron of system S_1^* is also valid.

3.2 Relation with stability

A well-known result in [4] is that if system S_0 is asymptotically stable, then it possesses positively invariant sets of the form

$$E(P,c) = \left\{x \in R^n : x^T Px \leq c\right\},$$

where $P \in R^{n\times n}$ is a symmetric positive-definite matrix and c is a positive real number. Furthermore, for a symmetric and positive-definite matrix $P \in R^{n\times n}$, the corresponding hyperellipsoid is a positively invariant set of system S_0 if and only if there exists a positive semidefinite matrix $Q \in R^{n\times n}$ such that $A^T PA - P = -Q$.

For the positive system S_1, the following theorem reveals the close connection between the Lyapunov stability and the existence of a positively invariant polyhedron.

Theorem 4 *Positive system S_1 in (2) possesses at least a positively invariant polyhedron $R^+[G,\gamma]$ with nonzero vector $\gamma \in R_+^n$ if and only if system S_1 is Lyapunov stable.*

Proof Necessity. Since $R^+[G,\gamma]$ is a closed convex set, it can be defined by the expression

$$R^+[G,\gamma] = \left\{x(k) \in R_+^n : V(x) \leq 1\right\},$$

where

$$V(x) = \max_{1\le i\le r}\left\{\frac{|(Gx)_i|}{\gamma_i}\right\},$$

then $|(Gx)_i| \le \gamma_i V(x)$ for any $i = 1, 2, \dots, r$ and $V(x) > 0$ for all $x \neq 0$.

$$\triangle V(x) = V(Ax) - V(x) = \max_{1\le i\le r}\left\{\frac{|(GAx)_i|}{\gamma_i}\right\} - \max_{1\le i\le r}\left\{\frac{|(Gx)_i|}{\gamma_i}\right\}.$$

From Theorem 1, there must exist a matrix $H \in R^{r\times r}$ such that $GA - HG \le 0$ and $(|H| - I)\gamma \le 0$. Accordingly,

$$\left|(GAx)_i\right| \le \left|(HGx)_i\right| = |H|\left|(Gx)_i\right| \le |H|\gamma_i V(x) \le \gamma_i V(x),$$

which implies that $V(Ax) \le V(x)$, that is, $\triangle V(x) \le 0$. Hence system S_1 is Lyapunov stable.

Sufficiency. Since system S_1 is Lyapunov stable, there must exist a nonzero vector $\gamma \in R_+^n$ such that $(A - I)\gamma \le 0$, which is equivalent to $[(A - I)\gamma]_i \le 0$. Due to A is a Schur matrix with all the eigenvalues in absolute value smaller than 1, that is, $\rho(A) < 1$ [20], then $A^k\gamma \le \gamma$, $k \in N_0$. Furthermore, taking into account the fact that A is nonnegative and $A^k \ge 0$, if $x_1 \le x_2$, then

$$x(k+1) = A^k x_1 \le A^k x_2 = x(k+2)$$

for all $k \in N_0$. Therefore, if $-\gamma \le Gx_0 \le \gamma$, then

$$-\gamma \le x(k) = GA^k x_0 \le A^k\gamma \le \gamma,$$

that is, $R^+[G,\gamma]$ is a positively invariant polyhedron of system S_1. □

Theorem 5 *Positive system S_1 in (2) possesses at least a positively invariant polyhedron $Q^+[G,\gamma_1,\gamma_2]$ with nonzero vector $\gamma_1, \gamma_2 \in R_+^n$ if and only if system S_1 is Lyapunov stable.*

Proof Necessity. Since $Q^+[G,\gamma_1,\gamma_2]$ is a closed convex set, it can be defined by the expression

$$Q^+[G,\gamma_1,\gamma_2] = \left\{x(k) \in R_+^n : V^*(x) \le 1\right\},$$

where

$$V^*(x) = \max_{1\le i\le r}\left\{\max\left(\frac{(Gx)_i}{(\gamma_2)_i}, -\frac{(Gx)_i}{(\gamma_1)_i}\right)\right\},$$

then $(Gx)_i \le (\gamma_2)_i V(x)$ for any $i = 1, 2, \dots, r$ and $V^*(x) > 0$ for all $x \neq 0$.

$$\begin{aligned}\triangle V^*(x) &= V^*(Ax) - V^*(x)\\ &= \max_{1\le i\le r}\left\{\max\left(\frac{(GAx)_i}{(\gamma_2)_i}, -\frac{(GAx)_i}{(\gamma_1)_i}\right)\right\} - \max_{1\le i\le r}\left\{\max\left(\frac{(Gx)_i}{(\gamma_2)_i}, -\frac{(Gx)_i}{(\gamma_1)_i}\right)\right\}.\end{aligned}$$

From Theorem 2, there must exist a matrix $H \in R^{r\times r}$ such that

$$GA - HG \leq 0,$$
$$\left(\begin{bmatrix} H^+ & -H^- \\ -(-H)^- & (-H)^+ \end{bmatrix} - I\right)\begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix} \leq 0.$$

Accordingly,

$$\begin{aligned}
(GAx)_i &\leq (HGx)_i \\
&= H(Gx)_i \\
&\leq H(\gamma_2)_i V^*(x) \\
&\leq \left(H^+ + H^-\right)(\gamma_2)_i V^*(x) \\
&= H^+(\gamma_2)_i V^*(x) + H^-(\gamma_2)_i V^*(x) \\
&\leq H^+(\gamma_2)_i V^*(x) - H^-(\gamma_1)_i V^*(x) \\
&= \left(H^+(\gamma_2)_i - H^-(\gamma_1)_i\right) V^*(x) \\
&\leq (\gamma_2)_i V^*(x),
\end{aligned}$$

which implies that $V(Ax) \leq V(x)$, that is, $\triangle V(x) \leq 0$. Hence system S_1 is Lyapunov stable.

Sufficiency can be obviously evaluated from the sufficient proof of Theorem 4 by assigning $-\gamma_1 = -\gamma$ and $\gamma_2 = \gamma$. □

4 Invariant polyhedron with exogenous inputs

In this section, we consider a positively invariant polyhedron for discrete-time positive linear systems with external inputs. Consider a discrete-time linear dynamical system described by the difference equations

$$\begin{aligned}
&S_2 : x(k+1) = Ax(k) + B_\omega \omega(k), \\
&\omega(k) \in \Omega \subset R_+^m,
\end{aligned} \tag{10}$$

where $x(k) \in \overline{R}_+^n$ is system state, $\omega(k) \in \overline{R}_+^m$ is an exogenous input signal, nonnegative matrix $A \in \overline{R}_+^{n\times n}$, nonzero matrix $B_\omega \in \overline{R}_+^{n\times m}$, Ω is a closed convex set, and $k \in N_0$.

A nonempty polyhedron $R^+[G,\gamma]$ is said to be a robustly positively invariant polyhedron of system S_2 with respect to Ω if for each initial state $x_0 \in R^+[G,\gamma]$ the motion emanating from x_0 remains in $R^+[G,\gamma]$ for all possible $\omega(k) \in \Omega$. When $\Omega = \{0\}$, the positive invariance is equivalent to the definition of positive invariance characterized in Sect. 3. The necessary and sufficient condition for the existence of positively invariant polyhedra $R^+[G,\gamma]$ based on $(\infty,1)$-norm is given below.

When $\omega(k) \in \Omega_{\infty,1} \overset{\triangle}{=} \{\omega(k) \in \overline{R}_+^m \mid \|\omega(k)\|_{\infty,1} \leq 1\}$, an interpretation of $\|\omega(k)\|_{\infty,1} \leq 1$ is that the sum of the components of $\omega(k)$ is not to exceed 1.

Theorem 6 *The polyhedron $R^+[G,\mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,1}$ if and only if there exists a matrix $H \in R^{r\times r}$ such that*

$$GA - HG \leq 0,$$

$$GB_\omega + \big(|H| - I\big)[\mathbf{1}] \leq 0,$$

in which $[\mathbf{1}]$ *is an* $r \times m$*-dimensional matrix with all elements being* 1.

Proof Necessity. An augmented system can be constructed from system S_2 in (10) as follows:

$$\begin{bmatrix} x(k+1) \\ \dot{\omega}(k) \end{bmatrix} = \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}.$$

The constraints $x(k) \in R^+[G, \mathbf{1}]$ and $\omega(k) \in \Omega_{\infty,1}$ can be rewritten as $\left[\begin{smallmatrix} x(k) \\ \omega(k) \end{smallmatrix}\right] \in \Pi_{\infty,1}$, which is defined as

$$\Pi_{\infty,1} \triangleq \left\{ \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \in \overline{R}_+^{n+m} : -\mathbf{1} \leq \begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \leq \mathbf{1} \right\}.$$

Emanating from any $\left[\begin{smallmatrix} x_0 \\ \omega_0 \end{smallmatrix}\right] \in \Pi_{\infty,1}$, where x_0 and ω_0 are the initial state and the disturbance vector of system S_2, there must exist a matrix

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in R^{(r+1)\times(r+1)}$$

with $H_{11} \in R^{r\times r}$, $H_{12} \in R^{r\times 1}$, $H_{21} \in R^{1\times r}$, $H_{22} \in R^1$ such that

$$\begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \leq 0,$$
$$\left(\left| \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \right| - I \right) \mathbf{1} \leq 0.$$

After a few algebraic manipulations that is identical to

$$GA - H_{11}G \leq 0, \tag{11}$$
$$GB_\omega - H_{12}\mathbf{1}^T \leq 0, \tag{12}$$
$$\big(|H_{11}| - I\big)\mathbf{1} + |H_{12}| \leq 0. \tag{13}$$

One can get the relationship as follows from the last two inequalities (12) and (13):

$$\max_{1\leq j\leq m} \big\{(GB_\omega)_{ij}\big\} \leq (H_{12})_i \leq -\big[\big(|H_{11}| - I\big)\mathbf{1}\big]_i,$$

which derives

$$GB_\omega + \big(|H_{11}| - I\big)[\mathbf{1}] \leq 0.$$

Now set $H = H_{11}$, one can get the conditions in the theorem as follows:

$$GA - HG \leq 0,$$
$$GB_\omega + \big(|H| - I\big)[\mathbf{1}] \leq 0.$$

Sufficiency. Denote a new variable

$$\mu(k) \triangleq \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix},$$

which is followed by a dynamical equation as follows:

$$\begin{aligned}
\mu(k+1) &= \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} GA & GB_\omega \\ -GA & -GB_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&\leq \begin{bmatrix} HG & H'\mathbf{1}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} H & 0 & H' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} H & 0 & H' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu(k),
\end{aligned}$$

with $H' = -(|H| - I)\mathbf{1} \geq 0$, since $-(|H| - I)[\mathbf{1}] \geq GB_\omega \geq 0$. And it satisfies

$$\left(\begin{bmatrix} H & 0 & H' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - I \right) \mathbf{1} \leq 0.$$

It follows from Lemma 1 in [9], in which G is assigned to the identity matrix I, that $\mu(k) \leq \mathbf{1}$ for any

$$\mu_0 = \begin{bmatrix} Gx_0 \\ -Gx_0 \\ \mathbf{1}^T \omega_0 \end{bmatrix} \leq \mathbf{1}.$$

Hence, $R^+[G, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,1}$. □

Theorem 7 *The polyhedron $Q^+[G, 0, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,1}$ if and only if there exists a matrix $H \in R^{r\times r}$ such that*

$$GA - HG \leq 0,$$

$$GB_\omega + \big(H^+ - I\big)[\mathbf{1}] \leq 0.$$

Proof Necessity. An augmented system can be established from system S_2 in (10) as follows:

$$\begin{bmatrix} x(k+1) \\ \dot{\omega}(k) \end{bmatrix} = \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}.$$

The constraints $x(k) \in Q^+[G,0,\mathbf{1}]$ and $\omega(k) \in \Omega_{\infty,1}$ can be rewritten as $\left[\begin{smallmatrix} x(k) \\ \omega(k) \end{smallmatrix}\right] \in \Pi_{\infty,1}$ defined as

$$\Pi_{\infty,1} \triangleq \left\{ \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \in \overline{R}_+^{n+m} : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq \begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\}.$$

Emanating from any $\left[\begin{smallmatrix} x_0 \\ \omega_0 \end{smallmatrix}\right] \in \Pi_{\infty,1}$, where x_0 and ω_0 are the initial state and the disturbance vector of system S_2, there must exist a matrix

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in R^{(r+1)\times(r+1)}$$

with $H_{11} \in R^{r\times r}$, $H_{12} \in R^{r\times 1}$, $H_{21} \in R^{1\times r}$, $H_{22} \in R^1$, such that

$$\begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \leq 0,$$
$$\left(\begin{bmatrix} H_{11}^+ & H_{12}^+ & -H_{11}^- & -H_{12}^- \\ H_{21}^+ & H_{22}^+ & -H_{21}^- & -H_{22}^- \\ -(-H_{11})^- & -(-H_{12})^- & (-H_{11})^+ & (-H_{12})^+ \\ -(-H_{21})^- & -(-H_{22})^- & (-H_{21})^+ & (-H_{22})^+ \end{bmatrix} - I \right) \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} \leq 0.$$

After a few algebraic manipulations, it can be obtained

$$GA - H_{11}G \leq 0, \tag{14}$$
$$GB_\omega - H_{12}\mathbf{1}^T \leq 0, \tag{15}$$
$$\left(H_{11}^+ - I\right)\mathbf{1} + H_{12}^+ \leq 0. \tag{16}$$

One can obtain the relationship as follows from the last two inequalities (15) and (16):

$$\max_{1\leq j\leq m} \left\{ (GB_\omega)_{ij} \right\} \leq (H_{12})_i \leq -\left[\left(H_{11}^+ - I\right)\mathbf{1} \right]_i,$$

which derives

$$GB_\omega + \left(H_{11}^+ - I\right)[\mathbf{1}] \leq 0.$$

One can obtain the conditions in the form of theorem as follows:

$$GA - HG \leq 0,$$
$$GB_\omega + \left(H^+ - I\right)[\mathbf{1}] \leq 0.$$

Sufficiency. Denote a new variable

$$\mu'(k) \triangleq \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix},$$

which is followed by a dynamic equation

$$\begin{aligned} \mu'(k+1) &= \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\ &= \begin{bmatrix} GA & GB_\omega \\ -GA & -GB_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\ &\leq \begin{bmatrix} HG & H''\mathbf{1}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\ &= \begin{bmatrix} H & 0 & H'' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & 0 \\ 0 & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\ &= \begin{bmatrix} H & 0 & H'' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu'(k), \end{aligned}$$

with $H'' = -(H^+ - I)\mathbf{1} \geq 0$ since $-(H^+ - I)[\mathbf{1}] \geq GB_\omega \geq 0$. And it satisfies

$$\left(\begin{bmatrix} H & 0 & H'' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - I \right) \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \end{bmatrix} \leq 0.$$

It follows from Lemma 1 in [9], in which G is assigned to the identity matrix I, that $\mu'(k) \leq [\mathbf{1}, 0, \mathbf{1}]^T$ for any

$$\mu'_0 = \begin{bmatrix} Gx_0 \\ -Gx_0 \\ \mathbf{1}^T\omega_0 \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \end{bmatrix}.$$

Hence, the polyhedron $Q^+[G, 0, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,1}$. □

When $\omega(k) \in \Omega_{\infty,\infty} \triangleq \{\omega(k) \in \overline{R}_+^m \mid \|\omega(k)\|_{\infty,\infty} \leq 1\}$, $\|\omega(k)\|_{\infty,\infty} \leq 1$ means that each component of $\omega(k)$ is not to exceed 1.

Theorem 8 *The polyhedron $R^+[G, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,\infty}$ if and only if there exist two matrices $H_1 \in R^{r\times r}$ and $H_2 \in R^{r\times m}$*

such that

$$GA - H_1 G \leq 0,$$
$$GB_\omega - H_2 \leq 0,$$
$$\big(|H_1| - I\big)\mathbf{1} + |H_2|\mathbf{1} \leq 0.$$

Proof Necessity. Consider the following system derived from system S_2:

$$\begin{bmatrix} x(k+1) \\ \dot{\omega}(k) \end{bmatrix} = \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}.$$

The constraints $x(k) \in R^+[G, \mathbf{1}]$ and $\omega(k) \in \Omega_{\infty,\infty}$ are identical to $\left[\begin{smallmatrix} x(k) \\ \omega(k) \end{smallmatrix}\right] \in \Pi_{\infty,\infty}$ defined as

$$\Pi_{\infty,\infty} \triangleq \left\{ \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \in \overline{R}_+^{n+m} : -\mathbf{1} \leq \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \leq \mathbf{1} \right\}.$$

Similar to the proof of Theorem 6, by virtue of Theorem 1, there must exist a matrix

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in R^{(r+m)\times(r+m)}$$

with $H_{11} \in R^{r\times r}$, $H_{12} \in R^{r\times m}$, $H_{21} \in R^{m\times r}$, $H_{22} \in R^{m\times m}$, such that

$$\begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \leq 0,$$
$$\left(\left| \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \right| - I \right) \mathbf{1} \leq 0.$$

After a few algebraic manipulations that is identical to

$$GA - H_{11} G \leq 0,$$
$$GB_\omega - H_{12} I \leq 0,$$
$$\big(|H_{11}| - I\big)\mathbf{1} + |H_{12}|\mathbf{1} \leq 0.$$

Then set $H_1 = H_{11}$, $H_2 = H_{12}$, one can get the conditions in the theorem as follows:

$$GA - H_1 G \leq 0,$$
$$GB_\omega - H_2 \leq 0,$$
$$\big(|H_1| - I\big)\mathbf{1} + |H_2|\mathbf{1} \leq 0.$$

Sufficiency. Denote a new variable

$$\xi(k) \triangleq \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix},$$

which is followed by a dynamic equation

$$\begin{aligned}
\xi(k+1) &= \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} GA & GB_\omega \\ -GA & -GB_\omega \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&\le \begin{bmatrix} H_1 G & H_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\
&= \begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi(k).
\end{aligned}$$

And it satisfies

$$\left(\begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - I \right) \mathbf{1} \le 0$$

since $(|H_1| - I)\mathbf{1} + |H_2|\mathbf{1} \le 0$. It follows from Lemma 1 in [9], in which G is assigned to the identity matrix I, that $\xi(k) \le \mathbf{1}$ for any

$$\xi_0 = \begin{bmatrix} Gx_0 \\ -Gx_0 \\ I\omega_0 \end{bmatrix} \le \mathbf{1}.$$

Hence, the polyhedron $R^+[G, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,\infty}$. □

$Q^+[G, 0, \mathbf{1}]$ is also a positively invariant polyhedron of system S_2 with respect to $\Omega_{\infty,\infty}$. Likewise, constraints $x(k) \in Q^+[G, 0, \mathbf{1}]$ and $\omega(k) \in \Omega_{\infty,\infty}$ are identical to $\left[\begin{smallmatrix} x(k) \\ \omega(k) \end{smallmatrix}\right] \in \Pi_{\infty,\infty}$ defined as

$$\Pi_{\infty,\infty} \triangleq \left\{ \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \in \overline{R}_+^{n+m} : \begin{bmatrix} 0 \\ 0 \end{bmatrix} \le \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \le \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \right\}.$$

Theorem 9 *The polyhedron $Q^+[G, 0, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,\infty}$ if and only if there exist two matrices $H_1 \in R^{r\times r}$ and $H_2 \in R^{r\times m}$ such that*

$$GA - H_1 G \le 0,$$

$$GB_\omega - H_2 \leq 0,$$
$$\left(H_1^+ - I\right)\mathbf{1} + H_2^+\mathbf{1} \leq 0.$$

Proof Necessity. Similar to the proof of Theorem 7, by virtue of Theorem 2, there exists a matrix

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in R^{(r+1)\times(r+1)}$$

with $H_{11} \in R^{r\times r}$, $H_{12} \in R^{r\times 1}$, $H_{21} \in R^{1\times r}$, $H_{22} \in R^1$, such that

$$\begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \leq 0,$$
$$\left(\begin{bmatrix} H_{11}^+ & H_{12}^+ & -H_{11}^- & -H_{12}^- \\ H_{21}^+ & H_{22}^+ & -H_{21}^- & -H_{22}^- \\ -(-H_{11})^- & -(-H_{12})^- & (-H_{11})^+ & (-H_{12})^+ \\ -(-H_{21})^- & -(-H_{22})^- & (-H_{21})^+ & (-H_{22})^+ \end{bmatrix} - I\right)\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} \leq 0.$$

Simplifying the proof above, we obtain

$$GA - H_{11}G \leq 0, \tag{17}$$
$$GB_\omega - H_{12}I \leq 0, \tag{18}$$
$$\left(H_{11}^+ - I\right)\mathbf{1} + H_{12}^+\mathbf{1} \leq 0. \tag{19}$$

Then setting $H_1 = H_{11}$, $H_2 = H_{12}$, we have the conditions in the theorem as follows:

$$GA - H_1G \leq 0,$$
$$GB_\omega - H_2 \leq 0,$$
$$\left(H_1^+ - I\right)\mathbf{1} + H_2^+\mathbf{1} \leq 0.$$

Sufficiency. Denote a new variable

$$\xi'(k) \triangleq \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix},$$

which is followed by a dynamic equation

$$\begin{aligned} \xi'(k+1) &= \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B_\omega \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \\ &= \begin{bmatrix} GA & GB_\omega \\ -GA & -GB_\omega \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} \end{aligned}$$

$$\leq \begin{bmatrix} H_1G & H_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$= \begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} G & 0 \\ -G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$= \begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi(k).$$

And it satisfies

$$\left(\begin{bmatrix} H_1 & 0 & H_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - I \right) \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \end{bmatrix} \leq 0$$

since $(H_1^+ - I)\mathbf{1} + H_2^+\mathbf{1} \leq 0$. It follows from Lemma 1 in [9], in which G is assigned to the identity matrix I, that $\xi'(k) \leq [\mathbf{1}, 0, \mathbf{1}]^T$ for any

$$\xi_0' = \begin{bmatrix} Gx_0 \\ -Gx_0 \\ I\omega_0 \end{bmatrix} \leq \begin{bmatrix} \mathbf{1} \\ 0 \\ \mathbf{1} \end{bmatrix}.$$

Hence, the polyhedron $Q^+[G, 0, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 in (10) with respect to $\Omega_{\infty,\infty}$. □

Remark 4 The conclusions and method in Theorem 1 to Theorem 9 can also be extended to Markovian positive systems [21].

5 Numerical examples

Example 1 Consider a two-dimensional positive system S_1 and a polyhedron $R^+[G, \gamma]$ with

$$A = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}; \qquad G = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}; \qquad \gamma = \begin{bmatrix} 3 \\ 12 \end{bmatrix}.$$

From Theorem 1, it can be verified that $R^+[G, \gamma]$ is a positively invariant polyhedron of system S_1 since there exists a matrix

$$H = \begin{bmatrix} 0.55 & -0.04 \\ 1.55 & -0.05 \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} - \begin{bmatrix} 0.55 & -0.04 \\ 1.55 & -0.05 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \leq 0,$$

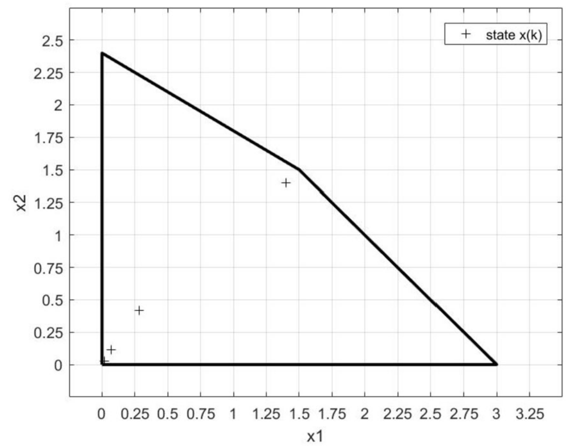

Figure 1 System trajectories

$$\left(\left|\begin{bmatrix} 0.55 & -0.04 \\ 1.55 & -0.05 \end{bmatrix}\right| - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 12 \end{bmatrix} \le 0.$$

Figure 1 indicates the trajectory of system state starting from $[1.4\ 1.4]^T$. The trajectory of system state starting from $[1.4\ 1.4]^T$ approaches the origin gradually. But it will never coincide with the origin. The system state trajectories are completely kept in this invariant polyhedron $R^+[G,\gamma]$. Set $\gamma_1 = \left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$ and $\gamma_2 = \gamma$, G and A remain unchanged, the result can also illustrate Theorem 2.

A counter-example is given in Example 2 for illustrating the necessity of Theorem 1.

Example 2 Consider a positive system S_1 and a polyhedron $R^+[G,\gamma]$ with

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}; \qquad G = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}; \qquad \gamma = \begin{bmatrix} 15 \\ 16 \end{bmatrix}.$$

According to Theorem 1, this polyhedron is not a positively invariant set of system S_1 since no feasible matrix H can be found. Figure 2 shows the trajectory of system state starting from $[0.1\ 0.1]^T$. The trajectory of system state is not in the given polyhedron after two iterations.

Example 3 Consider a positive system S_2 and a polyhedron $R^+[G,1]$ with

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}; \qquad B_\omega = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}; \qquad G = \begin{bmatrix} 0.1 & 0.3 \\ 0.15 & 0.2 \end{bmatrix}.$$

According to Theorem 6, the polyhedron $R^+[G,1]$ is a positively invariant set of system S_2 with respect to $\omega \in \Omega_{\infty,1}$ with a feasible matrix

$$H = \begin{bmatrix} -0.04 & 0.40 \\ -0.01 & 0.34 \end{bmatrix}$$

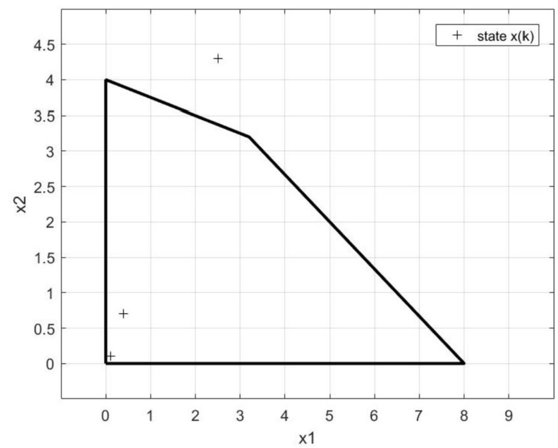

Figure 2 System trajectories

such that

$$\begin{bmatrix} 0.1 & 0.3 \\ 0.15 & 0.2 \end{bmatrix}\begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} - \begin{bmatrix} -0.04 & 0.40 \\ -0.01 & 0.34 \end{bmatrix}\begin{bmatrix} 0.1 & 0.3 \\ 0.15 & 0.2 \end{bmatrix} \le 0,$$
$$\begin{bmatrix} 0.1 & 0.3 \\ 0.15 & 0.2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \left(\left|\begin{bmatrix} -0.04 & 0.40 \\ -0.01 & 0.34 \end{bmatrix}\right| - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \le 0.$$

Take

$$\omega(k) = \begin{bmatrix} 0.25 + 0.25\sin(k) \\ 0.25 + 0.25\cos(k) \end{bmatrix}$$

and initial conditions $[0.1\ 0.1]^T$ to determine whether the given polyhedron is a positively invariant set of system S_2. The trajectory of system state starting from $[0.1\ 0.1]^T$ exhibits circular motion similar to an ellipse. And the ellipse stays in the given polyhedron in Fig. 3. Similarly, this example also satisfies

$$GA - HG \le 0,$$
$$GB_\omega + \left(H^+ - I\right)[\mathbf{1}] \le 0.$$

So the polyhedron $Q^+[G, 0, \mathbf{1}]$ is a positively invariant polyhedron of system S_2 with respect to $\Omega_{\infty,1}$.

To illustrate the necessity part of Theorem 6, a counter-example is given in Example 4.

Example 4 Consider a positive system S_2 and a polyhedron $R^+[G, 1]$ with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}; \qquad B_\omega = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}; \qquad G = \begin{bmatrix} 0.2 & 0.35 \\ 0.4 & 0.25 \end{bmatrix}.$$

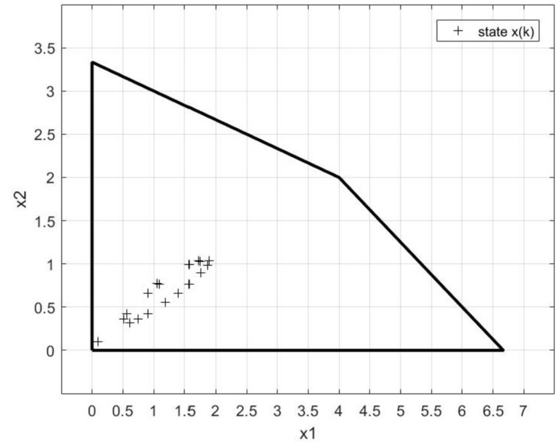

Figure 3 System trajectories

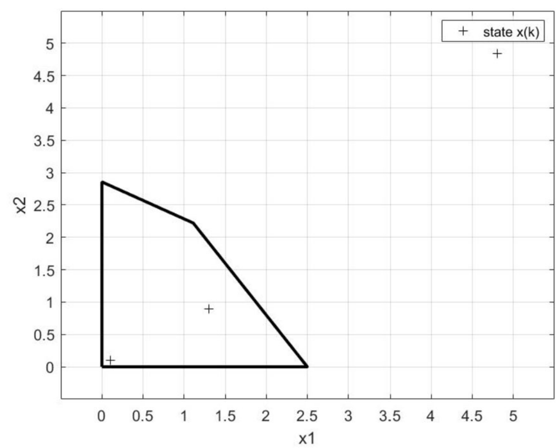

Figure 4 System trajectories

From Theorem 5, this polyhedron $R^{+}[G,1]$ is not a positively invariant set of system S_2 with respect to $\omega \in \Omega_{\infty,1}$. For the external disturbance

$$\omega(k) = \begin{bmatrix} 0.25 + 0.25\sin(k) \\ 0.25 + 0.25\cos(k) \end{bmatrix},$$

Figure 4 shows the system trajectory starting from point $[0.1\ 0.1]^T$. Figure 4 is similar to Fig. 2, the trajectory of system state starting from $[0.1\ 0.1]^T$ is not bounded inside the given polyhedron.

Example 5 Consider a positive system S_2 and a polyhedron $R^{+}[G,1]$ with the following parameters:

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}; \qquad B_\omega = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}; \qquad G = \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix}.$$

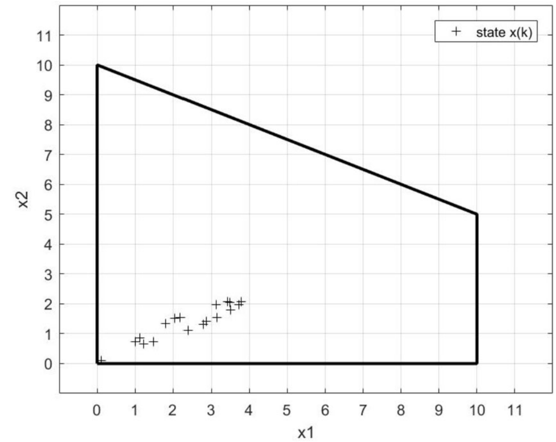

Figure 5 System trajectories

Based on Theorem 6, this polyhedron $R^+[G,1]$ is a positively invariant set of positive system S_2 with respect to $\omega \in \Omega_{\infty,\infty}$ since there exist two matrices

$$H_1 = \begin{bmatrix} -0.05 & 0.55 \\ 0.1 & 0.3 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.2 & 0.15 \\ 0.3 & 0.2 \end{bmatrix}$$

such that

$$\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix}\begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} - \begin{bmatrix} -0.05 & 0.55 \\ 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.15 \\ 0.3 & 0.2 \end{bmatrix} \le 0,$$

$$\left(\left|\begin{bmatrix} -0.05 & 0.55 \\ 0.1 & 0.3 \end{bmatrix}\right| - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left|\begin{bmatrix} 0.2 & 0.15 \\ 0.3 & 0.2 \end{bmatrix}\right|\begin{bmatrix} 1 \\ 1 \end{bmatrix} \le 0.$$

This conclusion can be showed in Fig. 5, which depicts the system trajectory with respect to

$$\omega(k) = \begin{bmatrix} 0.5 + 0.5\sin(k) \\ 0.5 + 0.5\cos(k) \end{bmatrix}^T$$

and the initial state $[0.1\ 0.1]^T$. Figure 5 is similar to Fig. 3, the system trajectory with respect to

$$\omega(k) = \begin{bmatrix} 0.5 + 0.5\sin(k) \\ 0.5 + 0.5\cos(k) \end{bmatrix}^T$$

and the initial state $[0.1\ 0.1]^T$ exhibits circular motion similar to an ellipse and stays in the given polyhedron.

For convenience, it is concluded that the polyhedron $Q^+[G,0,\mathbf{1}]$ is a positively invariant polyhedron of system S_2 with respect to $\Omega_{\infty,\infty}$ by Example 5.

6 Conclusion

Necessary and sufficient conditions for a polyhedral set to be a positively invariant set of a discrete-time positive linear system are presented in this paper. The relationship between Lyapunov stability and positively invariant polyhedra for discrete-time positive linear systems is also studied. Under two types of external perturbations whose $(\infty, 1)$-norm or (∞, ∞)-norm are bounded by a constant, the necessary and sufficient algebraic conditions for the positive invariant polyhedra are both investigated, which can be solved by a linear programming. The results obtained in this paper enrich and complete the results of positively invariant sets for positive linear systems with disturbances.

Acknowledgements
All authors are grateful to the respected reviewers for their valuable comments and constructive suggestions towards the improvement of the original paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Conceptualization, HY; formal analysis, YL; investigation, YL; supervision, HY; writing—original draft, YL; writing—review and editing, HY. All authors have read and agreed to the published version of the manuscript.

Received: 21 February 2023 Accepted: 11 July 2023 Published online: 11 August 2023

References

1. Shen, J.: Positive invariance of constrained affine dynamics and its applications to hybrid systems and safety verification. IEEE Trans. Autom. Control **57**(1), 3–18 (2011)
2. Hu, T., Lin, Z., Chen, B.M.: Disturbance rejection with saturating actuators for discrete-time linear systems. In: Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228), vol. 2, pp. 1723–1728 (2001)
3. Vassilaki, M., Hennet, J.C., Bitsoris, G.: Feedback control of linear discrete-time systems under state and control constraints. Int. J. Control **47**(6), 1727–1735 (1988)
4. Bitsoris, G.: Positively invariant polyhedral sets of discrete-time linear systems. Int. J. Control **47**(6), 1713–1726 (1988)
5. Castelan, E.B., Tarbouriech, S.: Positively invariant polyhedral sets for discrete-time singular systems with additive perturbations. In: Proceedings of 35th IEEE Conference on Decision and Control, 1, pp. 992–993 (1996)
6. Castelan, E.B., Tarbouriech, S.: Simple and weak delta-invariant polyhedral sets for discrete-time singular systems. SBA, Soc. Bras. Autom. **14**, 339–347 (2003)
7. Bitsoris, G.: On the positive invariance of polyhedral sets for discrete-time systems. Syst. Control Lett. **11**(3), 243–248 (1988)
8. Athanasopoulos, N., Bitsoris, G.: Invariant set computation for constrained uncertain discrete-time linear systems. In: 49th IEEE Conference on Decision and Control (CDC), pp. 5227–5232 (2010)
9. Ten Dam, A.A., Nieuwenhuis, J.W.: A linear programming algorithm for invariant polyhedral sets of discrete-time linear systems. Syst. Control Lett. **25**(5), 337–341 (1995)
10. Rubin, D., Nguyen, H.-N., Gutman, P.-O.: Computation of polyhedral positive invariant sets via linear matrix inequalities. In: 2018 European Control Conference (ECC), pp. 2941–2946 (2018)
11. Rubin, D.Y., Nguyen, H.-N., Gutman, P.-O.: Yet another algorithm for the computation of polyhedral positive invariant sets. In: 2018 IEEE Conference on Control Technology and Applications (CCTA), pp. 698–703 (2018)
12. Dorea, C.E.T., Hennet, J.-C.: Self-bounded (A, B)-invariant polyhedra of discrete-time systems. In: Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187), vol. 4, pp. 2163–3168 (2000)
13. Rakovic, S.V., Kerrigan, E.C., Kouramas, K., Mayne, D.Q.: Approximation of the minimal robustly positively invariant set for discrete-time LTI systems with persistent state disturbances. In: 42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475), vol. 4, pp. 3917–3918 (2003)
14. Rakovic, S.V., Grieder, P., Kvasnica, M., Mayne, D.Q., Morari, M.: Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances. In: 2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No. 04CH37601), vol. 2, pp. 1418–1423 (2004)
15. Liu, C., Tahir, F., Jaimoukha, I.M.: Full-complexity polytopic robust control invariant sets for uncertain linear discrete-time systems. Int. J. Robust Nonlinear Control **29**(11), 3587–3605 (2019)
16. Wu, L., Lam, J., Shu, Z., Du, B.: On stability and stabilizability of positive delay systems. Asian J. Control **11**(2), 226–234 (2009)

17. Zhao, X., Yin, Y., Liu, L., Sun, X.: Stability analysis and delay control for switched positive linear systems. IEEE Trans. Autom. Control **63**(7), 2184–2190 (2017)
18. Shu, Z., Lam, J., Gao, H., Du, B., Wu, L.: Positive observers and dynamic output-feedback controllers for interval positive linear systems. IEEE Trans. Circuits Syst. I, Regul. Pap. **55**(10), 3209–3222 (2008)
19. Du, B., Xu, S., Shu, Z., Chen, Y.: On positively invariant polyhedrons for continuous-time positive linear systems. J. Franklin Inst. **357**(17), 12571–12587 (2020)
20. Bof, N., Carli, R., Schenato, L.: Lyapunov theory for discrete time systems (2003). arXiv preprint, 2018. arXiv:1809.05289
21. Liu, X., Liu, W., Li, Y., Zhuang, J.: Finite-time H_∞ control of stochastic time-delay Markovian jump systems. J. Shandong Univ. Sci. Technol. Nat. Sci. **41**(201), 75–84, (2022)

Publisher's Note