Advances in Continuous and Discrete Models

RESEARCH Open Access

Integrable aspects, analytic solutions and their asymptotic analysis for a discrete relativistic Toda lattice system

Meng-Li Qin[1] and Xiao-Yong Wen[1*]

*Correspondence:
xiaoyongwen@163.com
[1]School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, China

Abstract

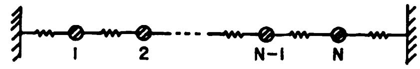
In this paper, we investigate a discrete relativistic Toda lattice ($\mathrm{dRTL}_{+}(\alpha)$) system, which may describe particle vibrations in lattices with an exponential interaction force. First, we construct its discrete generalized $(m, 2N-m)$-fold Darboux transformation, from which we can explicitly give its analytic solutions, such as discrete multi-soliton solutions, position controllable rational and semi-rational solutions and their hyperbolic-and-rational mixed solutions, whose properties and dynamics are analyzed and shown graphically. Second, the asymptotic behaviors of diverse exact solutions are analyzed, which shows that the interactions among different solutions are always elastic. In particular, the position of controllable rational solutions and asymptotic state analysis of discrete hyperbolic-and-rational mixed solutions are obtained and discussed for the first time. Finally, we study some integrable properties of this system, such as the integrable hierarchy and relevant Hamiltonian structures and conservation laws from a discrete spectral problem. These results may be helpful for understanding nonlinear lattice dynamics.

Keywords: Discrete relativistic Toda lattice system; Discrete generalized $(m, 2N-m)$-fold Darboux transformation; Hamiltonian structures; Analytic solutions; Asymptotic analysis

1 Introduction

In recent years, discrete nonlinear differential-difference equations (NDDEs), viewed as spatially discrete counterparts of nonlinear partial differential equations, have aroused increasing interest. NDDEs can model many interesting physical phenomena such as particle vibrations in lattices and pulses in biological chains, currents in electrical networks [1–6]. Some meaningful NDDEs have been proposed, such as the Ablowitz–Ladik lattice equation and its discrete nonlocal version [2–5], nonlinear self-dual network equation [3, 6, 7], discrete KdV equation [3, 8], Volterra lattice equation [6, 9], Toda lattice (TL) system [10–16] and its relativistic version [17–26] and so on. Among these NDDEs, the TL system equation is a very important class of NDDEs, which can describe a one-dimensional lattice dynamics of particles (see Fig. 1 in Ref. [6]). In order to better describe the nonlinear lattice dynamics, Ref. [26] proposed a new Hamiltonian function

Figure 1 A one-dimensional lattice with fixed ends (also see the first figure in Ref. [6])



$H = \sum_{n=1}^{N} \frac{e^{\alpha p_n} - 1 - \alpha p_n}{\alpha^2} + e^{x_{n+1} - x_n + \alpha p_n}$, whose corresponding Hamiltonian motion equation is the following discrete integrable relativistic Toda lattice (RTL) system with an arbitrary constant parameter α (see also Eq. (8.2.6) in Ref. [26]) as

$$\begin{cases} x_{n,t} = \frac{e^{\alpha p_n} - 1}{\alpha} + \alpha e^{x_{n+1} - x_n + \alpha p_n}, \\ p_{n,t} = e^{x_{n+1} - x_n + \alpha p_n} - e^{x_n - x_{n-1} + \alpha p_{n-1}}, \end{cases} \tag{1}$$

where $x_n = x(n,t), p_n = p(n,t)$ are the real functions of variables n, t. In Ref. [26], several kinds of RTL and modified TL systems have been proposed and investigated, all of which can be reduced to TL system. In this paper, we will only study Eq. (8.2.6) in Ref. [26], which is abbreviated as $\mathrm{dRTL}_+(\alpha)$ system by Suris. Therefore, in what follows, we still use this abbreviated name to call Eq. (1). When $\alpha \to 0$, Eq. (1) reduces to the famous TL system [6], which is the first integrable NDDE proposed by Toda in 1967. TL system has received extensive attention due to its potential application [6, 10–16, 26]. Equation (1) has Lax pair in the form [26]:

$$E\phi_n = U_n \phi_n = \begin{pmatrix} e^{\alpha p_n} - z^{-2} & \alpha z^{-1} e^{x_n} \\ -\alpha z^{-1} e^{-x_n + \alpha p_n} & 0 \end{pmatrix} \phi_n, \tag{2}$$

$$\phi_{n,t} = V_n \phi_n = \begin{pmatrix} \frac{z^{-2}}{2\alpha} - \frac{1}{\alpha} + \alpha e^{x_n - x_{n-1} + \alpha p_{n-1}} & -z^{-1} e^{x_n} \\ z^{-1} e^{-x_{n-1} + \alpha p_{n-1}} & -\frac{z^{-2}}{2\alpha} \end{pmatrix} \phi_n, \tag{3}$$

where $\phi_n = (\varphi_n, \psi_n)^T$ is an eigenfunction vector, z is the spectral parameter independent of time t, and E is the shift operator defined by $Ef(n,t) = f(n+1,t), E^{-1}f(n,t) = f(n-1,t)$. The integrability condition $U_{n,t} = (EV_n)U_n - U_n V_n$ between the spatial part (2) and time evolution part (3) of Lax pair yields Eq. (1). Here, we want to say that the above $\mathrm{dRTL}_+(\alpha)$ system (1) is different from ones described in the literature [17–25]. The Darboux transformation (DT) method is a very powerful tool for constructing soliton solutions of the Lax integrable NDDEs from a trivial seed [7, 12, 13, 23, 24, 27–30]. The above Lax pair (2) and (3) is inconvenient to construct DT due to exponential function, for the sake of later discussion, we take $u_n = e^{\alpha p_n}, v_n = e^{x_n}, \lambda = \frac{1}{z}$, then Eq. (1) is equivalent to the following equation

$$\begin{cases} u_{n,t} = \frac{\alpha u_n (u_n v_{n-1} v_{n+1} - u_{n-1} v_n^2)}{v_n v_{n-1}}, \\ v_{n,t} = \frac{\alpha^2 u_n v_{n+1} + u_n v_n - v_n}{\alpha}, \end{cases} \tag{4}$$

whose corresponding Lax pair is given from (2) and (3) as below:

$$E\phi_n = U_n(u,\lambda)\phi_n = \begin{pmatrix} -\lambda^2 + u_n & \alpha \lambda v_n \\ -\frac{\alpha \lambda u_n}{v_n} & 0 \end{pmatrix} \phi_n, \tag{5}$$

$$\phi_{n,t} = V_n \phi_n = \begin{pmatrix} \frac{\lambda^2}{2\alpha} + \frac{\alpha v_n u_{n-1}}{v_{n-1}} - \frac{1}{\alpha} & -\lambda v_n \\ \frac{\lambda u_{n-1}}{v_{n-1}} & -\frac{\lambda^2}{2\alpha} \end{pmatrix} \phi_n. \tag{6}$$

Recently, a discrete generalized $(m, 2N-m)$-fold DT has been proposed, compared with the usual DT, the main advantage of this technique is that it can give not only standard soliton solutions but also rational and semi-rational solutions and their mixed solutions [14, 17, 31]. To the best of our knowledge, the discrete generalized $(m, 2N-m)$-fold DT, diverse analytic solutions, asymptotic state analysis and dynamics, and associated integrable properties for Eq. (1) or (4) have not been investigated, in particular, the position-controlled rational solutions and asymptotic analysis of discrete hyperbolic-and-rational mixed solutions have not been reported before. Therefore, in this paper, we will study diverse analytic solutions of Eq. (4) by constructing the discrete generalized $(m, 2N-m)$-fold DT, and discuss their asymptotic state analysis and dynamics, then consider its integrable properties such as the conservation laws, lattice hierarchy, and relevant Hamiltonian structures via the Tu scheme [10]. The previous 2×2 Lax pair (5) and (6) of Eq. (4) is easier to construct the discrete generalized $(m, 2N-m)$-fold DT, so we first investigate Eq. (4), then we use the transformations $p_n = \frac{\ln u_n}{\alpha}, x_n = \ln v_n$ to give analytic solutions of Eq. (1).

The paper is divided into five sections. Section 2 is devoted to constructing the discrete generalized $(m, 2N-m)$-fold DT of Eq. (4) from its known Lax pair (5) and (6). Section 3 gives different types of analytic solutions of Eq. (4) using the special cases of the resulting DT and discusses their limit states via the asymptotic analysis technique. Section 4 investigates the integrable properties of Eq. (4), including the discrete integrable hierarchy, Hamiltonian structures, and infinite conservation laws. The final section is our conclusion.

2 Discrete generalized $(m, 2N-m)$-fold DT

In this section, we will proceed to establish the discrete generalized $(m, 2N-m)$-fold DT of Eq. (4). To achieve that, we consider the following gauge transformation

$$\tilde{\phi}_n = T_n \phi_n, \tag{7}$$

which can transform the Lax pair (5) and (6) into the same type Lax pair, namely,

$$\tilde{\phi}_{n+1} = \tilde{U}_n \tilde{\phi}_n, \qquad \tilde{\phi}_{n,t} = \tilde{V}_n \tilde{\phi}_n, \tag{8}$$

with $\tilde{U}_n = T_{n+1} U_n T_n^{-1}$ and $\tilde{V}_n = (T_{n,t} + V_n T_n) T_n^{-1}$. According to the knowledge of the Darboux transformation, we know that $\tilde{U}_n, \tilde{V}_n$ have the same forms as U_n, V_n in addition to replacing the old potentials u_n, v_n with the new potentials $\tilde{u}_n, \tilde{v}_n$. To achieve this special purpose, we must define a special matrix T_n as

$$T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} \lambda^{2N} + \sum_{j=0}^{N-1} a_n^{(2j)} \lambda^{2j} & \sum_{j=1}^{N} b_n^{(2j-1)} \lambda^{2j-1} \\ \sum_{j=1}^{N} c_n^{(2j-1)} \lambda^{2j-1} & \sum_{j=1}^{N} d_n^{(2j)} \lambda^{2j} + 1 \end{pmatrix}, \tag{9}$$

in which $\lambda_i (\lambda_i \neq \lambda_j), i \neq j, i = 1, 2, \ldots, 2N$ are $2N$ arbitrary parameters, $a_n^{(2j)}, b_n^{(2j-1)}, c_n^{(2j-2)}$ and $d_n^{(2j)}$ are some unknown functions of the variables n, t determined below.

From the definition of the matrix T_n, we know that $\det T_n$ is a $(4N)$-th order polynomial of λ. If we assume that $\lambda_i (\lambda_i \neq 0, i = 1, 2, \ldots m)$ $(1 \leq m \leq 2N)$ are the m roots of $\det T_n$. Let

$\phi_{i,n} = (\phi_{1,n}(\lambda_i), \phi_{2,n}(\lambda_i))^T$ be the solutions of spectral problem (5) and (6) with $\lambda = \lambda_i$ ($1 \leq m \leq 2N$), to determine $4N$ functions $a_n^{(2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}, d_n^{(2j)}$, for every λ_i, we expand

$$T(\lambda_i + \varepsilon)\phi_{i,n}(\lambda_i + \varepsilon) = \sum_{K=0}^{N-1} \sum_{j=0}^{k} T^{(j)}(\lambda_i)\phi_{i,n}^{(k-j)}(\lambda_i)\varepsilon^k \tag{10}$$

in which we expand $T_n(\lambda_i + \varepsilon)$ using binomial expansions as below

$$T(\lambda_i + \varepsilon) = T_n^{(0)} + T_n^{(1)}\varepsilon + \cdots + T_n^{(m_i)}\varepsilon^{m_i}, \tag{11}$$

and expand $\phi_{i,n}(\lambda_i + \varepsilon)$ by utilizing Taylor series around $\varepsilon = 0$ as

$$\phi_{i,n}(\lambda_i + \varepsilon) = \phi_{i,n}^{(0)}(\lambda_i) + \phi_{i,n}^{(1)}(\lambda_i)\varepsilon + \phi_{i,n}^{(2)}(\lambda_i)\varepsilon^2 + \phi_{i,n}^{(3)}(\lambda_i)\varepsilon^3 + \cdots, \tag{12}$$

where $\phi_n^{(k)}(\lambda_i) = \frac{1}{k!}\frac{\partial^k}{\partial \lambda_i^k}\phi_n(\lambda_i)$, and ε is a small parameter. Taking

$$\lim_{\varepsilon \to 0} \frac{T_n(\lambda_i + \varepsilon)\phi_n(\lambda_i + \varepsilon)}{\varepsilon^{k_i}} = 0, \quad \left(i = 1, 2, \ldots, m, k_i = 0, 1, \ldots, \nu_i, 2N = m + \sum_{i=1}^{m} \nu_i\right) \tag{13}$$

from which we can get $4N$ algebraic equations for $4N$ unknown functions $a_n^{(2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}, d_n^{(2j)}$, i.e.,

$$\begin{cases} T^{(0)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ T^{(0)}(\lambda_i)\phi_{i,n}^{(1)}(\lambda_i) + T^{(1)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ T_n^{(0)}(\lambda_i)\phi_{i,n}^{(2)}(\lambda_i) + T_n^{(1)}(\lambda_i)\phi_{i,n}^{(1)}(\lambda_i) + T_n^{(2)}(\lambda_i)\phi_{i,n}^{(0)}(\lambda_i) = 0, \\ \cdots, \\ \sum_{j=0}^{\nu_i} T^{(j)}(\lambda_i)\phi_{i,n}^{(\nu_i - j)}(\lambda_i) = 0.. \end{cases} \tag{14}$$

Here, the authors would like to say: the number m denotes the number of the distinct spectral parameter we use, the number $2N$ denotes the order number of DT, ν_i means the order number of the highest derivative in the Taylor series expansion for every $\phi_{i,n}(\lambda_i)$, and $2N - m = \sum_{i=1}^{m} \nu_i$ is the order number sum of the highest derivative of the Darboux matrix T_n or the vector eigenfunction $\phi_{i,n}(\lambda_i)$. When the m spectral parameters λ_i are suitably chosen, the determinant of the coefficients for system (14) is nonzero. In this way, the $4N$ undetermined functions $a_n^{(2j)}, b_n^{(2j-1)}, c_n^{(2j-1)}, d_n^{(2j)}$ in the Darboux matrix T_n can be uniquely determined by (14). From the above analysis, one can sum up the following generalized $(m, 2N - m)$-fold DT theorem:

Theorem 1 *Let $\phi_{i,n}(\lambda_i) = (\varphi_{i,n}, \psi_{i,n})^T$ be m column vector solutions of Lax pair (5) and (6) for the spectral parameters λ_i ($i = 1, 2, \ldots, m$) with the initial solutions u_n, v_n of Eq. (4), then the transformations of Eq. (4) from the old solutions u_n, v_n to the new solutions $\tilde{u}_n$, $\tilde{v}_n$ are given by*

$$\tilde{u}_n = \frac{u_n a_{n+1}^{(0)}}{a_n^{(0)}}, \qquad \tilde{v}_n = \frac{\alpha v_n + b_n^{(2N-1)}}{\alpha d_n^{(2N)}}, \tag{15}$$

where

$$a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1}, \qquad b_n^{(2N-1)} = \frac{\Delta b_n^{(2N-1)}}{\Delta_1}, \qquad d_n^{(2N)} = \frac{\Delta d_n^{(2N)}}{\Delta_2}. \tag{16}$$

with $\Delta_1 = (\Delta_1^{(1)}, \Delta_1^{(2)}, \dots, \Delta_1^{(m)})^T$, $\Delta_2 = (\Delta_2^{(1)}, \Delta_2^{(2)}, \dots, \Delta_2^{(m)})^T$, $\Delta_1^{(i)} = (\Delta_{1,j,s}^{(i)})_{2(\nu_i+1)\times 2N}$, $\Delta_2^{(i)} = (\Delta_{2,j,s}^{(i)})_{2(\nu_i+1)\times 2N}$ *in which* $\Delta_{1,j,s}^{(i)}$, $\Delta_{2,j,s}^{(i)}(1 \le j \le 2(\nu_i+1), 1 \le s \le 2N, i = 1, 2, \dots, m)$ *are expressed as*

$$\Delta_{1,j,s}^{(i)} = \begin{cases} \sum_{k=0}^{j-1} C_{2N-2s}^{k} \lambda_i^{2N-2s-k} \varphi_{i,n}^{(j-1-k)} \\ \quad for\ l + \sum_{i=1}^{l-1} \nu_i \le j \le l + \sum_{i=1}^{l} (1 \le l \le m), 1 \le s \le N, \\ \sum_{k=0}^{j-1} C_{4N-2s+1}^{k} \lambda_i^{4N-2s-k+1} \psi_{i,n}^{(j-1-k)} \\ \quad for\ l + \sum_{i=1}^{l-1} \nu_i \le j \le l + \sum_{i=1}^{l} (1 \le l \le m), N+1 \le s \le 2N, \end{cases}$$

$$\Delta_{2,j,s}^{(i)} = \begin{cases} \sum_{k=0}^{j-1} C_{2N-2s+1}^{k} \lambda_i^{2N-2s-k+1} \varphi_{i,n}^{(j-1-k)} \\ \quad for\ l + \sum_{i=1}^{l-1} \nu_i \le j \le l + \sum_{i=1}^{l} (1 \le l \le m), 1 \le s \le N, \\ \sum_{k=0}^{j-1} C_{4N-2s+2}^{k} \lambda_i^{4N-2s-k+2} \psi_{i,n}^{(j-1-k)} \\ \quad for\ l + \sum_{i=1}^{l-1} \nu_i \le j \le l + \sum_{i=1}^{l} (1 \le l \le m), N+1 \le s \le 2N, \end{cases}$$

where $\Delta a_n^{(0)}$ *and* $b_n^{(2N-1)}$ *are given from the determinant* Δ_1 *by replacing their N-th and* $(N+1)$*-th columns by the column vector* $(f_1^{(1)}, f_2^{(1)}, \dots, f_{(\nu_1+1)}^{(1)}, \dots, f_1^{(i)}, f_2^{(i)}, \dots, f_{(\nu_i+1)}^{(i)}, \dots, f_1^{(m)}, f_2^{(m)}, \dots, f_{(\nu_m+1)}^{(m)})$ *with* $f_j^{(i)} = -\lambda_i^{2N} \varphi_{i,n}^{(j-1)}$, *respectively, while* $\Delta d_n^{(2N)}$ *is obtained from the determinant* Δ_2 *by replacing* $(N+1)$*-th columns by the column vector* $(r_1^{(1)}, r_2^{(1)}, \dots, r_{(\nu_1+1)}^{(1)}, \dots, r_1^{(i)}, r_2^{(i)}, \dots, r_{(\nu_i+1)}^{(i)}, \dots, r_1^{(m)}, r_2^{(m)}, \dots, r_{(\nu_m+1)}^{(m)})$ *with* $r_j^{(i)} = -\psi_{i,n}^{(j-1)}(1 \le j \le (\nu_i+1), 1 \le i \le m)$.

Remark 1 Here we describe the transformations (7) and (15) using m distinct spectral parameters as the discrete generalized $(m, 2N-m)$-fold DT of Eq. (4). Now we discuss several kinds of special cases:

- If $m = 1$ and $m_i = 2N-1$, the discrete generalized $(m, 2N-m)$-fold DT reduces to the discrete generalized $(1, 2N-1)$-fold DT that is used to derive higher-order rational and semi-rational solutions;
- If $m = 2$ and $m_i = 2N-2$, the discrete generalized $(m, 2N-m)$-fold DT reduces to the discrete generalized $(2, 2N-2)$-fold DT that is used to obtain mixed solutions of usual soliton solutions and rational or semi-rational solutions;
- If $m = 2N$ and $m_i = 0$, the discrete generalized $(m, 2N-m)$-fold DT reduces to the discrete generalized $(2N, 0)$-fold DT that can include the discrete $2N$-fold DT if we do not make the Taylor series expansion for every $\phi_{i,n}(\lambda_i)$;
- If $2 < m < 2N$, we can derive the other discrete generalized DTs, which can give the new discrete mixed solutions and are not discussed in this paper.

3 Analytic solutions and their asymptotic analysis of Eq. (4)

In this section, we will obtain the discrete soliton solutions, rational and semi-rational solutions and their mixed solutions of Eq. (4) using the discrete generalized $(m, 2N-m)$-fold DT with three cases $m = 1, 2, 2N$. In what follows, we first give the solutions of Lax pair (5) and (6).

3.1 The solutions of Lax pair

Taking the seed solutions $u_n = \frac{1}{\alpha^2+1}, v_n = 1$ of Eq. (4) into Lax pair (5) and (6), with the aid of symbolic computation Maple, we can obtain the following eigenfunction solutions with regard to λ_i $(i = 1, 2, \ldots 2N)$:

$$\phi_{i,n} = \begin{pmatrix} \varphi_{i,n}(\lambda_i) \\ \psi_{i,n}(\lambda_i) \end{pmatrix} = C_{1,i} \begin{pmatrix} \tau_{1,i}^n e^{\rho_{1,i}t+\zeta(\varepsilon)} \\ -\frac{\alpha\lambda_i}{(1+\alpha^2)\tau_{1,i}} \tau_{1,i}^n e^{\rho_{1,i}t+\zeta(\varepsilon)} \end{pmatrix} + C_{2,i} \begin{pmatrix} \tau_{2,i}^n e^{\rho_{2,i}t-\zeta(\varepsilon)} \\ -\frac{\alpha\lambda_i}{(1+\alpha^2)\tau_{2,i}} \tau_{2,i}^n e^{\rho_{2,i}t-\zeta(\varepsilon)} \end{pmatrix}, \quad (17)$$

where

$$\tau_{1,i} = \frac{-\alpha^2\lambda_i^2 - \lambda_i^2 + 1 + \sqrt{\alpha^4\lambda_i^4 - 4\alpha^4\lambda_i^2 + 2\alpha^2\lambda_i^4 - 6\alpha^2\lambda_i^2 + \lambda_i^4 - 2\lambda_i^2 + 1}}{2(\alpha^2+1)},$$

$$\tau_{2,i} = \frac{-\alpha^2\lambda_i^2 - \lambda_i^2 + 1 - \sqrt{\alpha^4\lambda_i^4 - 4\alpha^4\lambda_i^2 + 2\alpha^2\lambda_i^4 - 6\alpha^2\lambda_i^2 + \lambda_i^4 - 2\lambda_i^2 + 1}}{2(\alpha^2+1)},$$

$$\rho_{1,i} = \frac{\alpha^2\lambda_i^2 + \lambda_i^2 - 1 - \sqrt{\alpha^4\lambda_i^4 - 4\alpha^4\lambda_i^2 + 2\alpha^2\lambda_i^4 - 6\alpha^2\lambda_i^2 + \lambda_i^4 - 2\lambda_i^2 + 1}}{2\alpha(\alpha^2+1)},$$

$$\rho_{2,i} = \frac{\alpha^2\lambda_i^2 + \lambda_i^2 - 1 + \sqrt{\alpha^4\lambda_i^4 - 4\alpha^4\lambda_i^2 + 2\alpha^2\lambda_i^4 - 6\alpha^2\lambda_i^2 + \lambda_i^4 - 2\lambda_i^2 + 1}}{2\alpha(\alpha^2+1)},$$

$$\zeta(\varepsilon) = \sqrt{\alpha^4\lambda_i^4 - 4\alpha^4\lambda_i^2 + 2\alpha^2\lambda_i^4 - 6\alpha^2\lambda_i^2 + \lambda_i^4 - 2\lambda_i^2 + 1} \sum_{j=0}^{2N} e_j \varepsilon^j,$$

in which e_j is the arbitrary real constant. It should be noted that these parameters e_j can control the position of the solution, which is also different from our previous work. Below, we first discuss the case where m takes both ends in the discrete generalized $(m, 2N - m)$-fold DT (i.e., $m = 1, 2N$) and then discuss the case where m takes the middle in the discrete generalized $(m, 2N - m)$-fold DT $(1 < m < 2N)$, and take $m = 2$ as an example.

3.2 Position controllable rational and semi-rational solutions and asymptotic analysis

In this subsection, we will investigate some rational and semi-rational solutions of Eq. (4) using the discrete generalized $(1, 2N - 1)$-fold DT (i.e., generalized $(m, 2N - m)$-fold DT with $m = 1$). To give rational and semi-rational solutions, we fix the spectral parameter $\lambda = \lambda_1 + \varepsilon$, then expand the vector function $\phi_{1,n}$ in (17) with $\lambda_1 = 1 + \frac{\alpha}{\sqrt{\alpha^2+1}}$ as two Taylor series around $\varepsilon = 0$.

$$\phi_{1,n}(h) = \phi_{1,n}^{(0)} + \phi_{1,n}^{(1)}\varepsilon + \phi_{1,n}^{(2)}\varepsilon^2 + \phi_{1,n}^{(3)}\varepsilon^3 + \phi_{1,n}^{(4)}\varepsilon^4 + \phi_{1,n}^{(5)}\varepsilon^5 + \cdots, \quad (18)$$

To give more abundant rational and semi-rational solutions, we will enumerate two kinds of different expansions:

- *Type I* Setting $\alpha = \frac{3}{4}$ (i.e., $\lambda_1 = \frac{8}{5}$), $C_{1,1} = C_{2,1} = 1$, we obtain

$$\phi_{1,n}^{(0)} = \begin{pmatrix} \varphi_{1,n}^{(0)} \\ \psi_{1,n}^{(0)} \end{pmatrix} = \begin{pmatrix} 2(-\frac{24}{25})^n e^{\frac{32}{25}t} \\ \frac{8}{5}(-\frac{24}{25})^n e^{\frac{32}{25}t} \end{pmatrix}, \qquad \phi_{1,n}^{(1)} = \begin{pmatrix} \varphi_{1,n}^{(1)} \\ \psi_{1,n}^{(1)} \end{pmatrix}, \qquad \phi_{1,n}^{(2)} = \begin{pmatrix} \varphi_{1,n}^{(2)} \\ \psi_{1,n}^{(2)} \end{pmatrix}, \quad (19)$$

in which

$$
\begin{aligned}
\varphi_{1,n}^{(1)} &= \frac{1}{300}\left(-\frac{24}{25}\right)^{n} \mathrm{e}^{\frac{32}{25}t}\bigl(625n^2 + 1600nt - 3750ne_0 + 1024t^2 - 4800te_0 + 5625e_0^2 \\
&\quad + 375n + 1280t\bigr), \\
\psi_{1,n}^{(1)} &= \frac{1}{375}\left(-\frac{24}{25}\right)^{n} \mathrm{e}^{\frac{32}{25}t}\bigl(625n^2 + 1600nt - 3750ne_0 + 1024t^2 - 4800te_0 + 5625e_0^2 \\
&\quad - 875n - 320t + 3750e_0 + 625\bigr), \\
\varphi_{1,n}^{(2)} &= \frac{1}{1{,}080{,}000}\left(-\frac{24}{25}\right)^{n} \mathrm{e}^{\frac{32}{25}t}\bigl(390{,}625n^4 + 2{,}000{,}000n^3t - 4{,}687{,}500n^3e_0 \\
&\quad + 3{,}840{,}000n^2t^2 - 18{,}000{,}000n^2te_0 + 21{,}093{,}750n^2e_0^2 + 3{,}276{,}800nt^3 \\
&\quad - 23{,}040{,}000nt^2e_0 + 54{,}000{,}000nte_0^2 - 42{,}187{,}500ne_0^3 + 1{,}048{,}576t^4 \\
&\quad - 9{,}830{,}400t^3e_0 + 34{,}560{,}000t^2e_0^2 - 54{,}000{,}000te_0^3 + 31{,}640{,}625e_0^4 \\
&\quad + 1{,}406{,}250n^3 + 8{,}400{,}000n^2t - 8{,}437{,}500n^2e_0 + 14{,}592{,}000nt^2 \\
&\quad - 39{,}600{,}000nte_0 + 12{,}656{,}250ne_0^2 + 7{,}864{,}320t^3 - 36{,}864{,}000t^2e_0 \\
&\quad + 43{,}200{,}000te_0^2 - 250{,}000n^2 + 6{,}760{,}000nt - 9{,}093{,}750ne_0 - 13{,}500{,}000ne_1 \\
&\quad + 10{,}982{,}400t^2 - 28{,}440{,}000te_0 - 17{,}280{,}000te_1 + 33{,}328{,}125e_0^2 \\
&\quad + 40{,}500{,}000e_0e_1 - 421{,}875n + 1{,}440{,}000t\bigr), \\
\psi_{1,n}^{(2)} &= \frac{1}{1{,}350{,}000}\left(-\frac{24}{25}\right)^{n} \mathrm{e}^{-\frac{32}{25}t}\bigl(390{,}625n^4 + 2{,}000{,}000n^3t - 4{,}687{,}500n^3e_0 \\
&\quad + 3{,}840{,}000n^2t^2 - 18{,}000{,}000n^2te_0 + 21{,}093{,}750n^2e_0^2 + 3{,}276{,}800nt^3 \\
&\quad - 23{,}040{,}000nt^2e_0 + 54{,}000{,}000nte_0^2 - 42{,}187{,}500ne_0^3 + 1{,}048{,}576t^4 \\
&\quad - 9{,}830{,}400t^3e_0 + 34{,}560{,}000t^2e_0^2 - 54{,}000{,}000te_0^3 + 31{,}640{,}625e_0^4 - 156{,}250n^3 \\
&\quad + 2{,}400{,}000n^2t + 5{,}625{,}000n^2e_0 + 6{,}912{,}000nt^2 - 3{,}600{,}000nte_0 \\
&\quad - 29{,}531{,}250ne_0^2 + 4{,}587{,}520t^3 - 13{,}824{,}000t^2e_0 - 10{,}800{,}000te_0^2 \\
&\quad + 42{,}187{,}500e_0^3 - 718{,}750n^2 - 440{,}000nt - 14{,}718{,}750ne_0 - 13{,}500{,}000ne_1 \\
&\quad + 2{,}534{,}400t^2 - 17{,}640{,}000te_0 - 17{,}280{,}000te_1 + 54{,}421{,}875e_0^2 \\
&\quad + 40{,}500{,}000e_0e_1 + 765{,}625n + 360{,}000t + 13{,}781{,}250e_0 + 13{,}500{,}000e_1 \\
&\quad - 281{,}250\bigr).
\end{aligned}
$$

The rest $(\varphi_{1,n}^{(j)}, \psi_{1,n}^{(j)})^{\mathrm{T}}$ $(j = 4, 5, \dots)$ are omitted here.

• *Type II* Setting $\alpha = \frac{3}{4}$, (i.e., $\lambda_1 = \frac{8}{5}$), $C_{1,1} = -C_{2,1} = \frac{1}{\varepsilon}$, we can give different Taylor expansions as follows:

$$
\phi_{1,n}^{(0)} = \begin{pmatrix} \varphi_{1,n}^{(0)} \\ \psi_{1,n}^{(0)} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{15}(-\frac{24}{25})^n \mathrm{e}^{\frac{32}{25}t}(25n + 32t - 75e_0) \\ -\frac{4\sqrt{3}}{75}(-\frac{24}{25})^n \mathrm{e}^{\frac{32}{25}t}(25n + 32t - 75e_0 - 25) \end{pmatrix}, \qquad \phi_{1,n}^{(1)} = \begin{pmatrix} \varphi_{1,n}^{(1)} \\ \psi_{1,n}^{(1)} \end{pmatrix}, \quad (20)
$$

in which

$$\varphi_{1,n}^{(1)} = \frac{\sqrt{3}}{108{,}000}\left(-\frac{24}{25}\right)^n e^{\frac{32}{25}t}\big(-62{,}500n^3 - 240{,}000n^2t + 562{,}500n^2e_0 - 307{,}200nt^2 + 1{,}440{,}000nte_0 - 1{,}687{,}500ne_0^2 - 131{,}072t^3 + 921{,}600t^2e_0 - 2{,}160{,}000te_0^2 + 1{,}687{,}500e_0^3 - 112{,}500n^2 - 528{,}000nt + 337{,}500ne_0 - 491{,}520t^2 + 1{,}152{,}000te_0 + 26{,}875n - 189{,}600t + 444{,}375e_0 + 540{,}000e_1\big),$$

$$\psi_{1,n}^{(1)} = \frac{\sqrt{3}}{135{,}000}\left(-\frac{24}{25}\right)^n e^{\frac{32}{25}t}\big(-62{,}500n^3 - 240{,}000n^2t + 562{,}500n^2e_0 - 307{,}200nt^2 + 1{,}440{,}000nte_0 - 1{,}687{,}500ne_0^2 - 131{,}072t^3 + 921{,}600t^2e_0 - 2{,}160{,}000te_0^2 + 1{,}687{,}500e_0^3 + 75{,}000n^2 - 48{,}000nt - 787{,}500ne_0 - 184{,}320t^2 - 288{,}000te_0 + 1{,}687{,}500e_0^2 - 48{,}125n - 45{,}600t + 1{,}006{,}875e_0 + 540{,}000e_1 + 35{,}625\big).$$

Case (1) Taking $N = 1$, the first-order position controllable rational solutions of Eq. (4) can be expressed as

$$\tilde{u}_n = \frac{a_{n+1}^{(0)}}{(1+\alpha^2)a_n^{(0)}}, \qquad \tilde{v}_n = \frac{\alpha + b_n^{(1)}}{\alpha d_n^{(2)}}, \tag{21}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1}, b_n^{(1)} = \frac{\Delta b_n^{(1)}}{\Delta_1}$ and $d_n^{(2)} = \frac{\Delta d_n^{(2)}}{\Delta_2}$, in which

$$\Delta_{1,n} = \begin{vmatrix} \varphi_1^{(0)} & \lambda_1\psi_{1,n}^{(0)} \\ \varphi_{1,n}^{(1)} & \lambda_1\psi_{1,n}^{(1)} + \psi_{1,n}^{(0)} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} \lambda_1\varphi_{1,n}^{(0)} & \lambda_1^2\psi_{1,n}^{(0)} \\ \lambda_1\varphi_{1,n}^{(1)} + \varphi_{1,n}^{(0)} & \lambda_1^2\psi_{1,n}^{(1)} + 2\lambda_1\psi_{1,n}^{(0)} \end{vmatrix},$$

$$\Delta b_n^{(1)} = \begin{vmatrix} \varphi_{1,n}^{(0)} & -\lambda_1^2\varphi_{1,n}^{(0)} \\ \varphi_{1,n}^{(1)} & -\lambda_1^2\varphi_{1,n}^{(1)} - 2\lambda_1\varphi_{1,n}^{(0)} \end{vmatrix},$$

$$\Delta a_n^{(0)} = \begin{vmatrix} -\lambda_1^2\varphi_{1,n}^{(0)} & \lambda_1\psi_{1,n}^{(0)} \\ -\lambda_1^2\varphi_{1,n}^{(1)} - 2\lambda_1\varphi_{1,n}^{(0)} & \lambda_1\psi_{1,n}^{(1)} + \psi_{1,n}^{(0)} \end{vmatrix}, \qquad \Delta d_n^{(2)} = \begin{vmatrix} \lambda_1\varphi_{1,n}^{(0)} & -\psi_{1,n}^{(0)} \\ \lambda_1\varphi_{1,n}^{(1)} + \varphi_{1,n}^{(0)} & -\psi_{1,n}^{(1)} \end{vmatrix}.$$

Using *Type I* expansion, direct calculation leads to specific analytical expressions of position controllable rational solution (21) as

$$\begin{aligned} \tilde{u}_n &= \frac{16}{25} - \frac{240}{(25n + 32t + 5 - 75e_0)(25n + 32t - 5 - 75e_0)}, \\ \tilde{v}_n &= 1 - \frac{2225n + 2848t + 1155 - 6675e_0}{625n + 800t - 125 - 1875e_0}, \end{aligned} \tag{22}$$

from which, we can see that $\tilde{u}_n$ possesses singularity at two paralleled straight lines, i.e., $L_1 : 25n + 32t + 5 - 75e_0 = 0$ and $L_2 : 25n + 32t - 5 - 75e_0 = 0$, while $\tilde{v}_n$ has singularity at one straight line, i.e., $L : 625n + 800t - 125 - 1875e_0 = 0$. It should be noted that there is an arbitrary constant e_0 in these singular lines, so we can change the position of the solution through it. Moreover, we can conclude that $\tilde{u}_n \to \frac{16}{25}, \tilde{v}_n \to 1$ as $n \to \pm\infty, t \to \pm\infty$.

Through the transformations $u_n = e^{\alpha p_n}, v_n = e^{x_n}$, we can give the solutions of (1) as

$$\tilde{p}_n = \frac{4}{3}\ln\left|\frac{16}{25} - \frac{240}{(25n + 32t + 5 - 75e_0)(25n + 32t - 5 - 75e_0)}\right|,$$

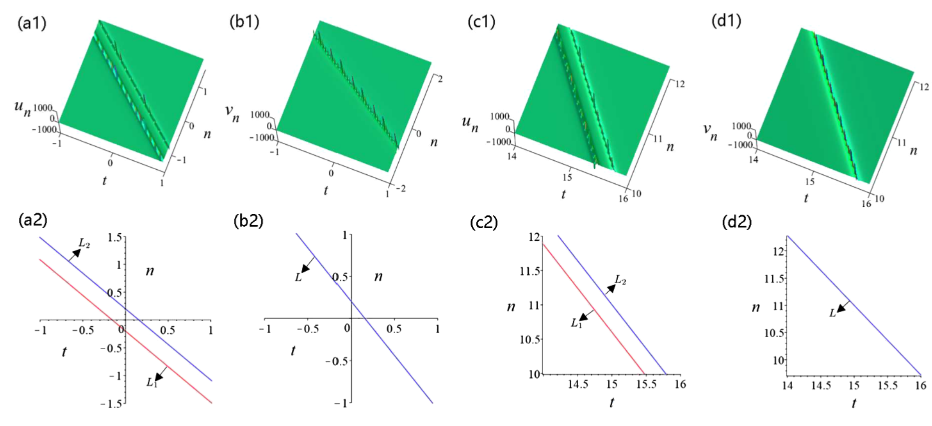


Figure 2 (Color online) First-order rational solutions: (**a1**)(**b1**) Three-dimensional plots of u_n and v_n with $e_0 = 0$ in (21); (**a2**)(**b2**) The trajectory plot of solutions u_n and v_n via expressions (22) with $e_0 = 0$ corresponding to (**a1**),(**b1**); (**c1**)(**d1**) Three-dimensional plots of u_n and v_n with $e_0 = 10$ in (21); (**c2**)(**d2**) The controllable moving trajectory plot of solutions u_n and v_n via expressions (22) with $e_0 = 10$ corresponding to (**c1**)(**d1**)

$$\tilde{x}_n = \ln\left|1 - \frac{2225n + 2848t + 1155 - 6675e_0}{625n + 800t - 125 - 1875e_0}\right|.$$

We draw the three-dimensional figures of solution (22) and its trajectory two-dimensional plots by choosing $e_0 = 0$ and $e_0 = 10$, as shown in Fig. 2.

It is important to note that we can derive the first-order semi-rational solutions if we fix the spectral parameter $\lambda = \lambda_1 + \varepsilon$ with $\lambda_1 \neq 1 + \frac{\alpha}{\sqrt{\alpha^2+1}}$, for example, here we choose $\lambda_1 = 2$ and expand the vector function $\phi_{1,n}$ in (17). Here we omit those expansions and just list the results of the first-order semi-rational solutions with $e_0 = e_1 = 0$ as follows:

$$\tilde{u}_n = \frac{a_{n+1}^{(0)}}{(1+\alpha^2)a_n^{(0)}} = \frac{Q_1}{Q_2}, \qquad \tilde{v}_n = \frac{\alpha + b_n^{(1)}}{\alpha d_n^{(2)}} = \frac{R_1}{R_2}, \tag{23}$$

with

$$\begin{aligned} Q_1 &= 96\big[80{,}000\sqrt{6}(7\sqrt{6}-12)\cosh\xi_2 + 2(-7+2\sqrt{6})\big(161{,}472n^2 + 979{,}968nt \\ &\quad + 1{,}486{,}848t^2 - 255{,}625\big) - 800(7\sqrt{6}-12)(319n + 968t)\sinh\xi_2 \\ &\quad + 31{,}250(-7+2\sqrt{6})\cosh(\xi_1+\xi_2)\big], \\ Q_2 &= 25\big[48(-7+2\sqrt{6})(29n + 88t + 4) + 2(192\sqrt{6} - 672)\cosh\xi_1 \\ &\quad + 2(1572 - 917\sqrt{6})\sinh\xi_1\big]\big[48(29n + 88t - 4) \\ &\quad + 262\sqrt{6}\sinh\xi_2 - 192\cosh\xi_2\big], \\ R_1 &= -192(29n + 88t + 25) - 4800\cosh\xi_2 - 2200\sqrt{6}\sinh\xi_2, \\ R_2 &= 48(29n + 88t - 4) + 131\sqrt{6}\sinh\xi_2 - 96\cosh\xi_2, \end{aligned}$$

where

$$\xi_1 = \frac{n}{2}\ln\frac{2628 + 1008\sqrt{6}}{2628 - 1008\sqrt{6}} + \frac{32\sqrt{6}}{25}t, \qquad \xi_2 = \frac{n}{2}\ln\frac{65{,}700 + 25{,}200\sqrt{6}}{65{,}700 - 125{,}200\sqrt{6}} + \frac{32\sqrt{6}}{25}t.$$

From (23), we can see that the semi-rational solutions are made up of polynomial and hyperbolic or exponential functions, which are different from the above rational solutions and soliton solutions to be discussed later.

Case (2) Taking $N = 2$, the second-order position controllable rational solutions of Eq. (4) can be expressed as

$$\tilde{u}_n = \frac{a_{n+1}^{(0)}}{(1+\alpha^2)a_n^{(0)}} = \frac{Q_1}{Q_2}, \qquad \tilde{v}_n = \frac{\alpha + b_n^{(3)}}{\alpha d_n^{(4)}} = \frac{R_1}{R_2}. \tag{24}$$

Here we omit the determinant representations of $a_n^{(0)}, b_n^{(3)}, d_n^{(4)}$, direct calculation yields the specific analytic expressions of solution (24) given by

$$\begin{aligned}
Q_1 = 16\big(&\eta^6 - 450\eta^5 e_0 + 84{,}375\eta^4 e_0^2 - 8{,}437{,}500\eta^3 e_0^3 + 474{,}609{,}375\eta^2 e_0^4 \\
&- 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625 e_0^6 - 120\eta^5 + 45{,}000\eta^4 e_0 \\
&- 6{,}750{,}000\eta^3 e_0^2 + 506{,}250{,}000\eta^2 e_0^3 - 18{,}984{,}375{,}000\eta e_0^4 \\
&+ 284{,}765{,}625{,}000 e_0^5 + 109{,}375\eta^3 n + 337{,}500\eta^3 e_1 - 24{,}609{,}375\eta^2 n e_0 \\
&- 75{,}937{,}500\eta^2 e_0 e_1 + 1{,}845{,}703{,}125\eta n e_0^2 + 5{,}695{,}312{,}500\eta e_0^2 e_1 \\
&- 46{,}142{,}578{,}125 n e_0^3 - 142{,}382{,}812{,}500 e_0^3 e_1 + 240{,}000\eta^3 - 6{,}562{,}500\eta^2 n \\
&- 54{,}000{,}000\eta^2 e_0 - 20{,}250{,}000\eta^2 e_1 + 984{,}375{,}000\eta n e_0 + 4{,}050{,}000{,}000\eta e_0^2 \\
&+ 3{,}037{,}500{,}000\eta e_0 e_1 - 36{,}914{,}062{,}500 n e_0^2 - 101{,}250{,}000{,}000 e_0^3 \\
&- 113{,}906{,}250{,}000 e_0^2 e_1 - 7{,}200{,}000\eta^2 + 242{,}578{,}125\eta n + 1{,}080{,}000{,}000\eta e_0 \\
&+ 499{,}921{,}875\eta e_1 - 303{,}750{,}000\eta e_2 - 2{,}392{,}578{,}125 n^2 - 18{,}193{,}359{,}375 n e_0 \\
&- 14{,}765{,}625{,}000 n e_1 - 40{,}500{,}000{,}000 e_0^2 - 37{,}494{,}140{,}625 e_0 e_1 \\
&+ 22{,}781{,}250{,}000 e_0 e_2 - 22{,}781{,}250{,}000 e_1^2 + 398{,}437{,}500 n + 6{,}201{,}562{,}500 e_1 \\
&+ 6{,}075{,}000{,}000 e_2\big)\big(\eta^6 - 450\eta^5 e_0 + 84{,}375\eta^4 e_0^2 - 8{,}437{,}500\eta^3 e_0^3 \\
&+ 474{,}609{,}375\eta^2 e_0^4 - 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625 e_0^6 + 120\eta^5 \\
&- 45{,}000\eta^4 e_0 + 6{,}750{,}000\eta^3 e_0^2 - 506{,}250{,}000\eta^2 e_0^3 + 18{,}984{,}375{,}000\eta e_0^4 \\
&- 284{,}765{,}625{,}000 e_0^5 + 109{,}375\eta^3 n + 337{,}500\eta^3 e_1 - 24{,}609{,}375\eta^2 n e_0 \\
&- 75{,}937{,}500\eta^2 e_0 e_1 + 1{,}845{,}703{,}125\eta n e_0^2 + 5{,}695{,}312{,}500\eta e_0^2 e_1 \\
&- 46{,}142{,}578{,}125 n e_0^3 - 142{,}382{,}812{,}500 e_0^3 e_1 - 240{,}000\eta^3 + 6{,}562{,}500\eta^2 n \\
&+ 54{,}000{,}000\eta^2 e_0 + 20{,}250{,}000\eta^2 e_1 - 984{,}375{,}000\eta n e_0 - 4{,}050{,}000{,}000\eta e_0^2 \\
&- 3{,}037{,}500{,}000\eta e_0 e_1 + 36{,}914{,}062{,}500 n e_0^2 + 101{,}250{,}000{,}000 e_0^3 \\
&+ 113{,}906{,}250{,}000 e_0^2 e_1 - 7{,}200{,}000\eta^2 + 242{,}578{,}125\eta n + 1{,}080{,}000{,}000\eta e_0 \\
&+ 499{,}921{,}875\eta e_1 - 303{,}750{,}000\eta e_2 - 2{,}392{,}578{,}125 n^2 - 18{,}193{,}359{,}375 n e_0 \\
&- 14{,}765{,}625{,}000 n e_1 - 40{,}500{,}000{,}000 e_0^2 - 37{,}494{,}140{,}625 e_0 e_1 \\
&+ 22{,}781{,}250{,}000 e_0 e_2 - 22{,}781{,}250{,}000 e_1^2 - 398{,}437{,}500 n - 6{,}201{,}562{,}500 e_1
\end{aligned}$$

$$
\begin{aligned}
&- 6{,}075{,}000{,}000e_2\big), \\
R_1 ={}& 4096\big(\eta^6 - 450\eta^5 e_0 + 84{,}375\eta^4 e_0^2 - 8{,}437{,}500\eta^3 e_0^3 + 474{,}609{,}375\eta^2 e_0^4 \\
&- 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625e_0^6 + 120\eta^5 - 45{,}000\eta^4 e_0 \\
&+ 6{,}750{,}000\eta^3 e_0^2 - 506{,}250{,}000\eta^2 e_0^3 + 18{,}984{,}375{,}000\eta e_0^4 - 284{,}765{,}625{,}000e_0^5 \\
&+ 109{,}375\eta^3 n + 337{,}500\eta^3 e_1 - 24{,}609{,}375\eta^2 n e_0 - 75{,}937{,}500\eta^2 e_0 e_1 \\
&+ 1{,}845{,}703{,}125\eta n e_0^2 + 5{,}695{,}312{,}500\eta e_0^2 e_1 - 46{,}142{,}578{,}125 n e_0^3 \\
&- 142{,}382{,}812{,}500e_0^3 e_1 - 240{,}000\eta^3 + 6{,}562{,}500\eta^2 n + 54{,}000{,}000\eta^2 e_0 \\
&+ 20{,}250{,}000\eta^2 e_1 - 984{,}375{,}000\eta n e_0 - 4{,}050{,}000{,}000\eta e_0^2 - 3{,}037{,}500{,}000\eta e_0 e_1 \\
&+ 36{,}914{,}062{,}500 n e_0^2 + 101{,}250{,}000{,}000e_0^3 + 113{,}906{,}250{,}000e_0^2 e_1 - 7{,}200{,}000\eta^2 \\
&+ 242{,}578{,}125\eta n + 1{,}080{,}000{,}000\eta e_0 + 499{,}921{,}875\eta e_1 - 303{,}750{,}000\eta e_2 \\
&- 2{,}392{,}578{,}125n^2 - 18{,}193{,}359{,}375 n e_0 - 14{,}765{,}625{,}000 n e_1 \\
&- 40{,}500{,}000{,}000e_0^2 - 37{,}494{,}140{,}625e_0 e_1 + 22{,}781{,}250{,}000e_0 e_2 \\
&- 22{,}781{,}250{,}000e_1^2 - 398{,}437{,}500n - 6{,}201{,}562{,}500e_1 - 6{,}075{,}000{,}000e_2\big), \\
Q_2 ={}& 25\big(\eta^6 - 450\eta^5 e_0 + 84{,}375\eta^4 e_0^2 - 8{,}437{,}500\eta^3 e_0^3 + 474{,}609{,}375\eta^2 e_0^4 \\
&- 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625e_0^6 + 30\eta^5 - 11{,}250\eta^4 e_0 + 1{,}687{,}500\eta^3 e_0^2 \\
&- 126{,}562{,}500\eta^2 e_0^3 + 4{,}746{,}093{,}750\eta e_0^4 - 71{,}191{,}406{,}250e_0^5 - 5625\eta^4 \\
&+ 109{,}375\eta^3 n + 1{,}687{,}500\eta^3 e_0 + 337{,}500\eta^3 e_1 - 24{,}609{,}375\eta^2 n e_0 \\
&- 189{,}843{,}750\eta^2 e_0^2 - 75{,}937{,}500\eta^2 e_0 e_1 + 1{,}845{,}703{,}125\eta n e_0^2 + 9{,}492{,}187{,}500\eta e_0^3 \\
&+ 5{,}695{,}312{,}500\eta e_0^2 e_1 - 46{,}142{,}578{,}125 n e_0^3 - 177{,}978{,}515{,}625e_0^4 \\
&- 142{,}382{,}812{,}500e_0^3 e_1 - 88{,}125\eta^3 + 1{,}640{,}625\eta^2 n + 19{,}828{,}125\eta^2 e_0 \\
&+ 5{,}062{,}500\eta^2 e_1 - 246{,}093{,}750\eta n e_0 - 1{,}487{,}109{,}375\eta e_0^2 - 759{,}375{,}000\eta e_0 e_1 \\
&+ 9{,}228{,}515{,}625 n e_0^2 + 37{,}177{,}734{,}375e_0^3 + 28{,}476{,}562{,}500e_0^2 e_1 - 450{,}000\eta^2 \\
&+ 119{,}531{,}250\eta n + 67{,}500{,}000\eta e_0 + 120{,}234{,}375\eta e_1 - 303{,}750{,}000\eta e_2 \\
&- 2{,}392{,}578{,}125n^2 - 8{,}964{,}843{,}750 n e_0 - 14{,}765{,}625{,}000 n e_1 - 2{,}531{,}250{,}000e_0^2 \\
&- 9{,}017{,}578{,}125e_0 e_1 + 22{,}781{,}250{,}000e_0 e_2 - 22{,}781{,}250{,}000e_1^2 + 33{,}750{,}000\eta \\
&- 714{,}843{,}750n - 2{,}531{,}250{,}000e_0 - 3{,}448{,}828{,}125e_1 - 1{,}518{,}750{,}000e_2\big) \\
&\times \big(\eta^6 - 450\eta^5 e_0 + 84{,}375\eta^4 e_0^2 - 8{,}437{,}500\eta^3 e_0^3 + 474{,}609{,}375\eta^2 e_0^4 \\
&- 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625e_0^6 - 30\eta^5 + 11{,}250\eta^4 e_0 - 1{,}687{,}500\eta^3 e_0^2 \\
&+ 126{,}562{,}500\eta^2 e_0^3 - 4{,}746{,}093{,}750\eta e_0^4 + 71{,}191{,}406{,}250e_0^5 - 5625\eta^4 \\
&+ 109{,}375\eta^3 n + 1{,}687{,}500\eta^3 e_0 + 337{,}500\eta^3 e_1 - 24{,}609{,}375\eta^2 n e_0 \\
&- 189{,}843{,}750\eta^2 e_0^2 - 75{,}937{,}500\eta^2 e_0 e_1 + 1{,}845{,}703{,}125\eta n e_0^2 + 9{,}492{,}187{,}500\eta e_0^3 \\
&+ 5{,}695{,}312{,}500\eta e_0^2 e_1 - 46{,}142{,}578{,}125 n e_0^3 - 177{,}978{,}515{,}625e_0^4 \\
&- 142{,}382{,}812{,}500e_0^3 e_1 + 88{,}125\eta^3 - 1{,}640{,}625\eta^2 n - 19{,}828{,}125\eta^2 e_0
\end{aligned}
$$

$$
\begin{aligned}
&- 5{,}062{,}500\eta^2e_1 + 246{,}093{,}750\eta n e_0 + 1{,}487{,}109{,}375\eta e_0^2 + 759{,}375{,}000\eta e_0e_1\\
&- 9{,}228{,}515{,}625ne_0^2 - 37{,}177{,}734{,}375e_0^3 - 28{,}476{,}562{,}500e_0^2e_1 - 450{,}000\eta^2\\
&+ 119{,}531{,}250\eta n + 67{,}500{,}000\eta e_0 + 120{,}234{,}375\eta e_1 - 303{,}750{,}000\eta e_2\\
&- 2{,}392{,}578{,}125n^2 - 8{,}964{,}843{,}750ne_0 - 14{,}765{,}625{,}000ne_1 - 2{,}531{,}250{,}000e_0^2\\
&- 9{,}017{,}578{,}125e_0e_1 + 22{,}781{,}250{,}000e_0e_2 - 22{,}781{,}250{,}000e_1^2 - 33{,}750{,}000\eta\\
&+ 714{,}843{,}750n + 2{,}531{,}250{,}000e_0 + 3{,}448{,}828{,}125e_1 + 1{,}518{,}750{,}000e_2\big),\\
R_2 ={}& 625\big(\eta^6 - 450\eta^5e_0 + 84{,}375\eta^4e_0^2 - 8{,}437{,}500\eta^3e_0^3 + 474{,}609{,}375\eta^2e_0^4\\
&- 14{,}238{,}281{,}250\eta e_0^5 + 177{,}978{,}515{,}625e_0^6 - 30\eta^5 + 11{,}250\eta^4e_0 - 1{,}687{,}500\eta^3e_0^2\\
&+ 126{,}562{,}500\eta^2e_0^3 - 4{,}746{,}093{,}750\eta e_0^4 + 71{,}191{,}406{,}250e_0^5 - 5625\eta^4\\
&+ 109{,}375\eta^3n + 1{,}687{,}500\eta^3e_0 + 337{,}500\eta^3e_1 - 24{,}609{,}375\eta^2ne_0\\
&- 189{,}843{,}750\eta^2e_0^2 - 75{,}937{,}500\eta^2e_0e_1 + 1{,}845{,}703{,}125\eta ne_0^2 + 9{,}492{,}187{,}500\eta e_0^3\\
&+ 5{,}695{,}312{,}500\eta e_0^2e_1 - 46{,}142{,}578{,}125ne_0^3 - 177{,}978{,}515{,}625e_0^4\\
&- 142{,}382{,}812{,}500e_0^3e_1 + 88{,}125\eta^3 - 1{,}640{,}625\eta^2n - 19{,}828{,}125\eta^2e_0\\
&- 5{,}062{,}500\eta^2e_1 + 246{,}093{,}750\eta ne_0 + 1{,}487{,}109{,}375\eta e_0^2 + 759{,}375{,}000\eta e_0e_1\\
&- 9{,}228{,}515{,}625ne_0^2 - 37{,}177{,}734{,}375e_0^3 - 28{,}476{,}562{,}500e_0^2e_1 - 450{,}000\eta^2\\
&+ 119{,}531{,}250\eta n + 67{,}500{,}000\eta e_0 + 120{,}234{,}375\eta e_1 - 303{,}750{,}000\eta e_2\\
&- 2{,}392{,}578{,}125n^2 - 8{,}964{,}843{,}750ne_0 - 14{,}765{,}625{,}000ne_1 - 2{,}531{,}250{,}000e_0^2\\
&- 9{,}017{,}578{,}125e_0e_1 + 22{,}781{,}250{,}000e_0e_2 - 22{,}781{,}250{,}000e_1^2 - 33{,}750{,}000\eta\\
&+ 714{,}843{,}750n + 2{,}531{,}250{,}000e_0 + 3{,}448{,}828{,}125e_1 + 1{,}518{,}750{,}000e_2\big),
\end{aligned}
$$

where $\eta = 25n + 32t$.

For better understanding the above second-order rational solutions, we do the asymptotic analysis of the rational solutions $\tilde{u}_n$ and $\tilde{v}_n$. Let$\xi_1 = \eta + (\frac{112}{25} + \frac{336}{125}\sqrt{5})^{\frac{1}{3}}t^{\frac{1}{3}}$, $\xi_2 = \eta + (\frac{112}{25} - \frac{336}{125}\sqrt{5})^{\frac{1}{3}}t^{\frac{1}{3}}$ and $c = (\frac{112}{25} + \frac{336}{125}\sqrt{5})^{\frac{1}{3}} - (\frac{112}{25} - \frac{336}{125}\sqrt{5})^{\frac{1}{3}} > 0$, then we can find that the solutions $\tilde{u}_n$ and $\tilde{v}_n$ have the following different asymptotic states when $|t| \to \infty$:

(i) If $\xi_1 = \eta + (\frac{112}{25} + \frac{336}{125}\sqrt{5})^{\frac{1}{3}}t^{\frac{1}{3}} = O(1)$, from $\xi_2 = \xi_1 - ct^{\frac{1}{3}}$, we have $\xi_2 \to \mp\infty$ when $t \to \pm\infty$, then calculating the limit states of solutions $\tilde{u}_n$ and $\tilde{v}_n$ in (24) gives the following asymptotic expressions as

$$
\begin{aligned}
\tilde{u}_n &\to u_1^{\pm} = \frac{16}{25} - \frac{240}{\xi_1^2 - 150e_0\xi_1 + 5625e_0^2 - 25},\\
\tilde{v}_n &\to v_1^{\pm} = 1 + \frac{3471\xi_1 - 260{,}325e_0 + 85{,}045}{625\xi_1 - 46{,}875e_0 - 3125}.
\end{aligned}
\tag{25}
$$

(ii) If $\xi_2 = \eta + (\frac{112}{25} - \frac{336}{125}\sqrt{5})^{\frac{1}{3}}t^{\frac{1}{3}} = O(1)$, from $\xi_1 = \xi_2 + ct^{\frac{1}{3}}$, we have $\xi_1 \to \pm\infty$ when $t \to \pm\infty$, then calculating the limits of solutions $\tilde{u}_n$ and $\tilde{v}_n$ in (24) produces the following

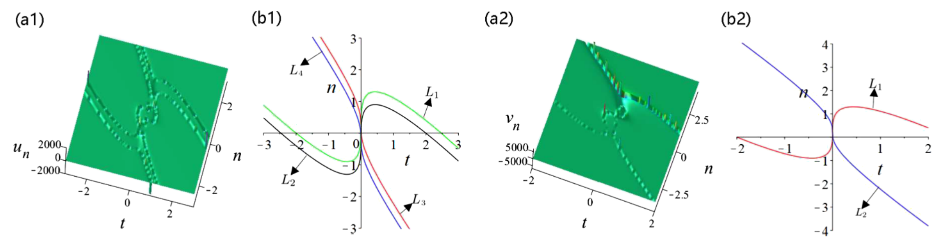


Figure 3 (Color online) Second-order rational solutions: (**a1**) Three dimensional plot of u_n with $e_0 = 0$ in (24); (**b1**) The trajectory plot of solution u_n by expressions (25) and (26) with $e_0 = 0$ corresponding to (**a1**); (**a2**) Three dimensional plot of v_n with $e_0 = 0$ in (24); (**b2**) The trajectory plot of solution v_n by expressions (25) and (26) with $e_0 = 0$ corresponding to (**a2**)

asymptotic expressions in the form

$$\begin{aligned} \tilde{u}_n \to u_2^{\pm} &= \frac{16}{25} - \frac{240}{\xi_2^2 - 150e_0\xi_2 + 5625e_0^2 - 25}, \\ \tilde{v}_n \to v_2^{\pm} &= 1 + \frac{3471\xi_2 - 260{,}325e_0 + 85{,}045}{625\xi_2 - 46{,}875e_0 - 3125}. \end{aligned} \tag{26}$$

It can be seen that $u_1^{\pm}$ and $u_2^{\pm}$ have singularity at four position controllable curves, i.e., $L_1 : \xi_1 - 75e_0 + 5 = 0$, $L_2 : \xi_1 - 75e_0 - 5 = 0$, $L_3 : \xi_2 - 75e_0 + 5 = 0$, $L_4 : \xi_2 - 75e_0 - 5 = 0$, which also are the four center trajectories of solution $\tilde{u}_n$, while $v_1^{\pm}$ and $v_2^{\pm}$ have singularity at two curves, i.e., $L_1 : 625\xi_1 - 46{,}875e_0 - 3125 = 0$, $L_2 : 625\xi_2 - 46{,}875e_0 - 3125 = 0$, which are also the two center trajectories of solution $\tilde{v}_n$. To show the correctness of our analysis results, we plot the rational solutions (24) and their trajectory plots, as shown in Fig. 3. Through comparison, we find that the singularity of rational solutions is completely consistent with these trajectories, showing the correctness of our asymptotic analysis results of second-order rational solutions. In addition, from the asymptotic expressions (25) and (26), we can also clearly see that the asymptotic expressions of second-order rational solutions are consistent with the expressions of the first-order rational solutions. The main difference is that the first-order rational solutions' trajectories are straight lines, while the trajectory lines of second-order rational solutions are curves.

When $N \geq 3$, we can give more complex rational solutions, which will not be discussed here. Below, we omit their analytical expressions and only summarize some mathematical properties of these higher-order rational solutions for Eq. (4). If we use the first kind of Taylor expansion *Type I*, the highest powers in the numerator and denominator for the rational solution u_n of order j are both $2j(2j - 1)$, while the highest powers in the numerator and denominator for the rational solution v_n of order j are both $j(2j - 1)$. If we use the first kind of Taylor expansion *Type II*, the highest powers in the numerator and denominator for the rational solution u_n of order j are both $2j(2j + 1)$, while the highest powers in the numerator and denominator for the rational solution v_n of order j are both $j(2j + 1)$. In either case, the background of u_n is $\frac{1}{1+\alpha^2}$, and the background of v_n is 1.

3.3 Bell-shaped and kink-shaped soliton solutions and dynamics

In this subsection, we will give the discrete soliton solutions of Eq. (4) by use of the discrete generalized $(m, 2N - m)$-fold DT with $m = 2N$ (i.e., the usual $2N$-fold DT), then discuss their dynamic behaviors via numerical simulations.

When $m = 2N$, the discrete generalized $(m, 2N - m)$-fold DT reduces to the discrete generalized $(2N, 0)$-fold DT, which includes the usual $2N$-fold DT if we do not make the Taylor expansion. Next, we will use the usual $2N$-fold DT to give multi-soliton solutions of Eq. (4) based on (17). Here, we take $\zeta(\varepsilon) = 0$. It is worth noting that higher-order soliton solutions will degenerate into lower-order soliton solutions if we take $\lambda_i = 1 + \frac{\alpha}{\sqrt{\alpha^2+1}}, 1 - \frac{\alpha}{\sqrt{\alpha^2+1}}, -1 + \frac{\alpha}{\sqrt{\alpha^2+1}}, -1 - \frac{\alpha}{\sqrt{\alpha^2+1}}$. Below, we uniformly choose $1 + \frac{\alpha}{\sqrt{\alpha^2+1}}$. Next, we only discuss the case of $N = 1$.

When $N = 1$, we need two spectral parameters $\lambda_i, i = 1, 2$, from (15), one can give the following exact solutions as

$$\tilde{u}_n = \frac{a_{n+1}^{(0)}}{(1+\alpha^2)a_n^{(0)}}, \qquad \tilde{v}_n = \frac{\alpha + b_n^{(1)}}{\alpha d_n^{(2)}}, \tag{27}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1}, b_n^{(1)} = \frac{\Delta b_n^{(1)}}{\Delta_1}$ and $d_n^{(2)} = \frac{\Delta d_n^{(2)}}{\Delta_2}$ in which

$$\Delta_{1,n} = \begin{vmatrix} \varphi_{1,n} & \lambda_1 \psi_{1,n} \\ \varphi_{2,n} & \lambda_2 \psi_{2,n} \end{vmatrix}, \qquad \Delta_{2,n} = \begin{vmatrix} \lambda_1 \varphi_{1,n} & \lambda_1^2 \psi_{1,n} \\ \lambda_2 \varphi_{2,n} & \lambda_2^2 \psi_{2,n} \end{vmatrix}, \qquad \Delta a_n^{(0)} = \begin{vmatrix} -\lambda_1^2 \varphi_{1,n} & \lambda_1 \psi_{1,n} \\ -\lambda_2^2 \varphi_{2,n} & \lambda_2 \psi_{2,n} \end{vmatrix},$$
$$\Delta b_n^{(1)} = \begin{vmatrix} \varphi_{1,n} & -\lambda_1^2 \varphi_{1,n} \\ \varphi_{2,n} & -\lambda_2^2 \varphi_{2,n} \end{vmatrix}, \qquad \Delta d_n^{(2)} = \begin{vmatrix} \lambda_1 \varphi_{1,n} & -\psi_{1,n} \\ \lambda_2 \varphi_{2,n} & -\psi_{2,n} \end{vmatrix}.$$

Direct calculation gives the analytical expressions of solution (27) as follows:

$$\begin{cases} \tilde{u}_n = ([\lambda_1 \cosh(\xi_1 + X_1)\cosh\xi_2 - \lambda_2 \cosh\xi_1 \cosh(\xi_2 + X_2)] \\ \quad \times [\lambda_1 \cosh(\xi_1 - X_1)\cosh\xi_2 - \lambda_2 \cosh\xi_1 \cosh(\xi_2 - X_2)]) \\ \quad /((1+\alpha^2)[\lambda_1 \cosh\xi_1 \cosh(\xi_2 + X_2) - \lambda_2 \cosh(\xi_1 + X_1)\cosh\xi_2] \\ \quad \times [\lambda_1 \cosh\xi_1 \cosh(\xi_2 - X_2) - \lambda_2 \cosh(\xi_1 - X_1)\cosh\xi_2]), \\ \tilde{v}_n = R_1/(\alpha\sqrt{1+\alpha^2}[\lambda_1 \cosh(\xi_1 - X_1)\cosh\xi_2 - \lambda_2 \cosh\xi_1 \cosh(\xi_2 - X_2)] \\ \quad \times [\lambda_1 \cosh\xi_1 \cosh(\xi_2 - X_2) - \lambda_2 \cosh(\xi_1 - X_1)\cosh\xi_2]), \end{cases} \tag{28}$$

in which

$$\begin{aligned} R_1 &= \lambda_1\lambda_2\big[\alpha\sqrt{1+\alpha^2}\lambda_1 \cosh(\xi_1 - X_1)\cosh\xi_2 - \alpha\sqrt{1+\alpha^2}\lambda_2 \cosh\xi_1 \cosh(\xi_2 - X_2) \\ &\quad + \left(\lambda_1^2 - \lambda_2^2\right)\cosh\xi_1 \cosh\xi_2\big]\big[\lambda_1 \cosh(\xi_1 - X_1)\cosh\xi_2 - \lambda_2 \cosh\xi_1 \cosh(\xi_2 - X_2)\big], \\ \xi_i &= \frac{1}{2}(\rho_{1,i} - \rho_{2,i})t + \frac{n}{2}(\ln\tau_{1,i} - \ln\tau_{2,i}) + \frac{1}{2}(\ln C_{1,i} - \ln C_{2,i}), X_i = \frac{1}{2}(\ln\tau_{1,i} - \ln\tau_{2,i}), \\ &\quad i = 1, 2. \end{aligned}$$

If one of the λ_1, λ_2 is $1 + \frac{\alpha}{\sqrt{\alpha^2+1}}$, we here set $\lambda_1 = 1 + \frac{\alpha}{\sqrt{\alpha^2+1}} = A$, and the solutions (28) can be rewritten as

$$\begin{cases} \tilde{u}_n = \frac{[A\cosh\xi_2 - \lambda_2 \cosh(\xi_2 + X_2)][A\cosh\xi_2 - \lambda_2 \cosh(\xi_2 - X_2)]}{(1+\alpha^2)[A\cosh(\xi_2 + X_2) - \lambda_2 \cosh\xi_2][A\cosh(\xi_2 - X_2) - \lambda_2 \cosh\xi_2]}, \\ \tilde{v}_n = \frac{A\lambda_2[\alpha\sqrt{1+\alpha^2}A\cosh\xi_2 - \alpha\sqrt{1+\alpha^2}\lambda_2 \cosh(\xi_2 - X_2) + (A^2 - \lambda_2^2)\cosh\xi_2][A\cosh\xi_2 - \lambda_2 \cosh(\xi_2 - X_2)]}{\alpha\sqrt{1+\alpha^2}[A\cosh\xi_2 - \lambda_2 \cosh(\xi_2 - X_2)][A\cosh(\xi_2 - X_2) - \lambda_2 \cosh\xi_2]}, \end{cases} \tag{29}$$

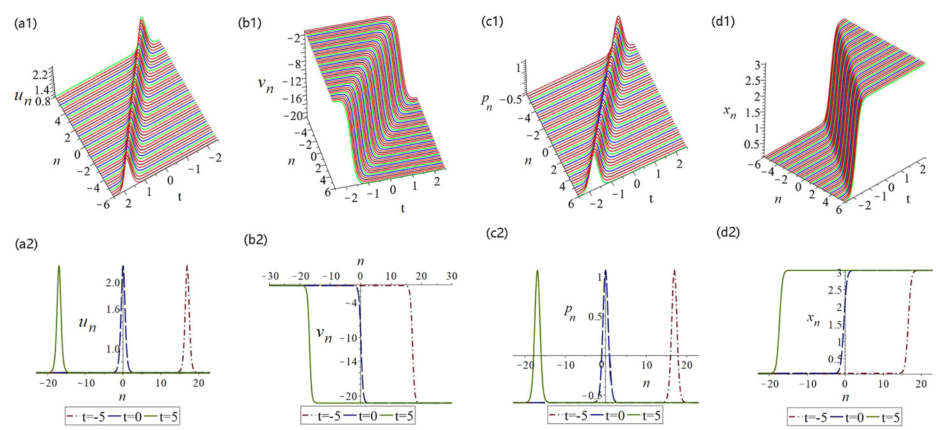


Figure 4 (Color online) Bell-shaped and kink-shaped one-soliton structures via solutions (27) with parameters $\lambda_1 = \frac{8}{5}, \lambda_2 = 3, \alpha = \frac{3}{4}, C_{1,1} = C_{2,1} = C_{1,2} = -C_{2,2} = 1$. (**a1**)–(**d1**) The profiles of one-soliton solutions $\tilde{u}_n, \tilde{p}_n, \tilde{v}_n$ and $\tilde{x}_n$. (**a2**)–(**d2**) The propagation processes for $\tilde{u}_n, \tilde{p}_n, \tilde{v}_n$ and $\tilde{x}_n$ at $t = -5$ (dash-doted line), $t = 0$ (long dashed line) and $t = 5$ (solid line)

whose corresponding evolution plots are shown in Fig. 4 from which can be seen that the solutions (29) are one-soliton solutions. Figures 4(a1)–(a2) present the bell-shaped anti-dark soliton structure of the component $\tilde{u}_n$ on nonzero seed background. Figures 4(b1)–(b2) show the anti-kink-shaped one-soliton structure for the component $\tilde{v}_n$. Figures 4(c1)–(c2) display the bell-shaped anti-dark soliton structures of the component $\tilde{p}_n$ of the original equation. Figures 4(d1)–(d2) show the kink-shaped one-soliton structure for the component $\tilde{x}_n$ of the original equation. From Fig. 4, we can clearly see that one-soliton keeps its same amplitude and shape during propagation.

When neither of λ_1 and λ_2 is $1 + \frac{\alpha}{\sqrt{\alpha^2+1}}$, the solutions (28) are two-soliton solutions. For solutions (28), without loss of generality, we assume that $\alpha > 0$. To exactly analyze the two-soliton solutions $\tilde{u}_n, \tilde{v}_n$ in (28), we perform their asymptotic analysis and arrive at the following four asymptotic patterns:

Before collision $t \to -\infty$:

(i) if ξ_1 is invariant, then $\xi_2 \to +\infty$:

$$
\begin{aligned}
\tilde{u}_n \to \xi^- &= L_1\left(r_{n1}^-\right) \\
&= \left(\left[\lambda_1 \cosh(\xi_1 + X_1) - \lambda_2 \mathrm{e}^{X_2} \cosh \xi_1\right]\left[\lambda_1 \cosh(\xi_1 - X_1) - \lambda_2 \mathrm{e}^{-X_2} \cosh \xi_1\right]\right) \\
&\quad /\left(\left(1 + \alpha^2\right)\left[\lambda_1 \mathrm{e}^{X_2} \cosh \xi_1 - \lambda_2 \cosh(\xi_1 + X_1)\right]\right. \\
&\quad \left.\times \left[\lambda_1 \mathrm{e}^{-X_2} \cosh \xi_1 - \lambda_2 \cosh(\xi_1 - X_1)\right]\right), \\
\tilde{v}_n \to v_{n1}^- &= \left(\lambda_1 \lambda_2\left[\alpha \sqrt{1 + \alpha^2} \lambda_1 \cosh(\xi_1 - X_1) - \alpha \sqrt{1 + \alpha^2} \lambda_2 \mathrm{e}^{-X_2} \cosh \xi_1\right.\right. \\
&\quad \left. + \left(\lambda_1^2 - \lambda_2^2\right) \cosh \xi_1\right]\left.\left[\lambda_1 \cosh(\xi_1 - X_1) - \lambda_2 \mathrm{e}^{-X_2} \cosh \xi_1\right]\right) \\
&\quad /\left(\alpha \sqrt{1 + \alpha^2}\left[\lambda_1 \cosh(\xi_1 - X_1) - \lambda_2 \mathrm{e}^{-X_2} \cosh \xi_1\right]\right. \\
&\quad \left.\times \left[\lambda_1 \mathrm{e}^{-X_2} \cosh \xi_1 - \lambda_2 \cosh(\xi_1 - X_1)\right]\right),
\end{aligned}
$$

(ii) if ξ_2 is invariant, then $\xi_1 \to +\infty$:

$$
\tilde{u}_n \to u_{n2}^- = \left(\left[\lambda_1 \mathrm{e}^{X_1} \cosh \xi_2 - \lambda_2 \cosh(\xi_2 + X_2)\right]\left[\lambda_1 \mathrm{e}^{-X_1} \cosh \xi_2 - \lambda_2 \cosh(\xi_2 - X_2)\right]\right)
$$

$$
\begin{aligned}
&/\big((1+\alpha^2)\big[\lambda_1\cosh(\xi_2+X_2)-\lambda_2 e^{X_1}\cosh\xi_2\big]\\
&\times\big[\lambda_1\cosh(\xi_2-X_2)-\lambda_2 e^{-X_1}\cosh\xi_2\big]\big),\\
\tilde{v}_n\to v_{n2}^- = {}&\big(\lambda_1\lambda_2\big[\alpha\sqrt{1+\alpha^2}\lambda_1 e^{-X_1}\cosh\xi_2-\alpha\sqrt{1+\alpha^2}\lambda_2\cosh(\xi_2-X_2)\\
&+\big(\lambda_1^2-\lambda_2^2\big)\cosh\xi_2\big]\big[\lambda_1 e^{-X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2-X_2)\big]\big)\\
&/\big(\alpha\sqrt{1+\alpha^2}\big[\lambda_1 e^{-X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2-X_2)\big]\\
&\times\big[\lambda_1\cosh(\xi_2-X_2)-\lambda_2 e^{-X_1}\cosh\xi_2\big]\big).
\end{aligned}
$$

After collision $t\to+\infty$:

(iii) if ξ_1 is invariant, then $\xi_2\to-\infty$:

$$
\begin{aligned}
\tilde{u}_n\to u_{n1}^+ = {}&\big(\big[\lambda_1\cosh(\xi_1-X_1)-\lambda_2 e^{X_2}\cosh\xi_1\big]\big[\lambda_1\cosh(\xi_1+X_1)-\lambda_2 e^{-X_2}\cosh\xi_1\big]\big)\\
&/\big((1+\alpha^2)\big[\lambda_1 e^{X_2}\cosh\xi_1-\lambda_2\cosh(\xi_1-X_1)\big]\\
&\times\big[\lambda_1 e^{-X_2}\cosh\xi_1-\lambda_2\cosh(\xi_1+X_1)\big]\big),\\
\tilde{v}_n\to v_{n1}^+ = {}&\big(\lambda_1\lambda_2\big[\alpha\sqrt{1+\alpha^2}\lambda_1\cosh(\xi_1-X_1)-\alpha\sqrt{1+\alpha^2}\lambda_2 e^{X_2}\cosh\xi_1\\
&+\big(\lambda_1^2-\lambda_2^2\big)\cosh\xi_1\big]\big[\lambda_1\cosh(\xi_1-X_1)-\lambda_2 e^{X_2}\cosh\xi_1\big]\big)\\
&/\big(\alpha\sqrt{1+\alpha^2}\big[\lambda_1\cosh(\xi_1-X_1)-\lambda_2 e^{X_2}\cosh\xi_1\big]\\
&\times\big[\lambda_1 e^{X_2}\cosh\xi_1-\lambda_2\cosh(\xi_1-X_1)\big]\big),
\end{aligned}
$$

(iv) if ξ_2 is invariant, then $\xi_1\to-\infty$:

$$
\begin{aligned}
\tilde{u}_n\to u_{n2}^+ = {}&\big(\big[\lambda_1 e^{X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2-X_2)\big]\big[\lambda_1 e^{-X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2+X_2)\big]\big)\\
&/\big((1+\alpha^2)\big[\lambda_1\cosh(\xi_2-X_2)-\lambda_2 e^{X_1}\cosh\xi_2\big]\\
&\times\big[\lambda_1\cosh(\xi_2+X_2)-\lambda_2 e^{-X_1}\cosh\xi_2\big]\big),\\
\tilde{v}_n\to v_{n2}^+ = {}&\big(\lambda_1\lambda_2\big[\alpha\sqrt{1+\alpha^2}\lambda_1 e^{X_1}\cosh\xi_2-\alpha\sqrt{1+\alpha^2}\lambda_2\cosh(\xi_2-X_2)\\
&\times+\big(\lambda_1^2-\lambda_2^2\big)\cosh\xi_2\big]\big[\lambda_1 e^{X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2-X_2)\big]\big)\\
&/\big(\alpha\sqrt{1+\alpha^2}\big[\lambda_1 e^{X_1}\cosh\xi_2-\lambda_2\cosh(\xi_2-X_2)\big]\\
&\times\big[\lambda_1\cosh(\xi_2-X_2)-\lambda_2 e^{X_1}\cosh\xi_2\big]\big).
\end{aligned}
$$

From the above analysis, we can see that the asymptotic expressions of two solitons for solution $\tilde{u}_n$ hardly change, only shift in phase, while the asymptotic expressions of two solitons for solution $\tilde{v}_n$ have changed obviously. So we can infer that the interaction between two solitons for solution $\tilde{u}_n$ is elastic, whereas the interaction between two solitons for solution $\tilde{v}_n$ is inelastic. Next we will draw their plots to verify our analysis results. The evolution structures of solution (28) is shown in Fig. 5. Figures 5(a1)–(a2) demonstrate the head-on elastic interaction between one bell-shaped anti-dark soliton and one dark soliton of the component $\tilde{u}_n$, from which we can clearly see that the amplitudes and shapes of two solitons have not changed. Figures 5(b1)–(b2) display the inelastic interaction between two kink-shaped solitons for the component $\tilde{v}_n$ from which we can clearly see that the amplitudes and shapes of two kink-shaped solitons have changed. Figures 5(c1)–(c2)

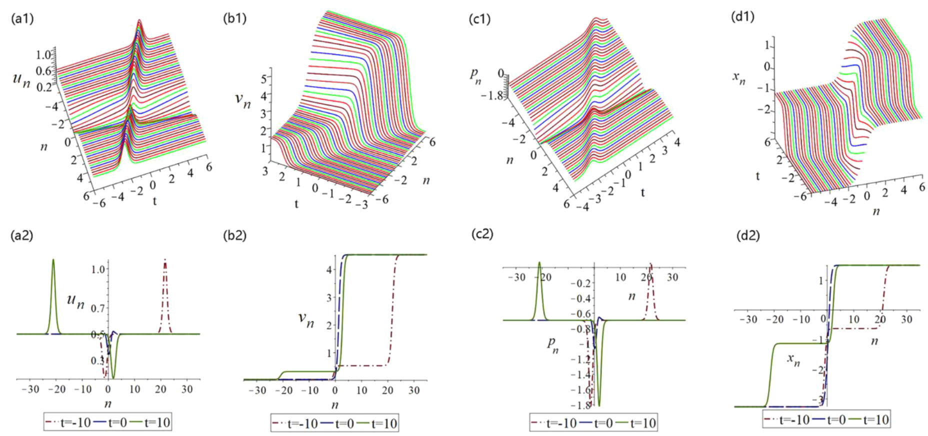


Figure 5 (Color online) Two-soliton interaction structures via the solutions (27) with parameters $\lambda_1 = \frac{1}{6}, \lambda_2 = \frac{5}{2}, \alpha = 1, C_{1,1} = 1, C_{2,1} = 2, C_{1,2} = 1, C_{2,2} = \frac{1}{2}$. (**a1**)–(**d1**) The profiles of two-soliton solutions $\tilde{u}_n, \tilde{p}_n, \tilde{v}_n$ and $\tilde{x}_n$. (**a2**)–(**d2**) The propagation processes for $\tilde{u}_n, \tilde{p}_n, \tilde{v}_n$ and $\tilde{x}_n$ at $t = -10$ (dash-dotted line), $t = 0$ (long dashed line) and $t = 10$ (solid line)

demonstrate the head-on elastic interaction between one bell-shaped anti-dark soliton and one dark soliton of the component $\tilde{p}_n$ of the original equation. Figures 5(d1)–(d2) exhibit the elastic interaction between two kink-shaped solitons for the component $\tilde{x}_n$ of the original equation. It should be noted that we find a very interesting phenomenon. The interaction of two kink-shaped solitons in Eq. (4) is inelastic, and the action of two torsional solitons in the original Eq. (1) is elastic. The main reason for this phenomenon is due to nonlinear transformation between $\tilde{x}_n$ and $\tilde{v}_n$. However, this nonlinear transformation has no effect on the elastic interaction of two bell-shaped solitons the components $\tilde{u}_n$ and $\tilde{p}_n$, and their interactions are still elastic before and after transformation. We think this is a very interesting phenomenon that deserves further study.

Next, we will illustrate the dynamical behaviors of the previous one- and two-soliton solutions of Eq. (4) using numerical simulations. Figures 6–7 exhibit the evolution behaviors of one- and two-soliton solutions, respectively. In Figs. 6–7, the first columns show exact soliton solutions corresponding to Figs. 4–5, the second columns present the numerical solutions without any noise by means of exact solutions as initial conditions of the difference scheme algorithm, and the last two columns present the perturbed numerical solutions through adding 2% and 8% small noises to exact solutions as initial conditions, respectively. From Fig. 6–7(a1),(b1)–(a2),(b2), we can clearly see that the wave evolutions of soliton solutions without any noise are almost identical to their corresponding exact soliton solutions which also show the accuracy of our numerical scheme. When a 2% small noise is added to both the initial exact solutions, the time evolutions are also almost the same as their corresponding exact soliton solutions for a relatively long time (see Figs. 6–7(c1)–(c2)). However, if an 8% noise is added to the initial exact solutions, the wave evolutions have an obviously small oscillation in a relatively short time (see Figs. 6–7(d1)–(d2)). In other words, the one- and two-soliton solutions have stable evolutions and are robust against a small noise.

When $N \geq 2$, we can give more higher-order soliton solutions, which will not be discussed here.

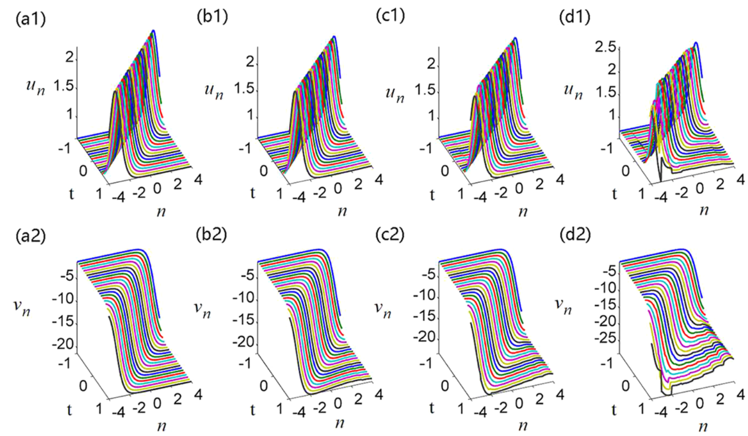


Figure 6 (Color online) One-soliton solutions (27) with the same parameters as Fig. 4. (**a1**)–(**a2**) Exact solutions. (**b1**)–(**b2**) Numerical solutions without noise. (**c1**)–(**c2**) Numerical solutions with a 2% noise. (**d1**)–(**d2**) Numerical solutions with an 8% noise

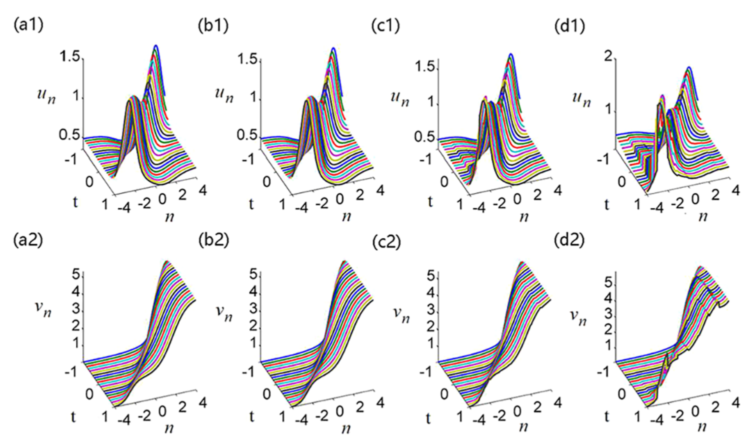


Figure 7 (Color online) Two-soliton solutions (27) with the same parameters as Fig. 5. (**a1**)(**a2**) Exact solutions. (**b1**)(**b2**) Numerical solutions without noise. (**c1**)(**c2**) Numerical solutions with a 2% noise. (**d1**)(**d2**) Numerical solutions with an 8% noise

3.4 Hyperbolic-and-rational mixed solutions with *m* = 2

In this subsection, we will give some hyperbolic-and-rational form mixed solutions of standard soliton and rational solutions of Eq. (4) using the discrete generalized $(2, 2N-2)$-fold DT with two spectral parameters (i.e., generalized $(2, 0)$-fold DT). Next, we will only discuss the case $N = 1$.

When $N = 1$, we set that $\lambda_1 = \frac{8}{5}$ (i.e., $\alpha = \frac{3}{4}$) and $\lambda_2 \neq \frac{8}{5}$ (e.g., $\lambda_2 = 3$), then we let the spectral parameter λ in (17) as $\lambda = \lambda_1 + \varepsilon$ and expand the vector function ϕ_n in (18) as Taylor series around $\varepsilon = 0$ by choosing $C_{1,1} = -C_{2,1} = \frac{1}{\varepsilon}$, and for λ_2, we choose $C_{1,2} = -C_{2,2} = 1$, based on the discrete generalized $(2, 0)$-fold DT, we can obtain the mixed solutions of standard soliton and rational solutions of Eq. (4) as

$$\tilde{u}_n = \frac{a_{n+1}^{(0)}}{(1+\alpha^2)a_n^{(0)}} = \frac{Q_1}{Q_2}, \qquad \tilde{v}_n = \frac{\alpha + b_n^{(1)}}{\alpha d_n^{(2)}} = \frac{R_1}{R_2}, \tag{30}$$

where $a_n^{(0)} = \frac{\Delta a_n^{(0)}}{\Delta_1}, b_n^{(1)} = \frac{\Delta b_n^{(1)}}{\Delta_1}$ and $d_n^{(2)} = \frac{\Delta d_n^{(2)}}{\Delta_2}$ in which

$$\Delta_{1,n} = \begin{vmatrix} \varphi_{1,n}^{(0)} & \lambda_1 \psi_{1,n}^{(0)} \\ \varphi_{2,n} & \lambda_2 \psi_{2,n} \end{vmatrix}, \qquad \Delta_{2,n} = \begin{vmatrix} \lambda_1 \varphi_{1,n}^{(0)} & \lambda_1^2 \psi_{1,n}^{(0)} \\ \lambda_2 \varphi_{2,n} & \lambda_2^2 \psi_{2,n} \end{vmatrix},$$
$$\Delta a_n^{(0)} = \begin{vmatrix} -\lambda_1^2 \varphi_{1,n}^{(0)} & \lambda_1 \psi_{1,n}^{(0)} \\ -\lambda_2^2 \varphi_{2,n} & \lambda_2 \psi_{2,n} \end{vmatrix}, \qquad \Delta b_n^{(1)} = \begin{vmatrix} \varphi_{1,n}^{(0)} & -\lambda_1^2 \varphi_{1,n}^{(0)} \\ \varphi_{2,n} & -\lambda_2^2 \varphi_{2,n} \end{vmatrix},$$
$$\Delta d_n^{(2)} = \begin{vmatrix} \lambda_1 \varphi_{1,n}^{(0)} & -\psi_{1,n}^{(0)} \\ \lambda_2 \varphi_{2,n} & -\psi_{2,n} \end{vmatrix}.$$

Through direct calculation, the simplified analytic expressions of solution (30) are given by

$$\begin{aligned} Q_1 &= 5625\big[30\big(3{,}864{,}000 \cosh \xi_2 - 6440\sqrt{35{,}581}\xi_1 \sinh \xi_2 + 25{,}921\xi_1^2 \cosh \xi_2\big) \\ &\quad + 161\big(30{,}751\xi_1^2 - 720{,}000\big)\big], \\ Q_2 &= 6750\big(13{,}584{,}375 \cosh \xi_2 - 6440\sqrt{35{,}581}\xi_1 \sinh \xi_2 + 25{,}921\xi_1^2 \cosh \xi_2\big) \\ &\quad + 322\big(595{,}217\xi_1^2 - 284{,}765{,}625\big), \\ R_1 &= 36\big[(\sqrt{35{,}581} + 161)(-3220 + 20\sqrt{35{,}581} - 161\xi_1)e^{\xi_2} - (\sqrt{35{,}581} - 161) \\ &\quad \times (3220 + 20\sqrt{35{,}581} + 161\xi_1)\big], \\ R_2 &= \big[(4\sqrt{35{,}581} + 161)(161\xi_1 - 805 + 20\sqrt{35{,}581}) - (4\sqrt{35{,}581} - 161) \\ &\quad \times (20\sqrt{35{,}581} - 161\xi_1 + 805)e^{\xi_2}\big], \end{aligned}$$

where $\xi_1 = 25n + 32t, \xi_2 = n \ln \frac{209+\sqrt{35{,}581}}{209-\sqrt{35{,}581}} + \frac{4\sqrt{35{,}581}}{75}t$. From the above expressions, we can see that the solutions are made up of hyperbolic and rational functions, and we call these solutions hyperbolic-and-rational mixed solutions. Next we will analyze these solutions using asymptotic analysis technique. The asymptotic expressions for solution (30) when $t \to \pm\infty$ are given as follows:

Before collision $t \to -\infty$:

(i) if ξ_1 is unchanged, then $\xi_2 \to -\infty$:

$$\begin{aligned} \tilde{u}_n \to u_{n1}^- &= \frac{16}{25} - \frac{130{,}410{,}000}{[(4\sqrt{35{,}581} + 161)\xi_1 + 16{,}875][(4\sqrt{35{,}581} - 161)\xi_1 + 16{,}875]}, \\ \tilde{v}_n \to v_{n1}^- &= 1 - \frac{5(8\sqrt{35{,}581} - 1127)\xi_1 + 60{,}075}{(4\sqrt{35{,}581} + 161)\xi_1 + 16{,}875}. \end{aligned}$$

(ii) if ξ_2 is unchanged, then $\xi_1 \to -\infty$:

$$\begin{aligned} \tilde{u}_n \to u_{n2}^- &= \frac{16}{25} + \frac{45{,}828{,}328}{25(543{,}375 \cosh \xi_2 + 595{,}217)}, \\ \tilde{v}_n \to v_{n2}^- & \\ &= 1 - \sqrt{\frac{139}{3}} \cosh\left[\frac{1}{2} \ln \frac{75(22{,}033 + 112\sqrt{35{,}581})}{139(3697 - 8\sqrt{35{,}581})}\right] \end{aligned}$$

$$-\sqrt{\frac{139}{3}}\sinh\left[\frac{1}{2}\ln\frac{75(22{,}033+112\sqrt{35{,}581})}{139(3697-8\sqrt{35{,}581})}\right]\tanh\left[\frac{1}{2}\xi_2+\frac{4\sqrt{35{,}581}-161}{\sqrt{543{,}375}}\right].$$

After collision $t\to+\infty$:

(iii) if ξ_1 is unchanged, then $\xi_2\to+\infty$:

$$\tilde{u}_n\to u_{n1}^{-}=\frac{16}{25}-\frac{130{,}410{,}000}{[(4\sqrt{35{,}581}-161)\xi_1-16{,}875][(4\sqrt{35{,}581}+161)\xi_1-16{,}875]},$$

$$\tilde{v}_n\to v_{n1}^{+}=1-\frac{5(8\sqrt{35{,}581}+1127)\xi_1+60{,}075}{(4\sqrt{35{,}581}-161)\xi_1-16{,}875}.$$

(iv) if ξ_2 is unchanged, then $\xi_1\to+\infty$:

$$\tilde{u}_n\to u_{n2}^{+}=\frac{16}{25}+\frac{45{,}828{,}328}{25(543{,}375\cosh\xi_2+595{,}217)},$$

$$\begin{aligned}\tilde{v}_n&\to v_{n2}^{+}\\&=1-\sqrt{\frac{139}{3}}\cosh\left[\frac{1}{2}\ln\frac{75(22{,}033+112\sqrt{35{,}581})}{139(3697-8\sqrt{35{,}581})}\right]\\&\quad-\sqrt{\frac{139}{3}}\sinh\left[\frac{1}{2}\ln\frac{75(22{,}033+112\sqrt{35{,}581})}{139(3697-8\sqrt{35{,}581})}\right]\tanh\left[\frac{1}{2}\xi_2+\frac{4\sqrt{35{,}581}-161}{\sqrt{543{,}375}}\right].\end{aligned}$$

For simplicity, we have converted the exponential function to the hyperbolic function in the asymptotic analysis results by taking advantage of the relationship between the exponential and hyperbolic functions. From the above analysis, we can observe that the solution $\tilde{u}_n$ in (30) is consisted of hyperbolic function soliton solutions and rational solution, while the solution $\tilde{v}_n$ in (30) is consisted by kink-shaped soliton solution and rational solution, just as shown in Fig. 8. Next, we analyze $\tilde{u}_n,\tilde{v}_n$ in (30), respectively:

- For solution $\tilde{u}_n$, before collision, $\tilde{u}_n$ has three trajectory lines: $L_1^-:4\sqrt{35{,}581}+161)\xi_1+16{,}875=0, L_2^-:(4\sqrt{35{,}581}-161)\xi_1+16{,}875=0$ and $\xi^-:\xi_2=0$. As $t\to-\infty$, the rational solution in $\tilde{u}_n$ possesses singularities in two trajectory lines L_1^-,L_2^-, and the bell-shaped soliton in $\tilde{u}_n$ has maximum. After collision $\tilde{u}_n$ three trajectory lines: $L_1^+:4\sqrt{35{,}581}-161)\xi_1-16{,}875=0, L_2^+:(4\sqrt{35{,}581}+161)\xi_1-16{,}875=0$ and $\xi^+:\xi_2=0$. As $t\to+\infty$, the solution $\tilde{u}_n$ possesses singularities in two trajectory lines L_1^+,L_2^+, and the bell-shaped soliton in $\tilde{u}_n$ has maximum. Before and after collisions, the rational solutions and hyperbolic-soliton in the hybrid solution $\tilde{u}_n$ keep their shapes and velocities, so their interactions are elastic.
- For solution $\tilde{v}_n$, before collision, $\tilde{v}_n$ has one singular trajectory line: $L_1^-:(4\sqrt{35{,}581}+161)\xi_1+16{,}875=0$. As $t\to-\infty$, the rational solution in $\tilde{v}_n$ possesses singularity in line L_1^-, and the kink-shaped soliton in $\tilde{v}_n$ has no trajectory. After collision, $\tilde{v}_n$ has one singular trajectory line $L_1^+:(4\sqrt{35{,}581}-161)\xi_1-16{,}875=0$. As $t\to+\infty$, the rational solution in $\tilde{v}_n$ possesses singularity in singular line L_1^+, and the kink-shaped soliton in $\tilde{v}_n$ has no trajectory. Before and after collisions, the rational solution and hyperbolic-soliton in the hybrid solution $\tilde{v}_n$ keep their shapes and velocities, so the interactions are elastic.

To show the correctness of our asymptotic analysis results, we draw the hybrid solutions $\tilde{u}_n$ and $\tilde{v}_n$, including their three-dimensional plots, propagation processes, two-dimensional density plot, and trajectory plots after asymptotic analysis, as shown in Fig. 8. Figures 8(a1)–(a2) exhibit the three-dimensional figures of solutions $\tilde{u}_n$ and

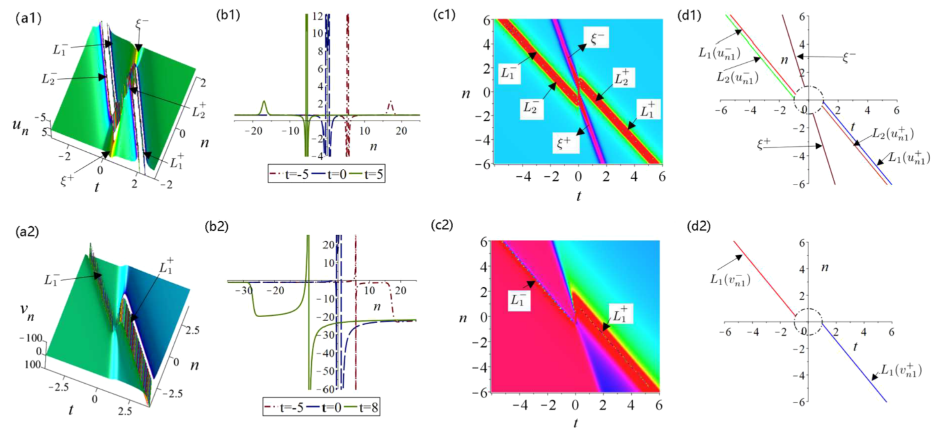


Figure 8 (Color online) Hybrid solutions of hyperbolic-and-rational solutions $\tilde{u}_n, \tilde{v}_n$ via expressions (30): (**a1**)(**a2**) Three-dimensional structure; (**b1**)(**b2**) Propagation processes at different times; (**c1**)(**c2**) Density plots; (**d1**)(**d2**) Trajectory line plots

$\tilde{v}_n$; Figs. 8(b1)–(b2) exhibit the propagation processes of solutions $\tilde{u}_n$ and $\tilde{v}_n$. Form Figs. 8(a1)(b1)–(a2)(b2), we can see that the solution $\tilde{u}_n$ is the mixed solution of one bell-shaped soliton solution and rational solution, while $\tilde{v}_n$ is the mixed solution of one kink-shaped soliton solution and rational solution. Figures 8(c1)–(c2) exhibit the two-dimensional density figures of solutions $\tilde{u}_n$ and $\tilde{v}_n$ corresponding to Figs. 8(a1)–(a2), respectively; Figs. 8(d1)–(d2) exhibit the trajectory plots of solutions $\tilde{u}_n$ and $\tilde{v}_n$ after asymptotic analysis, $L_1(u_{n1}^-)$ and $L_2(u_{n1}^-)$ are the trajectory curve lines of the rational solution in $\tilde{u}_n$ before collision, $L_1(u_{n1}^+)$ and $L_2(u_{n1}^+)$ are the trajectory curve lines after collision, while ξ^- and ξ^+ are the same straight line, which also means that the soliton in solution $\tilde{u}_n$ does not change its propagation direction in the interaction with the rational solutions when $t \to \pm\infty$. This new property is completely different from the interaction between two usual solitons with changing their phases after the collision. From Fig. 8, we can clearly see that the trajectory lines and density plots are completely consistent with our asymptotic analysis results, which also show the correctness of our analysis.

Remark 2 From the above analysis, we can clearly see that the solitons in the hyperbolic-and-rational mixed solutions u_n and v_n are bell-shaped and kink-shaped, respectively, which are also completely consistent with the soliton solutions in the above subsection. These mixed solutions are consistent with the analysis results of the individual soliton or rational solutions. These rational solutions are singular before and after asymptotic analysis, and the shapes and structures of the rational solutions and soliton solutions remain unchanged. From this respect, the interaction of mixed solutions can be considered as elastic.

4 Integrable properties of Eq. (4)

In this section, we will study some integrable aspects of Eq. (4), such as Hamiltonian structures and conservation laws.

4.1 A hierarchy associated with of Eq. (4) and its Hamiltonian structures

In this subsection, we will use the Tu scheme [10] to construct the lattice hierarchy of Eq. (4) and then construct its Hamiltonian structures. We first solve the following stationary discrete zero-curvature equation

$$P_{n+1}U_n - U_nP_n = 0, \tag{31}$$

with

$$P_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}.$$

Substituting the following expansions

$$A_n = \sum_{j=0}^{\infty} A_n^{(j)}\lambda^{-2j}, \qquad B_n = \sum_{j=0}^{\infty} B_n^{(j)}\lambda^{-2j+1}, \qquad C_n = \sum_{j=0}^{\infty} C_n^{(j)}\lambda^{-2j+1},$$

into (31) yields

$$\begin{cases} u_n(A_{n+1}^{(j)} - A_n^{(j)}) - A_{n+1}^{(j+1)} + A_n^{(j+1)} - \frac{\alpha u_n}{v_n}B_{n+1}^{(j+1)} - \alpha v_n C_n^{(j+1)} = 0, \\ \alpha v_n(A_{n+1}^{(j)} + A_n^{(j)}) + B_n^{(j+1)} - u_n B_n^{(j)} = 0, \\ \frac{\alpha u_n}{v_n}(A_{n+1}^{(j)} + A_n^{(j)}) - C_{n+1}^{(j+1)} + u_n C_{n+1}^{(j)} = 0, \\ \alpha v_n C_{n+1}^{(j)} + \frac{\alpha u_n}{v_n}B_n^{(j)} = 0, \end{cases} \tag{32}$$

where $A_n^{(j)}$, $B_n^{(j)}$ and $C_n^{(j)}$ are the functions of u_n, v_n. Now we choose the initial condition $A_n^{(0)} = \frac{1}{2\alpha}, B_n^{(0)} = C_n^{(0)} = 0$ using the recursion relations (32), the following formulae can be obtained as

$$\begin{aligned}
B_n^{(1)} &= -v_n, C_n^{(1)} = \frac{u_{n-1}}{v_{n-1}}, \qquad A_n^{(1)} = \frac{\alpha v_n u_{n-1}}{v_{n-1}} - \frac{1}{\alpha}, \\
B_n^{(2)} &= -u_n v_n - \frac{\alpha^2 v_n^2 u_{n-1}}{v_{n-1}} - \alpha^2 v_{n+1}u_n + 2v_n, \\
C_n^{(2)} &= \frac{u_{n-1}^2}{v_{n-1}} + \frac{\alpha^2 v_n u_{n-1}^2}{v_{n-1}^2} + \frac{\alpha^2 u_{n-1}u_{n-2}}{v_{n-2}} - \frac{2u_{n-1}}{v_{n-1}}, \\
A_n^{(2)} &= \frac{\alpha v_n u_{n-1}u_n}{v_{n-1}} + \frac{\alpha v_n u_{n-1}^2}{v_{n-1}} + \frac{\alpha^3 v_n^2 u_{n-1}^2}{v_{n-1}^2} + \frac{\alpha^3 v_n u_{n-1}u_{n-2}}{v_{n-2}} \\
&\quad + \frac{\alpha^3 v_{n+1}u_n u_{n-1}}{v_{n-1}} - \frac{2\alpha v_n u_{n-1}}{v_{n-1}}, \\
B_n^{(3)} &= -u_n^2 v_n - 2\alpha^2 u_n^2 v_{n+1} + 2u_n v_n - \frac{2u_n v_n^2 \alpha^2 u_{n-1}}{v_{n-1}} - \alpha^2 u_n u_{n+1}v_{n+1} + 2\alpha^2 u_n v_{n+1} \\
&\quad - \frac{\alpha^4 u_n^2 v_{n+1}^2}{v_n} - \frac{2\alpha^4 v_n u_n u_{n-1} v_{n+1}}{v_{n-1}} - \alpha^4 u_n u_{n+1} v_{n+2} - \frac{\alpha^2 v_n^2 u_{n-1}^2}{v_{n-1}} + \frac{2v_n^2\alpha^2 u_{n-1}}{v_{n-1}} \\
&\quad - \frac{\alpha^4 v_n^3 u_{n-1}^2}{v_{n-1}^2} - \frac{\alpha^4 v_n^2 u_{n-1}u_{n-2}}{v_{n-2}}, \\
C_n^{(3)} &= \frac{u_{n-1}^3}{v_{n-1}} + \frac{2\alpha^2 u_{n-1}^3 v_n}{v_{n-1}^2} - \frac{2u_{n-1}^2}{v_{n-1}} + \frac{2u_{n-1}^2\alpha^2 u_{n-1}}{v_{n-2}} + \frac{\alpha^2 u_{n-1}^2 u_n v_n}{v_{n-1}^2} -
\end{aligned} \tag{33}$$

$$\frac{2\alpha^2 u_{n-1}^2 v_n}{v_{n-1}^2} + \frac{\alpha^4 u_{n-1}^3 v_n^2}{v_{n-1}^3} + \frac{2\alpha^4 u_{n-1}^2 u_{n-2} v_n}{v_{n-1} v_{n-1}} + \frac{\alpha^4 u_{n-1}^2 u_n v_{n+1}}{v_{n-1}^2} + \frac{u_{n-1}\alpha^2 u_{n-2}^2}{v_{n-2}}$$
$$- \frac{2u_{n-1}\alpha^2 u_{n-2}}{v_{n-2}} + \frac{v_{n-1} u_{n-1}\alpha^4 u_{n-2}^2}{v_{n-2}^2} + \frac{u_{n-1}\alpha^4 u_{n-1} u_{n-3}}{v_{n-3}} \cdots .$$

Now we truncate P_n as

$$P_n^{(m)} = \lambda^{2m} P_n = \begin{pmatrix} \sum_{j=0}^m A_n^{(j)} \lambda^{2m-2j} & \sum_{j=0}^m B_n^{(j)} \lambda^{2m-2j+1} \\ \sum_{j=0}^m C_n^{(j)} \lambda^{2m-2j+1} & -\sum_{j=0}^m A_n^{(j)} \lambda^{2m-2j} \end{pmatrix}, \quad m \geq 0,$$

from Eq. (31) together with (33), we arrive at

$$EP_n^{(m)} U_n - U_n P_n^{(m)} = \begin{pmatrix} u_n(A_{n+1}^{(m)} - A_n^{(m)}) & -\lambda B_n^{(m+1)} \\ \lambda C_{n+1}^{(m+1)} & 0 \end{pmatrix}. \tag{34}$$

To get the lattice hierarchy of Eq. (4), we need to change $P_n^{(m)}$ in (34), here we set

$$V_n^{(m)} = P_n^{(m)} + \begin{pmatrix} 0 & 0 \\ 0 & A_n^{(m)} \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^m A_n^{(j)} \lambda^{2m-2j} & \sum_{j=0}^m B_n^{(j)} \lambda^{2m-2j+1} \\ \sum_{j=0}^m C_n^{(j)} \lambda^{2m-2j+1} & -\sum_{j=0}^m A_n^{(j)} \lambda^{2m-2j} + A_n^{(m)} \end{pmatrix}, \quad m \geq 0,$$

then we have

$$EV_n^{(m)} U_n - U_n V_n^{(m)} = \begin{pmatrix} u_n(A_{n+1}^{(m)} - A_n^{(m)}) & -\lambda B_n^{(m+1)} - \alpha\lambda v_n A_n^m \\ \lambda C_{n+1}^{(m+1)} - \alpha\lambda \frac{u_n}{v_n} A_{n+1}^{(m)} & 0 \end{pmatrix}. \tag{35}$$

Assuming that the time evolution of ϕ_n satisfies $\phi_{n,t_m} = V_n^{(m)} \phi_n$, then the compatibility condition $E\phi_{n,t_m} = (E\phi_n)_{t_m}$ implies

$$U_{n,t_m} = \big(EV_n^{(m)}\big) U_n - U_n V_n^{(m)}, \quad m \geq 0, \tag{36}$$

which yields the following integrable lattice hierarchy:

$$\begin{cases} u_{n,t_m} = u_n(A_{n+1}^{(m)} - A_n^{(m)}), \\ v_{n,t_m} = -\frac{(B_n^{(m+1)} + \alpha v_n A_n^{(m)})}{\alpha}. \end{cases} \tag{37}$$

The first few equations of this hierarchy can be obtained using (33).

(1) Taking $m = 0$, the hierarchy (37) reduces to the following trivial equation

$$\begin{cases} u_{n,t_0} = u_n(A_{n+1}^{(0)} - A_n^{(0)}) = 0, \\ v_{n,t_0} = -\frac{(B_n^{(1)} + \alpha v_n A_n^{(0)})}{\alpha} = \frac{v_n}{\alpha}, \end{cases} \tag{38}$$

whose time part of Lax pair is

$$\phi_{n,t_0} = V_n^{(0)} \phi_n = \begin{pmatrix} \frac{1}{2\alpha} & 0 \\ 0 & 0 \end{pmatrix} \phi_n. \tag{39}$$

(2) Taking $m = 1$, the hierarchy (37) reduces to Eq. (4), i.e.,

$$\begin{cases} u_{n,t_1} = u_n(A_{n+1}^{(1)} - A_n^{(1)}) = \frac{\alpha u_n(u_n v_{n-1} v_{n+1} - u_{n-1} v_n^2)}{v_n v_{n-1}}, \\ v_{n,t_1} = -\frac{(B_n^{(2)} + \alpha v_n A_n^{(1)})}{\alpha} = \frac{\alpha^2 u_n v_{n+1} + u_n v_n - v_n}{\alpha}, \end{cases} \tag{40}$$

whose time part of Lax pair is

$$\phi_{n,t_1} = V_n^{(1)} \phi_n = \begin{pmatrix} \frac{\lambda^2}{2\alpha} + \frac{\alpha v_n u_{n-1}}{v_{n-1}} - \frac{1}{\alpha} & -\lambda v_n \\ \frac{\lambda u_{n-1}}{v_{n-1}} & -\frac{\lambda^2}{2\alpha} \end{pmatrix} \phi_n. \tag{41}$$

(3) Taking $m = 2$, the hierarchy (37) reduces to the following new equation

$$\begin{cases} u_{n,t_2} = u_n(A_{n+1}^{(2)} - A_n^{(2)}) \\ \quad = u_n\Big(\frac{\alpha u_n u_{n+1} v_{n+1}}{v_n} + \frac{\alpha v_{n+1} u_n^2}{v_n} - \frac{2\alpha v_{n+1} u_n}{v_n} + \frac{\alpha^3 u_n^2 v_{n+1}^2}{v_n^2} \\ \quad + \frac{\alpha^3 u_n u_{n+1} v_{n+2}}{v_n} - \frac{u_n \alpha v_n u_{n-1}}{v_{n-1}} - \frac{\alpha v_n u_{n-1}^2}{v_{n-1}} + \frac{2\alpha v_n u_{n-1}}{v_{n-1}} \\ \quad - \frac{\alpha^3 v_n^2 u_{n-1}^2}{v_{n-1}^2} - \frac{\alpha^3 v_n u_{n-1} u_{n-2}}{v_{n-2}}\Big), \\ v_{n,t_2} = -\frac{(B_n^{(3)} + \alpha v_n A_n^{(2)})}{\alpha} = \frac{\alpha u_n v_n^2 u_{n-1}}{v_{n-1}} + \frac{\alpha^3 v_n u_n u_{n-1} v_{n+1}}{v_{n-1}} + \frac{u_n^2 v_n}{\alpha} \\ \quad + 2\alpha u_n^2 v_{n+1} - \frac{2 u_n v_n}{\alpha} + \alpha u_n u_{n+1} v_{n+1} - 2\alpha u_n v_{n+1} \\ \quad + \frac{\alpha^3 u_n^2 v_{n+1}^2}{v_n} + \alpha^3 u_n u_{n+1} v_{n+2}, \end{cases} \tag{42}$$

whose time part of Lax pair is

$$\phi_{n,t_2} = V_n^{(2)} \phi_n = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \phi_n \tag{43}$$

in which

$$\begin{aligned} V_{11} &= \frac{\lambda^4}{2\alpha} - \frac{\lambda^2}{\alpha} + \frac{\alpha^3 v_n u_{n-1} u_{n-2}}{v_{n-2}} \\ &\quad + \frac{u_{n-1}}{v_{n-1}}\left(\alpha\lambda^2 v_n + \alpha u_n v_n + \alpha v_n u_{n-1} - 2\alpha v_n + \frac{\alpha^3 v_n^2 u_{n-1}}{v_{n-1}} + \alpha^3 u_n v_{n+1}\right), \\ V_{12} &= -\lambda^3 v_n - \lambda u_n v_n - \lambda\alpha^2 u_n v_{n+1} + 2\lambda v_n - \frac{\lambda v_n^2 \alpha^2 u_{n-1}}{v_{n-1}}, \\ V_{21} &= \frac{u_{n-1}}{v_{n-1}}\left(\lambda^3 + \lambda u_{n-1} + \frac{\lambda\alpha^2 u_{n-1} v_n}{v_{n-1}} - 2\lambda\right) + \frac{\lambda u_{n-1}\alpha^2 u_{n-2}}{v_{n-2}}, \\ V_{22} &= -\frac{\lambda^4}{2\alpha} + \frac{\lambda^2}{\alpha} - \frac{\lambda^2 \alpha v_n u_{n-1}}{v_{n-1}}. \end{aligned}$$

We call Eq. (42) the second-order relativistic Toda lattice system, which is a new discrete system that deserves further study.

Our next target is to write the lattice hierarchy (37) into its Hamiltonian form. First of all, we need to understand the meaning of the symbol. The variational derivative of the scalar function f_n with regard to u_i is defined as $\frac{\delta f_n}{\delta u_i} = \sum_{k\in Z} E^{-k}\frac{\partial f_n}{\partial u_{i+k}}$. The formula $(f_n, g_n) = \sum_{n\in Z}\sum_{i=0}^{p} f_{i,n}g_{i,n}$ denotes the inner product between vector functions $f_n = (f_{1,n}, f_{2,n}, \dots, f_{p,n})^{\mathrm{T}}$ and $g_n = (g_{1,n}, g_{2,n}, \dots, g_{p,n})^{\mathrm{T}}$. The Poisson bracket [10] for the Hamiltonian operator J between functions f_n and g_n is defined by $\{f_n, g_n\} = (J\frac{\delta f_n}{\delta u}, \frac{\delta g_n}{\delta u})$. The operator J^* defined by $(f_n, J^* g_n) = (Jf_n, g_n)$ is called the adjoint operator of J with respective to the inner product in which J is described as the skew-symmetric operator if $J = -J^*$.

Define $\langle U, V\rangle = \mathrm{tr}(UV)$, where U and V are arbitrary square matrices. Set

$$V_n = P_n U_n^{-1} = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}\begin{pmatrix} 0 & -\frac{v_n}{\alpha\lambda u_n} \\ \frac{1}{\alpha\lambda v_n} & \frac{-\lambda^2+u_n}{\alpha^2\lambda^2 u_n} \end{pmatrix} = \begin{pmatrix} \frac{B_n}{\alpha\lambda v_n} & -\frac{v_n A_n}{\alpha\lambda u_n} + \frac{(-\lambda^2+u_n)B_n}{\alpha^2\lambda^2 u_n} \\ \frac{-A_n}{\alpha\lambda v_n} & -\frac{v_n C_n}{\alpha\lambda u_n} - \frac{(-\lambda^2+u_n)A_n}{\alpha^2\lambda^2 u_n} \end{pmatrix}, \tag{44}$$

then we have

$$\begin{aligned} &\left\langle V_n, \frac{\partial U_n}{\partial\lambda}\right\rangle = -\frac{(\lambda^2+u_n)B_n}{\alpha\lambda^2 v_n}, \qquad \left\langle V_n, \frac{\partial U_n}{\partial u_n}\right\rangle = \frac{\lambda B_n}{\alpha v_n u_n} + \frac{A_n}{u_n}, \\ &\left\langle V_n, \frac{\partial U_n}{\partial v_n}\right\rangle = \frac{u_n B_n}{\alpha\lambda v_n^2} - \frac{\lambda B_n}{\alpha v_n^2} - \frac{2A_n}{v_n}. \end{aligned} \tag{45}$$

Using the trace identity [10]

$$\frac{\delta}{\delta u}\sum_{n\in Z}\left\langle V_n, \frac{\partial U_n}{\partial\lambda}\right\rangle = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\right)\left\langle V_n, \frac{\partial U_n}{\partial u_i}\right\rangle, \quad i = 1, 2, \tag{46}$$

we have

$$\frac{\delta}{\delta u}\sum_{n\in Z}\left[-\frac{(\lambda^2+u_n)B_n}{\alpha\lambda^2 v_n}\right] = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\right)\begin{pmatrix} \frac{\lambda B_n}{\alpha v_n u_n} + \frac{A_n}{u_n} \\ \frac{u_n B_n}{\alpha\lambda v_n^2} - \frac{\lambda B_n}{\alpha v_n^2} - \frac{2A_n}{v_n} \end{pmatrix}. \tag{47}$$

Direct calculations and equating the coefficients of λ^{-2m-1} on both sides of Eq. (47) yield

$$\begin{pmatrix} \frac{\delta}{\delta u_n} \\ \frac{\delta}{\delta v_n} \end{pmatrix}\cdot\sum_{n\in Z}\left(-\frac{B_n^{(m+1)}}{\alpha v_n} - \frac{u_n B_n^{(m)}}{\alpha v_n}\right) = (\varepsilon - 2m)\begin{pmatrix} \frac{B_n^{(m+1)}}{\alpha v_n u_n} + \frac{A_n^{(m)}}{u_n} \\ \frac{u_n B_n^{(m)}}{\alpha v_n^2} - \frac{B_n^{(m+1)}}{\alpha v_n^2} - \frac{2A_n^{(m)}}{v_n} \end{pmatrix}. \tag{48}$$

To fix the constant ε, we simply set $m = 0$, from Eq. (48), we have $\varepsilon = 0$. Let $H_n^{(m)} = \sum_{n\in Z}\frac{B_n^{(m+1)}+u_n B_n^{(m)}}{2m\alpha v_n}$, then

$$\frac{\delta H_n^{(m)}}{\delta u} = \begin{pmatrix} \frac{B_n^{(m+1)}}{\alpha v_n u_n} + \frac{A_n^{(m)}}{u_n} \\ \frac{u_n B_n^{(m)}}{\alpha v_n^2} - \frac{B_n^{(m+1)}}{\alpha v_n^2} - \frac{2A_n^{(m)}}{v_n} \end{pmatrix}, \tag{49}$$

if we set $f_n^{(m)} = \frac{B_n^{(m+1)}}{\alpha v_n u_n} + \frac{A_n^{(m)}}{u_n}$, $g_n^{(m)} = \frac{u_n B_n^{(m)}}{\alpha v_n^2} - \frac{B_n^{(m+1)}}{\alpha v_n^2} - \frac{2A_n^{(m)}}{v_n}$, then we have

$$\begin{aligned} &A_n^{(m)} = (E-1)^{-1}v_n g_n^{(m)}, \qquad B_n^{(m)} = \frac{\alpha v_n u_n f_n^{(m)} + \alpha v_n (E-1)^{-1}v_{n+1}g_{n+1}^{(m+1)}}{u_n}, \\ &C_n^{(m)} = -\frac{\alpha u_{n-1}E^{-1}f_n^{(m)} + \alpha(E-1)^{-1}v_n g_n^{(m)}}{v_{n-1}}. \end{aligned} \tag{50}$$

Then Eq. (37) can be rewritten as the following Hamiltonian form:

$$U_{t_m}=\begin{pmatrix}u_{n,t_m}\\ v_{n,t_m}\end{pmatrix}=J\frac{\delta H_n^{(m)}}{\delta u}=J\begin{pmatrix}f_n^{(m)}\\ g_n^{(m)}\end{pmatrix}, \tag{51}$$

with

$$J=\begin{pmatrix}0 & u_nv_n\\ -u_nv_n & 0\end{pmatrix}$$

from which we can see that the matrix J is skew-symmetric. Taking $\eta=\begin{pmatrix}\eta_{11}&\eta_{12}\\ \eta_{21}&\eta_{22}\end{pmatrix}$ to satisfy $\frac{\delta H_n^{(m)}}{\delta u}=\eta\frac{\delta H_n^{(m-1)}}{\delta u}$, then by recursion relations (32), we have

$$\begin{aligned}
\eta_{11}&=u_n-\frac{(E-1)^{-1}}{u_n}\left(\frac{\alpha^2u_n^2v_{n+1}}{v_n}-\frac{\alpha^2v_{n+2}u_{n+1}u_{n+2}E^2}{v_{n+1}}\right),\\
\eta_{12}&=(E-1)^{-1}v_{n+1}E-(E+1)(E-1)^{-1}v_n\\
&\quad-\frac{1}{u_n}(E-1)^{-1}\left[u_{n+1}v_{n+1}E-\frac{\alpha^2u_{n+1}v_{n+2}(E-1)^{-1}v_{n+3}E^3}{v_{n+1}}\right.\\
&\quad+\frac{\alpha^2u_{n+1}v_{n+2}(E+1)(E-1)^{-1}v_{n+2}E^2}{v_{n+1}}\\
&\quad\left.-\frac{\alpha^2u_nv_{n+1}(E+1)(E-1)^{-1}v_n}{v_n}+\frac{\alpha^2u_nv_{n+1}(E-1)^{-1}v_{n+1}E}{v_n}\right],\\
\eta_{21}&=-\frac{\alpha^2u_nv_{n+1}u_{n+1}E}{v_n^2}+\frac{\alpha^2u_{n-1}^2E^{-1}}{v_{n-1}},\\
\eta_{22}&=u_n-\frac{\alpha^2u_nv_{n+1}(E-1)^{-1}v_{n+2}E^2}{v_n^2}+\frac{\alpha^2u_nv_{n+1}(E+1)(E-1)^{-1}v_{n+1}E}{v_n^2}\\
&\quad-\frac{\alpha^2u_{n-1}(E+1)(E-1)^{-1}v_{n-1}E^{-1}}{v_{n-1}}\\
&\quad+\frac{\alpha^2q_{n-1}(E-1)^{-1}r_n}{r_{n-1}}.
\end{aligned}$$

Therefore, we have rewritten the lattice hierarchy (37) into the following Hamiltonian form:

$$U_{t_m}=J\frac{\delta H_n^{(m)}}{\delta u}=J\begin{pmatrix}f_n^{(m)}\\ g_n^{(m)}\end{pmatrix}=J\eta\begin{pmatrix}f_n^{(m-1)}\\ g_n^{(m-1)}\end{pmatrix}=\cdots=J\eta^m\begin{pmatrix}f_n^{(0)}\\ g_n^{(0)}\end{pmatrix}=J\eta^m\begin{pmatrix}-\frac{1}{2\alpha u_n}\\ 0\end{pmatrix}. \tag{52}$$

It can be verified that J and $J\eta$ are skew-symmetric operators, and, moreover, the Hamiltonian functions $H_n^{(m)}$ ($m\geq 0$) denoted by Eq. (49) are pairwise involutory with respect to the Poisson bracket.

4.2 An infinite number of conservation laws of Eq. (4)

As is known, the conservation law plays a very important role in the study of integrable systems [6, 32]. So, in this subsection, we will present an infinite number of conservation laws of Eq. (4) based on its known Lax pair (5) and (6). From the 2×2 matrix spectral problem (5), we have

$$\begin{cases} \varphi_{n+1} = (-\lambda^2 + u_n)\varphi_n + \alpha\lambda v_n \psi_n, \\ \psi_{n+1} = -\frac{\alpha\lambda u_n}{v_n}\varphi_n, \end{cases} \tag{53}$$

Set $\theta_n = \frac{\varphi_n}{\psi_n}$, from (53), we can get

$$(-\lambda^2 + u_n + \alpha\lambda v_n \theta_n)\theta_{n+1} + \frac{\alpha\lambda u_n}{v_n} = 0. \tag{54}$$

Inserting $\theta_n = \sum_{j=0}^{n} \theta_n^{(j)}\lambda^j$ into (54) and collecting the coefficients of same powers of λ, we obtain the following recursion relations:

$$\begin{aligned} &\theta_{n+1}^{(0)} = 0, \qquad \theta_{n+1}^{(1)} = -\frac{\alpha}{v_n}, \qquad \theta_{n+1}^{(2)} = 0, \\ &\theta_{n+1}^{(3)} = -\frac{\alpha}{u_n}\left(\frac{1}{v_n} + \frac{\alpha^2}{v_{n-1}}\right), \qquad \theta_{n+1}^{(4)} = 0, \\ &\theta_{n+1}^{(5)} = -\frac{\alpha}{u_n^2 v_n} - \frac{\alpha^3}{u_n^2 v_{n-1}} \\ &\qquad - \frac{\alpha^3 v_n}{u_n}\left(\frac{1}{v_{n-1}v_n u_n} + \frac{\alpha^2}{v_{n-1}^2 u_n} + \frac{1}{v_{n-1}v_n u_{n-1}} + \frac{\alpha^2}{v_{n-2}v_n u_{n-1}}\right), \ldots, \\ &\theta_{n+1}^{(2j)} = 0, \qquad \theta_{n+1}^{(2j+1)} = \frac{\theta_{n+1}^{(2j-1)} - \alpha v_n \sum_{i=0}^{2j} \theta_n^{(i)}\theta_{n+1}^{(2j-i)}}{u_n}, \qquad j \geq 3. \end{aligned} \tag{55}$$

At the same time, from Eqs. (41) and (53), a straightforward calculation yields conservation laws for Eq. (4) as

$$\left[\ln\left(-\lambda^2 + u_n + \alpha\lambda v_n \theta_n\right)\right]_t = (E-1)\left(\frac{\lambda^2}{2\alpha} + \frac{\alpha v_n u_{n-1}}{v_{n-1}} - \frac{1}{\alpha} + \lambda v_n \theta_n\right). \tag{56}$$

Substituting the expressions (55) into (56) and comparing the same powers of λ on both sides of (56), we can get an infinite number of conservation laws for Eq. (4). The first three conservation laws usually stand for the energy conservation, momentum conservation, and Hamiltonian conservation, which are listed as follows:

$$[\ln u_n]_t = (E-1)\left(\frac{\alpha v_n u_{n-1}}{v_{n-1}} - \frac{1}{\alpha}\right), \tag{57}$$

$$\left[-\frac{1}{u_n}\left(\frac{\alpha^2 v_n}{v_{n-1}} - 1\right)\right]_t = (E-1)\left(\frac{1}{2\alpha} - \frac{\alpha v_n}{v_{n-1}}\right), \tag{58}$$

$$\begin{aligned} &\left[-\frac{\alpha^2 v_n}{u_{n-1}u_n}\left(\frac{1}{v_{n-1}} + \frac{\alpha^2}{v_{n-2}}\right) - \frac{1}{2u_n^2}\left(\frac{\alpha^4 v_n^2}{v_{n-1}^2} + \frac{2\alpha^2 v_n}{v_{n-1}} + 1\right)\right]_t \\ &\quad = (E-1)\left(-\frac{\alpha^3 v_n}{u_{n-1}v_{n-2}} - \frac{\alpha v_n}{u_{n-1}v_{n-1}}\right). \end{aligned} \tag{59}$$

5 Conclusions

In this paper, we have studied the relativistic Toda lattice equation (4), which might explain particle vibrations in lattices. The main achievements of this paper are as follows: (i) Based on the known Lax pair of Eq. (4), we have constructed its discrete $(m, 2N - m)$-fold DT for the first time; (ii) By using the special cases of the resulting DT, various analytic solutions such as the rational and semi-rational solutions, soliton solutions and their mixed solutions of Eq. (4), and the asymptotic analysis technique is used to discuss their limit states. We have especially discussed the elastic and inelastic interactions of two-soliton solutions. And numerical simulations are used to illustrate the dynamical behaviors of one- and two-soliton solutions, showing that the evolutions are robust against a small noise. It is a very interesting phenomenon that there are both elastic and inelastic interactions in the same equation, which is worthy of further study. In addition, we also have summarized some mathematical features of different-order rational solutions of Eq. (4). Through the asymptotic state analysis of rational solutions, we find that the singularities of rational solutions are completely consistent with the trajectories of their asymptotic state expressions, from which we can better understand the characteristics of these rational solutions; (iii) We have investigated some integrable aspects of Eq. (4) such as the infinitely many conservation laws, relevant discrete integrable hierarchy, and Hamiltonian structures via the Tu scheme, which can better help us understand this equation. The results presented in this paper might help understand some physical phenomena in lattice dynamics.

Acknowledgements
We would like to express our sincere thanks to other members of our discussion group for their valuable comments.

Funding
This work has been partially supported by Beijing Natural Science Foundation under Grant No. 1202006. M. L. Qin is supported by Postgraduate Science and Technology Innovation Project of Beijing Information Science and Technology University under Grant No. 5112111017.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contributions
MQ performed the theory analysis, performed the computations, and prepared the manuscript. XW participated in the design of the study and the theory analysis, and moreover helped to revise and improve the manuscript. All authors have read and approved the final manuscript.

Received: 8 December 2021 Accepted: 6 May 2023 Published online: 15 May 2023

References

1. Toda, M.: Theory of Nonlinear Lattices. Springer, Berlin (1989)
2. Ablowitz, M.J., Ladik, J.: Nonlinear differential-difference equation. J. Math. Phys. **16**, 598–603 (1975)
3. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
4. Ablowitz, M.J., Musslimani, Z.H.: Integrable discrete PT symmetric model. Phys. Rev. E **90**, 032912 (2014)
5. Ablowitz, M.J., Musslimani, Z.H.: Integrable nonlocal nonlinear equations. Stud. Appl. Math. **139**, 7–59 (2017)
6. Wadati, M.: Transformation theories for nonlinear discrete systems. Prog. Theor. Phys. Suppl. **59**, 36–63 (1976)
7. Wen, X.Y.: Elastic interaction and conservation laws for the nonlinear self-dual network equation in electric circuit. J. Phys. Soc. Jpn. **81**, 114006 (2012)
8. Ohta, Y., Hirota, R.: A discrete KdV equation and its Casorati determinant solution. J. Phys. Soc. Jpn. **60**, 2095 (1991)
9. Wen, X.Y., Hu, X.Y.: *N*-fold Darboux transformation and solitonic interactions for a Volterra lattice system. Adv. Differ. Equ. **2014**, 213 (2014)
10. Tu, G.Z.: A trace identity and its application to the theory of discrete integrable systems. J. Phys. A, Math. Gen. **23**, 3903–3922 (1990)

11. Suris, Y.B.: On some integrable systems related to the Toda lattice. J. Phys. A, Math. Gen. **30**, 2235–2249 (1997)
12. Zhang, X.Q., Yang, H.X., Zhao, J.C., Xu, X.X.: Soliton solution of the Toda lattice equation by the Darboux transformation. Chin. J. Phys. **44**, 109–116 (2006)
13. Wen, X.Y.: *N*-fold Darboux transformation and soliton solutions for Toda lattice equation. Rep. Math. Phys. **68**, 211–223 (2011)
14. Liu, N., Wen, X.Y.: Dynamics of bright and dark multi-soliton solutions for two higher-order Toda lattice equations for nonlinear waves. Adv. Differ. Equ. **2018**, 289 (2018)
15. Ma, W.X., You, Y.C.: Rational solutions of the Toda lattice equation in Casoratian form. Chaos Solitons Fractals **22**, 395–406 (2004)
16. Ma, W.X., Maruno, K.I.: Complexiton solutions of the Toda lattice equation. Physica A **343**, 219–237 (2004)
17. Qin, M.L., Wen, X.Y., Yuan, C.L.: Integrability, multi-soliton and rational solutions, and dynamical analysis for a relativistic Toda lattice system with one perturbation parameter. Commun. Theor. Phys. **73**, 065003 (2021)
18. Ruijsenaars, S.N.M.: Relativistic Toda systems. Commun. Math. Phys. **133**, 217–247 (1990)
19. Fan, F.C., Xu, Z.G., Shi, S.Y.: *N*-fold Darboux transformations and exact solutions of the combined Toda lattice and relativistic Toda lattice equation. Anal. Math. Phys. **10**, 31 (2020)
20. Bruschi, M., Ragnisco, O.: Lax representation and complete integrability for the periodic relativistic Toda lattice. Phys. Lett. A **134**, 365–370 (1989)
21. Yang, H.X., Shen, D., Zhu, L.L.: A hierarchy of Hamiltonian lattice equations associated with the relativistic Toda type system. Phys. Lett. A **373**, 2695–2703 (2009)
22. Yang, H.X., Xu, X.X., Sun, Y.P., Ding, H.H.: Integrable relativistic Toda type lattice hierarchies, associated coupling systems and the Darboux transformation. J. Phys. A, Math. Gen. **39**, 3933–3947 (2006)
23. Andrew, P., Zhu, Z.N.: Darboux–Bäcklund transformation and explicit solutions to a hybrid lattice of the relativistic Toda lattice and the modified Toda lattice. Phys. Lett. A **378**, 1510–1513 (2014)
24. Zhou, R.G., Jiang, Q.Y.: A Darboux transformation and an exact solution for the relativistic Toda lattice equation. J. Phys. A **38**, 7735–7742 (2005)
25. Darvishi, M.T., Khani, F.: New exact solutions of a relativistic Toda lattice system. Chin. Phys. Lett. **29**, 094101 (2012)
26. Suris, Y.B.: The Problem of Integrable Discretization: Hamiltonian Approach. Birkhäuser, Basel (2003)
27. Yu, F.J., Feng, S.: Explicit solution and Darboux transformation for a new discrete integrable soliton hierarchy with 4×4 Lax pairs. Math. Methods Appl. Sci. **40**, 5515–5525 (2017)
28. Yu, F.J.: Dynamics of nonautonomous discrete rogue wave solutions for an Ablowitz–Musslimani equation with PT-symmetric potential. Chaos **27**, 023108 (2017)
29. Yu, F.J., Yu, J.M., Li, L.: Some discrete soliton solutions and interactions for the coupled Ablowitz–Ladik equations with branched dispersion. Wave Motion **94**, 102500 (2020)
30. Matveev, V.B., Salle, M.A.: Darboux Transformations and Solitons. Springer, Berlin (1991)
31. Yuan, C.L., Wen, X.Y.: Integrability, discrete kink multi-soliton solutions on an inclined plane background and dynamics in the modified exponential Toda lattice equation. Nonlinear Dyn. **105**, 643–669 (2021)
32. Wadati, M., Sanuki, H., Konno, K.: Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws. Prog. Theor. Phys. **53**, 419–436 (1975)

Publisher's Note