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Time-averaging principle for G-SDEs based on Lyapunov condition

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Abstract

In this paper, we tame the uncertainty about the volatility in time-averaging principle for stochastic differential equations driven by G-Brownian motion (G-SDEs) based on the Lyapunov condition. That means we treat the time-averaging principle for stochastic differential equations based on the Lyapunov condition in the presence of a family of probability measures, each corresponding to a different scenario for the volatility. The main tool for the mathematical analysis is the G-stochastic calculus, which is introduced in the book by Peng (Nonlinear Expectations and Stochastic Calculus Under Uncertainty. Springer, Berlin, 2019). We show that the solution of a standard equation converges to the solution of the corresponding averaging equation in the sense of sublinear expectation with the help of some properties of G-stochastic calculus. Numerical results obtained using PYTHON illustrate the efficiency of the averaging method.

MSC: 70K70; 65C30

Keywords: Volatility uncertainty; Averaging principle; Stochastic differential equation; G-Brownian motion; Sublinear expectation

1 Introduction

In mathematical finance, the traditional way of representing random fluctuations of financial quantities is to use a Brownian motion, which is typically scaled by the volatility constant. However, there is plenty of empirical evidence, derived from market prices, showing that the volatility of financial quantities is not constant and even not deterministic. The presence of volatility uncertainty leads to mathematical difficulties since the family of probability measures representing volatility uncertainty contains mutually singular measures. The canonical process B has a different volatility under each probability measure in a family of probability measures $\mathcal{P}$. Thus the quadratic variation process $\langle B\rangle = (\langle B\rangle_t)_{t\geq 0}$ differs among the probability measures in $\mathcal{P}$. For example, we consider the probability measures $P^{\bar{\sigma}}$ and $P^{\underline{\sigma}}$, induced by the constant volatilities $\bar{\sigma}$ and $\underline{\sigma}$, respectively, and we have

$$P^{\underline{\sigma}}\big(\langle B\rangle_t = \underline{\sigma}^2 t\big) = 1 \neq 0 = P^{\bar{\sigma}}\left(\langle B\rangle_t = \underline{\sigma}^2 t\right).$$

Therefore, the set $\mathcal{P}$ contains mutually singular probability measures, that is, there are probability measures in the set of probability measures that have different null sets. This causes mathematical problems since many results from probability theory and stochastic calculus only hold up to null sets of the underlying measure. Important examples include the time consistent conditional expectations and stochastic integrals.

The averaging principle is an important property in the study of the dynamical behavior for nonlinear dynamical systems. The key technique of the averaging principle is time-scales separation. In particular, the averaging principle provides a powerful tool for simplifying dynamical systems and obtaining approximate solutions to differential equations arising from mechanics, mathematics, physics, control, and other areas. Averaging principles for stochastic systems were proposed by Stratonovich [19, 20] to examine nonlinear oscillation problems in the presence of random noise. Since then there has been a big amount of papers devoted to the study of the averaging principle for stochastic (partial) differential equations, see Khasminskii [12], Freidlin and Wentzell [2], Givon [7], Fu and Liu [3], Xu, Duan and Xu [22], Fu, Wan, Liu [4], Xu, Miao, Liu [21], etc.

From the point of view of fully nonlinear parabolic partial differential equations, in Hu and Wang [10] the authors perfectly established the averaging principle for stochastic differential equations driven by G-Brownian motion, where the condition on the coefficients is the global Lipschitz condition, see assumption (H1) in Hu and Wang [10]. From the same point of view, Hu, Jiang, and Wang [9] extended the one of Hu and Wang [10] to the forward-backward stochastic differential equations driven by G-Brownian motion with global Lipschitz condition. To the authors knowledge, averaging principles for stochastic differential equations with locally Lipschitz coefficients based on a Lyapunov condition under volatility uncertainty, that is, stochastic differential equations driven by G-Brownian motion (G-SDEs) under sublinear expectation, have not been considered. Therefore in this paper we consider these averaging principles in a family of probability measures $\mathcal{P}$, where the coefficients in the stochastic differential equations have no global Lipschitz assumption. Another important difference from Hu, Jiang, and Wang [9] is that the convergence mode is different. As the reviewer pointed out to the author, the global Lipschitz is not essential, the main difference is that the author obtained a strong convergence result instead of weak convergence compared with Hu et al. [9], but the author needed quite a strong condition (B). Mao, Chen, and You [15] obtained an excellent result about the averaging principle for stochastic differential equations driven by G-Brownian motion with global non-Lipschitz coefficients. Recent important progress in the theory of volatility uncertainty/G-Brownian motion is reviewed by Peng [16] with comments on its explanation, theory, and significance.

In this paper, we study the averaging principle for the following stochastic differential equation with locally Lipschitz coefficients based on a Lyapunov condition under volatility uncertainty:

$$\begin{aligned} X_t^\epsilon = X_0^\epsilon &+ \int_0^t b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) ds + \int_0^t h\left(\frac{s}{\epsilon}, X_s^\epsilon\right) d\langle B\rangle_s \\ &+ \int_0^t \sigma\left(\frac{s}{\epsilon}, X_s^\epsilon\right) dB_s, \quad t \in [0,T], \end{aligned}$$

where the canonical process B is a 1-dimensional G-Brownian motion, which was introduced by Peng [16] and is used to represent the uncertainty about the volatility, and $\langle B\rangle$.

is a quadratic variation process of the G-Brownian motion B. From the construction of G-Brownian motion, Peng proved that the quadratic variation process $(\langle B\rangle_t)_{t\geq 0}$ is an increasing stochastic process with $(\langle B\rangle_t)_0 = 0$, and $\langle B\rangle_t$ is not a deterministic process unless $(B_t)_{t\geq 0}$ is a classical Brownian motion. For more details on G-Brownian motion, we refer to Peng [16].

The main difficulty of this paper is dealing with the local Lipschitz coefficients based on a Lyapunov condition. We shall apply the localization technique to approximate the solution of stochastic differential equation with locally Lipschitz coefficients. Based on the work of Li, Peng [14], Hu, Wang, Zheng [11], the space of suitable integrands is essentially expanded, which requires less regularity and only local integrability. Particularly, we will define the truncated G-SDEs that are uniform Lipschitz and carefully choose the stopping times to construct a consistent localized sequence. However, the Lyapunov-type condition ensures that the G-SDEs with the local Lipschitz coefficients can be approximated pathwisely by the truncated G-SDEs.

This paper is organized as follows. Section 2 introduces G-Brownian motion to represent the volatility uncertainty and related stochastic calculus briefly. In Sect. 3, we prove the averaging principle for stochastic differential equations in a family of probability measures. In Sect. 4, we present numerical simulation of stochastic differential equations under volatility uncertainty and give three examples to demonstrate the averaging method using PYTHON.

2 Preliminaries

We introduce in this section some of the basic notions relating to volatility uncertainty and G-Brownian motion and then recall some preliminary results in G-Brownian motion, which are needed in the sequel. More details can be found in Peng [16].

2.1 *G*-Brownian motion

Definition 2.1 Given a set Ω and a linear space $\mathcal{H}$ of real-valued functions defined on Ω. Moreover, if $X_i \in \mathcal{H}, i = 1,2,\ldots,d$, then $\varphi(X_1,\ldots,X_d) \in \mathcal{H}$ for all $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d)$, where $C_{b,\mathrm{Lip}}(\mathbb{R}^d)$ is the space of all bounded real-valued Lipschitz continuous functions. A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,

(i) Monotonicity: If $X \leq Y$, then $\hat{\mathbb{E}}[X] \leq \hat{\mathbb{E}}[Y]$;
(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c$ for any $c \in \mathbb{R}$;
(iii) Subadditivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
(iv) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X]$ for any $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Denote by $\Omega = C_0(\mathbb{R}^+)$ the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$d(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\Big[\max_{t\in[0,i]}\big|\omega_t^1 - \omega_t^2\big| \wedge 1\Big].$$

$\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω. For each $t \in [0,\infty)$, we introduce the following spaces:

- $\Omega_t := \{\omega(\cdot \wedge t) : \omega \in \Omega\}, \mathcal{F}_t := \mathcal{B}(\Omega_t)$;

- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $L^0(\Omega_t)$: the space of all $\mathcal{F}_t$-measurable real functions;
- $B_b(\Omega)$: all bounded elements in $L^0(\Omega)$, $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$;
- $C_b(\Omega)$: all continuous elements in $B_b(\Omega)$, $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$.
- $C_{b,\mathrm{Lip}}(\mathbb{R}^n)$: the space of all bounded $\mathbb{R}$-valued Lipschitz continuous functions on $\mathbb{R}^n$.

Let $\Omega = C_0$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\geq 0}$ starting from origin, equipped with local uniformity, and let $B_t(\omega) = \omega_t$ be the canonical process. For each $t \in [0,\infty)$, define $\Omega_t = \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$. Set

$$\mathrm{Lip}(\Omega_t) = \bigl\{\varphi(B_{t_1},\dots,B_{t_n}) : n \geq 1, t_1,\dots,t_n \in 0,t, \varphi \in C_{b,\mathrm{Lip}(\mathbb{R}^n)}\bigr\},$$

and $\mathrm{Lip}(\Omega) = \bigcup_{t\geq 0} \mathrm{Lip}(\Omega_t)$. For each $x \in \mathbb{R}$, we consider any given monotonic and sublinear function

$$G(x) := \frac{1}{2}\bigl(\bar{\sigma}^2 x^+ - \underline{\sigma}^2 x^-\bigr). \tag{1}$$

Here, we assume that G is nondegenerate, i.e., $0 < \underline{\sigma}^2 \leq \bar{\sigma}^2 < \infty$. In Peng [16], a G-Brownian motion is constructed on a sublinear expectation space $(\Omega, \mathrm{Lip}(\Omega), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t\geq 0})$, which is called G-expectation space. In this space the corresponding canonical process $B_t(\omega) = \omega_t$ is a G-Brownian motion.

Let $\mathbb{L}_G^p(\Omega)$ (respectively $L_*^p(\Omega)$) be the completion of $\mathrm{Lip}(\Omega)$ (respectively $B_b(\Omega)$) under the natural norm $\|X\|_p = \hat{\mathbb{E}}[|X|^p]^{1/p}$. Denis, Hu, and Peng [1] proved that

$$C_b(\Omega) \subset \mathbb{L}_G^p(\Omega) \subset L_*^p(\Omega),$$

and there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X] \quad \text{for any } X \in \mathbb{L}_G^1(\Omega).$$

Then $L_*^p(\Omega)$ and $\mathbb{L}_G^p(\Omega)$ can be characterized as follows:

$$L_*^p(\Omega) = \Bigl\{X \in L^0(\Omega) \big| \lim_{x\to\infty} \hat{\mathbb{E}}\bigl[|X|^p I_{|X|\geq x}\bigr] = 0\Bigr\}$$

and

$$\mathbb{L}_G^p(\Omega) = \bigl\{X \in L_*^p(\Omega) \,|\, X \text{ has a quasi-continuous version}\bigr\}.$$

We will introduce two natural capacities:

$$\mathbb{V}(A) := \hat{\mathbb{E}}[I_A] = \sup_{P\in\mathcal{P}} E_P[I_A] = \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega)$$

and

$$v(A) := -\hat{\mathbb{E}}[-I_A] = -\sup_{P\in\mathcal{P}} E_P[-I_A] = \inf_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 2.2 A set $A \in \mathcal{B}(\Omega)$ is polar if $V(A) = 0$. The property holds quasi-surely (q.s.) if it holds outside a polar set.

In what follows, we do not distinguish between two random variables X and Y if $X = Y$ q.s.

The following inequality is a capacity version of the Markov inequality.

Proposition 2.3 *Let $X \in L^0(\Omega)$ and $\hat{\mathbb{E}}[|X|^p] < \infty, p > 0$. For any $x > 0$, then*

$$\mathbb{V}\big(|X| \geq x\big) \leq \frac{\hat{\mathbb{E}}[|X|^p]}{x^p}.$$

For the proof, see Lemma 6.1.17 in Peng [16].

2.2 *G*-stochastic calculus

Peng [16] also introduced the related stochastic calculus of Itô type with respect to G-Brownian motion. Now we recall Peng's G-stochastic calculus from Li and Peng [14] or Chap. 8 in Peng [16], and let $T > 0$ be fixed.

Definition 2.4 Consider the following simple type of processes:

$$M_{b,0}(0,T) = \Bigg\{\eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t) \ \forall N > 0, \\ 0 = t_0 < \cdots < t_N = T, \xi_j \in B_b(\Omega_{t_j}), j = 0,1,2,\ldots,N-1\Bigg\}.$$

For an element $\eta \in M_{b,0}(0,T)$ with $\eta_t = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t)$, the related Bochner integral is

$$\int_0^T \eta_t(\omega)\,dt = \sum_{j=0}^{N-1} \xi_i(\omega)(t_{j+1} - t_j).$$

Definition 2.5 For each $p \geq 1$, we denote by $M_*^p(0,T)$ the completion of $M_{b,0}(0,T)$ under the norm

$$\|\eta\|_{M^p(0,T)} = \left(\hat{\mathbb{E}}\left[\int_0^T \big|\eta(t)\big|^p\,dt\right]\right)^{1/p}.$$

Definition 2.6 For each $\eta \in M_{b,0}(0,T)$ of the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t),$$

define

$$I(\eta) = \int_0^T \eta_s\,dB_s := \sum_{j=0}^{N-1} \xi_j\big(B_{t^N_{j+1}} - B_{t^N_j}\big).$$

The mapping $I : M_{b,0}(0,T) \to L_*^2(\Omega_T)$ can be continuously extended to $I : M_*^2(0,T) \to L_*^2(\Omega_T)$. For each $\eta \in M_*^2(0,T)$, the stochastic integral is defined by

$$I(\eta) = \int_0^T \eta_s \, dB_s, \quad \eta \in M_*^2(0,T).$$

Definition 2.7 For fixed $p \geq 1$, a stochastic process $(\eta_t)_{t\geq 0}$ is said to be in $M_w^p(0,T)$ if it is associated with a sequence of increasing stopping times $\{\sigma_n\}_{n\in\mathbb{N}}$ such that

$$\{\eta_t I_{[0,\sigma_n]}(t)\}_{t\in[0,T]} \in M_*^p(0,T), \quad \forall n \in \mathbb{N},$$

and if $\Omega^{(n)} := \{\omega \in \Omega : \sigma_n(\omega) \wedge T + T\}$ and $\hat{\Omega} := \lim_{n\to\infty} \Omega^{(n)}$, then $V(\hat{\Omega}^c) = 0$.

Given $\eta \in M_w^2(0,T)$ associated with $\{\sigma_n\}_{n\in\mathbb{N}}$, we note $\tau_n := \sigma_n \wedge T$ and consider the continuous modification of $(\int_0^t \eta_s I_{[0,\tau_n]}(s)\, dB_s)_{0\leq t\leq T}$. For each $m, n \in \mathbb{N}, n > m$, we can find a polar set $\hat{A}^{m,n}$ such that for all $\omega \in (\hat{A}^{m,n})^c$ the following equality holds:

$$\int_0^{t\wedge\tau_m} \eta_s \, dB_s(\omega) = \int_0^t \eta_s I_{[0,\tau_m]}(s) I_{[0,\tau_n]}(s) \, dB_s(\omega) = \int_0^{t\wedge\tau_m} \eta_s I_{[0,\tau_n]}(s) \, dB_s(\omega)$$

for $0 \leq t \leq T$. Define a polar set $\hat{A} := \bigcup_{m=1}^\infty \bigcup_{n=m+1}^\infty \hat{A}^{m,n}$. For each $n \in \mathbb{N}$ and $(\omega, t) \in \Omega \times [0,T]$, we set

$$X_t^n(\omega) := \begin{cases} \int_0^t \eta_s I_{[0,\tau_n]} \, dB_s(\omega), & \omega \in \hat{A}^c \cap \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

For each $\omega \in \hat{A}^c$ and $m, n \in \mathbb{N}, n > m, X^n(\omega) \equiv X^m(\omega)$ on $[0, \tau_m(\omega)]$. Therefore we can define unambiguously a process by stipulating that it is equal to X^m on $[0, \tau_m(\omega)]$.

Definition 2.8 Let $\eta \in M_w^2(0,T)$ for each $(\omega, t) \in \Omega \times [0,T]$, we define

$$\int_0^t \eta_s \, dB_s(\omega) := \lim_{n\to\infty} X_t^n(\omega).$$

It is important that the quadratic process of G-Brownian motion B is not always a deterministic process, and it can be formulated by

$$\langle B \rangle_t := \lim_{N\to\infty} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s \, dB_s,$$

where $t_i^N = (jT)/N$ for each integer $N \geq 1$.

Definition 2.9 Define a mapping $Q : M_{b,0}(0,T) \to \mathbb{L}_*^1(\Omega_T)$:

$$Q(\eta) = \int_0^T \eta_s \, d\langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j \big(\langle B \rangle_{t_{j+1}^N} - \langle B \rangle_{t_j^N} \big).$$

Then Q can be uniquely extended to $M_w^1(0,T)$, we also denote this mapping by

$$Q(\eta) = \int_0^T \eta_s \, d\langle B\rangle_s, \quad \eta \in M_w^1(0,T).$$

In view of the dual formulation of sublinear expectation as well as the properties of the quadratic variation process $\langle B\rangle$ in the framework of sublinear expectation, we can generalize the following BDG-type inequalities in Gao [5] to $\eta \in M_w^p([0,T])$.

Lemma 2.10

(1) *For each $p \geq 1$ and $\eta \in M_w^p(0,T)$,*

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t \eta_s \, d\langle B\rangle_s\right|^p\right] \leq \bar{\sigma}^{2p} T^{p-1} \int_0^T \hat{\mathbb{E}}\left[|\eta_s|^p\right] ds.$$

(2) *For each $p \geq 2$ and $\eta \in M_w^p(0,T)$, there exists some constant C_p depending only on p and T such that*

$$\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t \eta_s \, dB_s\right|^p\right] \leq C_p \hat{\mathbb{E}}\left[\left|\int_0^T |\eta_s|^2 \, ds\right|^{\frac{p}{2}}\right].$$

2.3 *G*-stochastic differential equation

Consider the following SDE driven by a 1-dimensional G-Brownian motion:

$$X_t = X_0 + \int_0^t b(s,X_s) \, ds + \int_0^t h(s,X_s) \, d\langle B\rangle_s + \int_0^t \sigma(s,X_s) \, dB_s, \quad t \in [0,T], \tag{2}$$

where the initial condition $X_0 \in \mathbb{R}$ is a given constant.

We will consider this GSDE, whose coefficients satisfy both a locally Lipschitz condition and a Lyapunov-type condition.

(A1) $b, h, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given deterministic functions satisfying continuous in t and locally Lipschitz in x, i.e., for each $x, y \in B_0(R) := \{a||a| \leq R\}$, there exists a positive constant C_R that depends only on R such that for each $t \in [0,T]$,

$$\left|\psi(t,x) - \psi(t,y)\right| \leq C_R|x-y|$$

and

$$\sup_{t\in[0,T]}\left|\psi(t,0)\right| \leq L,$$

where $\psi = b, h, \sigma$, respectively.

(A2) There exists a deterministic nonnegative Lyapunov function $V \in C^{1,2}([0,T] \times \mathbb{R})$ such that

$$\inf_{|x|\geq R} \inf_{t\in[0,T]} V(t,x) \to \infty, \quad \text{as } R \to \infty,$$

and for some constant $C_L > 0$ and all $(t,x) \in [0,T] \times \mathbb{R}$,

$$\mathcal{L}V(t,x) \leq C_L V(t,x),$$

where $\mathcal{L}$ is a differential operator defined by

$$\mathcal{L}V = \partial_t V + \partial_x Vb + G\big(2\partial_x Vh + \partial_{x^2}^2 V\sigma^2\big).$$

Here, $G(\cdot)$ is a sublinear function defined in (1).

In Li, Lin, and Lin [13], they established the existence and uniqueness of the solution for the above GSDE with locally Lipschitz and Lyapunov conditions through the localization methods. We review their excellent results in what follows.

Theorem 2.11 *Under assumptions* $(A1)$ *and* $(A2)$, *there exists a unique solution* $X \in M_w^2(0,T;\mathbb{R})$ *to the G-stochastic differential equation* (2) *and* X *has t-continuous paths on* $[0,T]$.

We notice that the domain of coefficients here is a little larger than the one in Peng [16], where Peng [16] states the following result. We recall the following standard linear growth and Lipschitz assumption:

(H1) There exists some constant L such that

$$\big|\psi(t,x) - \psi(t,y)\big| \leq L|x-y|$$

for each $t \in [0,T], x, y \in \mathbb{R}$, and

$$\sup_{t\in[0,T]} \big|\psi(t,0)\big| \leq L,$$

where $\psi = b, h, \sigma$, respectively.

(H2) $b, h, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given functions satisfying, for each $x \in \mathbb{R}$, $b(\cdot,x), h(\cdot,x), \sigma(\cdot,x) \in M_G^2(0,T)$ and

$$\big|b(\cdot,x)\big|^2 + \big|h(\cdot,x)\big|^2 + \big|\sigma(\cdot,x)\big|^2 < C\big(1+|x|^2\big).$$

We also recall the following excellent results from Peng [16].

Theorem 2.12 *Under assumptions* $(H1)$ *and* $(H2)$, *there exists a unique solution* $X \in M_G^2(0,T)$ *to the G-stochastic differential equation* (2). *Denote by* X_t *the solution starting with* $X_0 \in \mathbb{R}$, *then there exists* $C > 0$ *that depends on* T *such that*

$$\hat{\mathbb{E}}\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] \leq C\big(1+|X_0|^2\big). \tag{3}$$

The following corollary can be deduced from (3), see also Corollary 5.3.2 in Peng [16].

Corollary 2.13 *Assume that Lipschitz condition* $(H1)$ *and linear growth condition* $(H2)$ *hold*, *then we have*

$$\hat{\mathbb{E}}\big[|X_t - X_s|^2\big] \leq C|t-s|,$$

where the constant C *depends only on the Lipschitz constant and the initial value* X_0.

To apply this theorem, we need to assume that (H2) holds throughout this paper, and C may be a positive constant whose value may change in different occasions.

3 Averaging principle under volatility uncertainty

We now study an averaging principle for a stochastic differential equation driven by a G-Brownian motion in $\mathbb{R}$:

$$X_t^\epsilon = X_0 + \int_0^t b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) ds + \int_0^t h\left(\frac{s}{\epsilon}, X_s^\epsilon\right) d\langle B\rangle_s + \int_0^t \sigma\left(\frac{s}{\epsilon}, X_s^\epsilon\right) dB_s, \quad t \in [0,T], \tag{4}$$

where $\epsilon \in (0,1)$ and the initial condition $X_0 \in \mathbb{R}$ is a given constant. When the functions b, h, σ satisfy the conditions as in (A1)–(A2), then, by Theorem 2.11, equation (4) has a unique solution $X^\epsilon \in M_w^2(0,T;\mathbb{R})$.

Our objective is to show that the solution $X^\epsilon \in M_w^2(0,T;\mathbb{R})$ could be approximated by the solution of some simplified equations. For this, we associate the above stochastic differential equation with the following averaged stochastic differential equation driven by a G-Brownian motion:

$$\bar{X}_t = X_0 + \int_0^t \bar{b}(\bar{X}_s)\, ds + \int_0^t \bar{h}(\bar{X}_s)\, d\langle B\rangle_s + \int_0^t \bar{\sigma}(\bar{X}_s)\, dB_s, \quad t \in [0,T]. \tag{5}$$

Here, the functions $\bar{b}, \bar{h}, \bar{\sigma}$ are called time averaged functions, and the locally Lipschitz condition and the Lyapunov-type condition (A1) and (A2) are satisfied without time term, where the differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}V = \partial_t V + \partial_x V \bar{b} + G\big(2\partial_x V \bar{h} + \partial_{x^2}^2 V \bar{\sigma}^2\big).$$

By Theorem 2.11, the above averaged G-stochastic differential equation has a unique solution $\bar{X} \in M_w^2(0,T)$ to (5).

To get the averaging principle, we need the following time averaging conditions for functions b, h, σ and $\bar{b}, \bar{h}, \bar{\sigma}$:

(B) For any $T_1 \in [0,T]$ and all x, there exists a function φ such that

$$\sup_{t\geq 0}\left|\frac{1}{T_1}\int_t^{t+T_1}\big(b(s,x) - \bar{b}(x)\big)\, ds\right|^2 \leq \varphi(T_1)\big(1+|x|^2\big), \tag{6}$$

$$\sup_{t\geq 0}\frac{1}{T_1}\int_t^{t+T_1}\big|h(s,x) - \bar{h}(x)\big|^2\, ds \leq \varphi(T_1)\big(1+|x|^2\big),$$

and

$$\sup_{t\geq 0}\frac{1}{T_1}\int_t^{t+T_1}\big|\sigma(s,x) - \bar{\sigma}(x)\big|^2\, ds \leq \varphi(T_1)\big(1+|x|^2\big).$$

Here, $\varphi(T_1)$ is a positive bounded function with $\lim_{T_1\to\infty}\varphi(T_1)=0$. The averaged functions $\bar{f}$ (where $\bar{f}=\bar{b},\bar{h},\bar{\sigma}$) have been given many different definitions in the literature, for instance, we can choose $\bar{f}(x)=\frac{1}{T}\int_0^T f(s,x)\,ds$.

In fact, it follows from assumption (B) that the averaged functions $\bar{f}$ (where $\bar{f}=\bar{b},\bar{h},\bar{\sigma}$) satisfy (locally) Lipschitz conditions with the same Lipschitz constant as f, provided that f satisfy (locally) Lipschitz conditions. We only state that the function $\bar{b}$ satisfies the locally Lipschitz condition as b. In fact, for every $x,y\in B_0(R)$ and every $T>0$, we have

$$\begin{aligned}
\left|\bar{b}(x)-\bar{b}(y)\right|^2 &\le \left|\frac{1}{T}\int_0^T\left[b(s,x)-\bar{b}(x)\right]ds\right|^2+\left|\frac{1}{T}\int_0^T\left[b(s,y)-\bar{b}(y)\right]ds\right|^2\\
&\quad+\left|\frac{1}{T}\int_0^T\left[b(s,x)-b(s,y)\right]ds\right|^2\\
&\le 2C\varphi(T)\left(1+|x|^2+|y|^2\right)+C_R^2|x-y|^2.
\end{aligned}$$

Then, taking T tending to infinity in the inequality, we get the function $\bar{b}$ is locally Lipschitz. Similar discussion and the following remark can be found in Gao [6], Guo, Lv, Wei [8], Shen, Song, Wu [17], and Shen, Xiang, Wu [18].

Remark 3.1 Due to

$$\sup_{t\ge0}\left|\frac{1}{T_1}\int_t^{t+T_1}\left(b(s,x)-\bar{b}(x)\right)ds\right|^2\le\sup_{t\ge0}\frac{1}{T_1}\int_t^{t+T_1}\left|\left(b(s,x)-\bar{b}(x)\right)\right|^2ds,$$

we can claim that (6) in assumption (B) is weaker than the following traditional averaging condition:

$$\sup_{t\ge0}\frac{1}{T_1}\int_t^{t+T_1}\left|\left(b(s,x)-\bar{b}(x)\right)\right|^2ds\le\varphi(T_1)\left(1+|x|^2\right).$$

With several preliminary assumptions at our hands, we are in a position to present our main results. We first introduce a lemma which is important for our averaging principle, and then we consider the averaging principle of the GSDE with standard linear growth and Lipschitz assumption. Finally, we extend those assumptions to a locally Lipschitz condition and a Lyapunov-type condition.

Lemma 3.2 *Suppose that assumptions* $(H1)$, $(H2)$, *and* (B) *are satisfied. Then*

$$\lim_{\epsilon\to0}\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t\left[b\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right|^2\right]=0. \tag{7}$$

Proof Let $\{t_1,t_2,\dots,t_N\}$ be a partition of $[0,T]$:

$$t_i=i\sqrt{\epsilon},\quad i=0,1,2,\dots,N-1;t_N=T,$$

and

$$0<T-t_{N-1}\le\sqrt{\epsilon}.$$

Then it is easy to obtain that $T \leq N\sqrt{\epsilon} < T + \sqrt{\epsilon}$. Let

$$Z_i := \int_{t_i}^{t_{i+1}} \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds,$$

then we have

$$\begin{aligned}
&\left| \int_0^t \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 \\
&\quad \leq N \left| \int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^t \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 + N \sum_{i=0}^{N-2} |Z_i|^2.
\end{aligned} \tag{8}$$

Using Hölder's inequality and linear growth assumption on b and $\bar{b}$, we get

$$\begin{aligned}
&\left| \int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^t \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 \\
&\quad \leq 2\left(t - \left[\frac{t}{\epsilon}\right]\sqrt{\epsilon} \right) \int_{[\frac{t}{\sqrt{\epsilon}}]\sqrt{\epsilon}}^t \left[\left| b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) \right|^2 + \left| \bar{b}\left(X_s^\epsilon\right) \right|^2 \right] ds \\
&\quad \leq C \left| t - \left[\frac{t}{\sqrt{\epsilon}}\right]\sqrt{\epsilon} \right|^2 \left(1 + \sup_{0 \leq t \leq T} \left| X_s^\epsilon \right|^2 \right) \\
&\quad \leq C\epsilon \left(1 + \sup_{0 \leq t \leq T} \left| X_s^\epsilon \right|^2 \right).
\end{aligned} \tag{9}$$

By Theorem 2.12, (8), and (9), we have

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \left[b\left(s, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 \right] \\
&\quad \leq C\epsilon N + N\hat{\mathbb{E}}\left[\sum_{i=0}^{N-2} |Z_i|^2 \right] \\
&\quad \leq C\epsilon(T + \sqrt{\epsilon}) + N \sum_{i=0}^{N-2} \hat{\mathbb{E}}\left[|Z_i|^2 \right].
\end{aligned} \tag{10}$$

Here, in the last inequality, we have used the subadditivity of sublinear expectation. By assumption (B), the Lipschitz conditions of b and $\bar{b}$, we obtain

$$\begin{aligned}
|Z_i|^2 &= \left| \int_{t_i}^{t_{i+1}} \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 \\
&\leq 3 \left| \int_{t_i}^{t_{i+1}} \left[b\left(\frac{s}{\epsilon}, X_s^\epsilon\right) - b\left(\frac{s}{\epsilon}, X_{t_i}^\epsilon\right) \right] ds \right|^2 + 3 \left| \int_{t_i}^{t_{i+1}} \left[b\left(\frac{s}{\epsilon}, X_{t_i}^\epsilon\right) - \bar{b}\left(X_{t_i}^\epsilon\right) \right] ds \right|^2 \\
&\quad + 3 \left| \int_{t_i}^{t_{i+1}} \left[\bar{b}\left(X_{t_i}^\epsilon\right) - \bar{b}\left(X_s^\epsilon\right) \right] ds \right|^2 \\
&\leq 3 \left| \epsilon \int_{t_i/\epsilon}^{t_{i+1}/\epsilon} \left[b\left(s, X_{t_i}^\epsilon\right) - \bar{b}\left(X_{t_i}^\epsilon\right) \right] ds \right|^2 + 6L\sqrt{\epsilon} \int_{t_i}^{t_{i+1}} \left| X_s^\epsilon - X_{t_i}^\epsilon \right|^2 ds
\end{aligned}$$

$$\leq C\epsilon\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\left(1+\sup_{0\leq t\leq T}\left|X_t^\epsilon\right|^2\right)+6L\sqrt{\epsilon}\int_{t_i}^{t_{i+1}}\left|X_s^\epsilon-X_{t_i}^\epsilon\right|^2 ds. \tag{11}$$

Hence, by Corollary 2.13, we have

$$\begin{aligned}
N\sum_{i=0}^{N-2}\hat{\mathbb{E}}\left[|Z_i|^2\right] &\leq C\epsilon N\sum_{i=0}^{N-2}\hat{\mathbb{E}}\left[\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\left(1+\sup_{0\leq t\leq T}\left|X_t^\epsilon\right|^2\right)\right] \\
&\quad + 6l_1\sqrt{\epsilon}N\sum_{i=0}^{N-2}\hat{\mathbb{E}}\left[\int_{t_i}^{t_{i+1}}\left|X_s^\epsilon-X_{t_i}^\epsilon\right|^2 ds\right] \\
&\leq C\epsilon N^2\left[\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)+\sqrt{\epsilon}\right] \\
&\leq C(T+\sqrt{\epsilon})^2\left[\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)+\sqrt{\epsilon}\right].
\end{aligned} \tag{12}$$

Then, substituting (12) in (10), we finally obtain the estimate

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right|^2\right] \\
&\quad\leq C\epsilon(T+\sqrt{\epsilon})+C(T+\sqrt{\epsilon})^2\left[\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)+\sqrt{\epsilon}\right].
\end{aligned}$$

Finally, the required inequality follows by letting ϵ tend to zero in the inequality. □

Lemma 3.3 *Suppose that assumptions* ($H1$), ($H2$), *and* (B) *are satisfied. Then*

$$\lim_{\epsilon\to 0}\hat{\mathbb{E}}\left[\int_0^T\left|h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right|^2 ds\right]=0. \tag{13}$$

Proof Let $\{t_1,t_2,\ldots,t_N\}$ be a partition of $[0,T]$, $t_i=i\sqrt{\epsilon}, i=0,1,2,\ldots,N-1; t_N=T, 0<T-t_{N-1}\leq\sqrt{\epsilon}$. Hence $T\leq N\sqrt{\epsilon}<T+\sqrt{\epsilon}$. Let

$$H_i:=\int_{t_i}^{t_{i+1}}\left|h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right|^2 ds,$$

then, by subadditivity of the sublinear expectation, we have

$$\hat{\mathbb{E}}\left[\int_0^T\left|h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right|ds\right]\leq N\sum_{i=0}^{N-1}\hat{\mathbb{E}}H_i. \tag{14}$$

By assumptions (H1), (H2), and (B), we get

$$\begin{aligned}
\hat{\mathbb{E}}H_i &= \hat{\mathbb{E}}\int_{t_i}^{t_{i+1}}\left|h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right|^2 ds \\
&\leq 3\int_{t_i}^{t_{i+1}}\left|h\left(\frac{s}{\epsilon},X_{t_i}^\epsilon\right)-\bar{h}\left(X_{t_i}^\epsilon\right)\right|^2 ds+3\int_{t_i}^{t_{i+1}}\left|h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-h\left(\frac{s}{\epsilon},X_{t_i}^\epsilon\right)\right|^2 ds \\
&\quad + 3\int_{t_i}^{t_{i+1}}\left|\bar{h}\left(X_{t_i}^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq 3\epsilon \int_{\frac{t_i}{\epsilon}}^{\frac{t_{i+1}}{\epsilon}} \left|h\left(s, X_{t_i}^{\epsilon}\right) - \bar{h}\left(X_{t_i}^{\epsilon}\right)\right|^2 d\langle B\rangle_s + 6L \int_{t_i}^{t_{i+1}} \left|X_s^{\epsilon} - X_{t_i}^{\epsilon}\right|^2 ds \\
&\leq 3\sqrt{\epsilon}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\left(1 + \sup_{t\in[0,T]}\left|X_t^{\epsilon}\right|^2\right) + 6L \int_{t_i}^{t_{i+1}} \left|X_s^{\epsilon} - X_{t_i}^{\epsilon}\right|^2 ds. \qquad (15)
\end{aligned}$$

Hence, using Corollary 2.13 again, we have

$$\begin{aligned}
\sum_{i=0}^{N-1} \hat{\mathbb{E}}[H_i] &\leq 3\sqrt{\epsilon} \sum_{i=0}^{N-1} \varphi\left(\frac{1}{\sqrt{\epsilon}}\right)\left(1 + \hat{\mathbb{E}}\left[\sup_{t\in[0,T]}\left|X_t^{\epsilon}\right|^2\right]\right) \\
&\quad + 6L \sum_{i=0}^{N-1} \hat{\mathbb{E}}\left[\int_{t_i}^{t_{i+1}} \left|X_s^{\epsilon} - X_{t_i}^{\epsilon}\right|^2 ds\right] \\
&\leq CN\sqrt{\epsilon}\varphi\left(\frac{1}{\sqrt{\epsilon}}\right) + CN\sqrt{\epsilon}L\epsilon. \qquad (16)
\end{aligned}$$

Then, substituting (16) in (14), we obtain the following estimate:

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\int_0^T \left|h\left(s, X_s^{\epsilon}\right) - \bar{h}\left(X_s^{\epsilon}\right)\right|^2 ds\right] \\
&\quad \leq C(T + \sqrt{\epsilon})\left[\varphi\left(\frac{1}{\sqrt{\epsilon}}\right) + L\epsilon\right].
\end{aligned}$$

Finally, the required inequality follows by letting ϵ tend to zero in the inequality. □

Lemma 3.4 *Suppose that assumptions* $(H1)$, $(H2)$, *and* (B) *are satisfied. Then*

$$\lim_{\epsilon\to 0} \hat{\mathbb{E}}\left[\int_0^T \left|\sigma\left(\frac{s}{\epsilon}, X_s^{\epsilon}\right) - \bar{\sigma}\left(X_s^{\epsilon}\right)\right|^2 ds\right] = 0. \qquad (17)$$

The proof is the same as Lemma 3.3, here we omit the proof.

Now, we present our first main result, the averaging principle of the GSDE with standard linear growth and Lipschitz assumption.

Theorem 3.5 *Assume that assumptions* $(H1)$, $(H2)$, *and* (B) *are satisfied. Then*

$$\lim_{\epsilon\to 0} \hat{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|X_t^{\epsilon} - \bar{X}_t\right|^2\right] = 0.$$

Proof Starting with

$$\begin{aligned}
X_t^{\epsilon} - \bar{X}_t &= \int_0^t \left[b\left(\frac{s}{\epsilon}, X_s^{\epsilon}\right) - \bar{b}(\bar{X}_s)\right] ds + \int_0^t \left[h\left(\frac{s}{\epsilon}, X_s^{\epsilon}\right) - \bar{h}(\bar{X}_s)\right] d\langle B\rangle_s \\
&\quad + \int_0^t \left[\sigma\left(\frac{s}{\epsilon}, X_s^{\epsilon}\right) - \bar{\sigma}(\bar{X}_s)\right] dB_s \qquad (18)
\end{aligned}$$

and employing the simple arithmetic inequality

$$|x_1 + x_2 + \cdots x_m|^2 \leq m\left(|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2\right),$$

we arrive at

$$\begin{aligned}\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2 &\le 3\sup_{0\le t\le T}\left(\int_0^t\left[b\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{b}(\bar{X}_s)\right]ds\right)^2\\&\quad+3\sup_{0\le t\le T}\left(\int_0^t\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}(\bar{X}_s)\right]d\langle B\rangle_s\right)^2\\&\quad+3\sup_{0\le t\le T}\left(\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]dB_s\right)^2\\&=:I_1+I_2+I_3,\end{aligned}\tag{19}$$

where $u\in[0,T]$, and $I_i, i=1,2,3$, denote the above terms respectively. Now we present some useful estimates for $I_i, i=1,2,3$.

Firstly, we apply the arithmetic inequality and Hölder's inequality to obtain

$$\begin{aligned}I_1&=3\sup_{0\le t\le T}\left(\int_0^t\left[b\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{b}(\bar{X}_s)\right]ds\right)^2\\&=6\sup_{0\le t\le T}\left(\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right)^2\\&\quad+6\sup_{0\le t\le T}\left|\int_0^t\left[\bar{b}(\bar{X}_s)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right|^2\\&\le 6\sup_{0\le t\le T}\left(\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right)^2\\&\quad+6T\int_0^T\left|\bar{b}(\bar{X}_s)-\bar{b}\left(X_s^\epsilon\right)\right|^2ds\\&\le 6\sup_{0\le t\le T}\left(\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right)^2\\&\quad+6T\bar{L}\int_0^T\left|X_s^\epsilon-X_s\right|^2ds.\end{aligned}\tag{20}$$

Here, in the last inequality, we have used the Lipschitz condition of $\bar{b}$.

Second, for I_2, we take the expectation on I_2:

$$\begin{aligned}\hat{\mathbb{E}}[I_2]&\le 3\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}(\bar{X}_s)\right]d\langle B\rangle_s\right)^2\right]\\&\le 6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right]d\langle B\rangle_s\right)^2\right]\\&\quad+6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[\bar{h}\left(X_s^\epsilon\right)-\bar{h}(\bar{X}_s)\right]d\langle B\rangle_s\right)^2\right]\\&\le 6\bar{\sigma}^4\hat{\mathbb{E}}\left[\int_0^T\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right]^2ds\right]\\&\quad+6\bar{\sigma}^4\hat{\mathbb{E}}\left[\int_0^T\left[\bar{h}\left(X_s^\epsilon\right)-\bar{h}(\bar{X}_s)\right]^2ds\right]\end{aligned}$$

$$\begin{aligned}
&\le 6\bar{\sigma}^4\hat{\mathbb{E}}\left[\int_0^T\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right]^2 ds\right]\\
&\quad + 6\bar{\sigma}^4\bar{L}\hat{\mathbb{E}}\left[\int_0^T\left|X_s^\epsilon - X_s\right|^2 ds\right]. \qquad (21)
\end{aligned}$$

Here, we have used the BDG-type inequality (see Lemma 2.10).

Finally, we take expectation on I_3 using the BDG-type inequality in Lemma 2.10(2):

$$\begin{aligned}
\hat{\mathbb{E}}[I_3] &= 3\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]dB_s\right)^2\right]\\
&\le 6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s^\epsilon)\right]dB_s\right)^2\right]\\
&\quad + 6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[\bar{\sigma}(\bar{X}_s^\epsilon)-\bar{\sigma}(\bar{X}_s)\right]dB_s\right)^2\right]\\
&\le 6C\hat{\mathbb{E}}\left[\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]^2 ds\right]\\
&\quad + 6C\hat{\mathbb{E}}\left[\int_0^t\left[\bar{\sigma}(\bar{X}_s)-\bar{\sigma}(\bar{X}_s)\right]^2 ds\right]\\
&\le 6C\hat{\mathbb{E}}\left[\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]^2 ds\right]\\
&\quad + 6C\bar{L}\hat{\mathbb{E}}\left[\int_0^T\left|X_s^\epsilon - X_s\right|^2 ds\right]. \qquad (22)
\end{aligned}$$

Here, in the last inequality, we have used the Lipschitz condition of $\bar{\sigma}$.

Therefore, taking sublinear expectation on (19), substituting (20)–(22) in (19), we get

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left|X_t^\epsilon - \bar{X}_t\right|^2\right]\\
&\le 6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right)^2\right]\\
&\quad + 6\bar{\sigma}^4\hat{\mathbb{E}}\left[\int_0^T\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right]^2 ds\right]\\
&\quad + 6C\hat{\mathbb{E}}\left[\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]^2 ds\right]\\
&\quad + \left(6T\bar{L}+6\bar{\sigma}^4\bar{L}+6C\bar{L}\right)\hat{\mathbb{E}}\left[\int_0^T\left|X_s^\epsilon - X_s\right|^2 ds\right]\\
&\le 6\hat{\mathbb{E}}\left[\sup_{0\le t\le T}\left(\int_0^t\left[b\left(s,X_s^\epsilon\right)-\bar{b}\left(X_s^\epsilon\right)\right]ds\right)^2\right]\\
&\quad + 6\bar{\sigma}^4\hat{\mathbb{E}}\left[\int_0^T\left[h\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{h}\left(X_s^\epsilon\right)\right]^2 ds\right]\\
&\quad + 6C\hat{\mathbb{E}}\left[\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^\epsilon\right)-\bar{\sigma}(\bar{X}_s)\right]^2 ds\right]\\
&\quad + \left(6T\bar{L}+6\bar{\sigma}^4\bar{L}+6C\bar{L}\right)\int_0^T\hat{\mathbb{E}}\left[\sup_{0\le r\le s}\left|X_r^\epsilon - X_r\right|^2\right]ds. \qquad (23)
\end{aligned}$$

An application of the Gronwall inequality in (23) implies that

$$\begin{aligned}
&\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\big|X_t^\epsilon-\bar{X}_t\big|^2\Big]\\
&\quad\le e^{(6T\bar{L}+6\bar{\sigma}^4\bar{L}+6C\bar{L})T}\Bigg\{6\hat{\mathbb{E}}\Bigg[\sup_{0\le t\le T}\bigg(\int_0^t\big[b\big(s,X_s^\epsilon\big)-\bar{b}\big(X_s^\epsilon\big)\big]\,ds\bigg)^2\Bigg]\\
&\qquad+6\bar{\sigma}^4\hat{\mathbb{E}}\Bigg[\int_0^T\bigg[h\bigg(\frac{s}{\epsilon},X_s^\epsilon\bigg)-\bar{h}\big(X_s^\epsilon\big)\bigg]^2\,ds\Bigg]\\
&\qquad+6C\hat{\mathbb{E}}\Bigg[\int_0^t\bigg[\sigma\bigg(\frac{s}{\epsilon},X_s^\epsilon\bigg)-\bar{\sigma}(\bar{X}_s)\bigg]^2\,ds\Bigg]\Bigg\}.
\end{aligned}\tag{24}$$

Consequently, the required result follows by applying Lemmas 3.2–3.4. The proof is complete. □

Next we turn our attention to the time-averaging principle of the GSDE with a locally Lipschitz condition and a Lyapunov-type condition. Our method of argument of the time-averaging principle is based on the localization method.

Theorem 3.6 *Assume that assumptions* ($A1$, $A2$) *and* (B) *are satisfied. Then*

$$\lim_{\epsilon\to 0}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\big|X_t^\epsilon-\bar{X}_t\big|^2\Big]=0.$$

Proof We first consider the following truncated GSDE of (4) and (5), respectively, with coefficients that satisfy assumptions (A1, A2) and (B) for each $N\in\mathbb{N}$, $0\le t\le T$:

$$\begin{aligned}
X_t^{\epsilon,N}&=X_0+\int_0^t b^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)\,ds+\int_0^t h^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)\,d\langle B\rangle_s\\
&\quad+\int_0^t\sigma^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)\,dB_s
\end{aligned}\tag{25}$$

and

$$\begin{aligned}
\bar{X}_t^N&=X_0+\int_0^t\bar{b}^N\big(\bar{X}_s^N\big)\,ds+\int_0^t\bar{h}^N\big(\bar{X}_s^N\big)\,d\langle B\rangle_s\\
&\quad+\int_0^t\bar{\sigma}^N\big(\bar{X}_s^N\big)\,dB_s,
\end{aligned}\tag{26}$$

and then we consider the following truncated GSDEs of (18):

$$\begin{aligned}
X_t^{\epsilon,N}-\bar{X}_t^N&=\int_0^t\bigg[b^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)-\bar{b}^N\big(\bar{X}_s^N\big)\bigg]\,ds+\int_0^t\bigg[h^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)-\bar{h}^N\big(\bar{X}_s^N\big)\bigg]\,d\langle B\rangle_s\\
&\quad+\int_0^t\bigg[\sigma^N\bigg(\frac{s}{\epsilon},X_s^{\epsilon,N}\bigg)-\bar{\sigma}^N\big(\bar{X}_s^N\big)\bigg]\,dB_s,
\end{aligned}\tag{27}$$

where $b^N,h^N,\sigma^N,\bar{b}^N,\bar{h}^N,\bar{\sigma}^N$ are defined in the following way:

$$f^N(\cdot,x)=\begin{cases}f(\cdot,x) & \text{if } |x|\le N;\\ f(\cdot,\frac{Nx}{|x|}) & \text{if } |x|>N.\end{cases}$$

It is easy to verify that $b^N, h^N, \sigma^N, \bar{b}^N, \bar{h}^N, \bar{\sigma}^N$ are all bounded functions and uniformly Lipschitz in x. Then, by the result of Lipschitz GSDE with coefficients in $M_G^p(0,T;\mathbb{R})$ in Theorem 3.5, for the truncated GSDEs (25) and (26), we obtain the following result:

$$\lim_{\epsilon\to 0}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\big|X_t^{\epsilon,N}-\bar{X}_t^N\big|^2\Big]=0. \tag{28}$$

Define two sequences of stopping times by

$$\tau_N := \inf\{t : |X_t^{\epsilon,N}| \ge N\} \wedge T$$

and

$$\bar{\tau}_N := \inf\{t : |\bar{X}_t^{N}| \ge N\} \wedge T,$$

which satisfy $\{\tau_N \le t\} \cup \{\bar{\tau}_N \le t\} \in \mathcal{F}_t$. Based on stopping times, we can deduce from (25) and (26) that

$$\begin{aligned}
X_{t\wedge\tau_N}^{\epsilon,N} &= X_0 + \int_0^t b^N\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,ds + \int_0^t h^N\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \sigma^N\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,dB_s \\
&= X_0 + \int_0^t b^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,ds + \int_0^t h^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \sigma^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N}\Big) I_{[0,\tau_N]}\,dB_s
\end{aligned}$$

and

$$\begin{aligned}
\bar{X}_{t\wedge\bar{\tau}_N}^{N} &= X_0 + \int_0^t \bar{b}^N(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,ds + \int_0^t \bar{h}^N(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \bar{\sigma}^N(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,dB_s \\
&= X_0 + \int_0^t \bar{b}^{N+1}(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,ds + \int_0^t \bar{h}^{N+1}(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \bar{\sigma}^{N+1}(\bar{X}_s^N) I_{[0,\bar{\tau}_N]}\,dB_s.
\end{aligned}$$

On the other hand, by the representations of $X^{N+1}, \bar{X}^{N+1}$ and the continuity of process $X^{N+1}, \bar{X}^{N+1}$, we have

$$\begin{aligned}
X_{t\wedge\tau_N}^{\epsilon,N+1} &= X_0 + \int_0^t b^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N+1}\Big) I_{[0,\tau_N]}\,ds + \int_0^t h^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N+1}\Big) I_{[0,\tau_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \sigma^{N+1}\Big(\frac{s}{\epsilon}, X_s^{\epsilon,N+1}\Big) I_{[0,\tau_N]}\,dB_s
\end{aligned}$$

and

$$\bar{X}_{t\wedge\bar{\tau}_N}^{N+1} = X_0 + \int_0^t \bar{b}^{N+1}(\bar{X}_s^{N+1}) I_{[0,\bar{\tau}_N]}\,ds + \int_0^t \bar{h}^{N+1}(\bar{X}_s^{N+1}) I_{[0,\bar{\tau}_N]}\,d\langle B\rangle_s$$

$$+\int_0^t \bar{\sigma}^{N+1}(\bar{X}_s^{N+1})I_{[0,\bar{\tau}_N]}\,dB_s.$$

By the uniqueness of the solution to the truncated GSDEs (25) and (26), for each $N \in \mathbb{N}$, X^N and X^{N+1} are distinguishable on $[0,\tau_n]$; at the same time, $\bar{X}^N$ and $\bar{X}^{N+1}$ are distinguishable on $[0,\bar{\tau}_n]$. This also implies that the sequences $\{\tau_N\}_{N\in\mathbb{N}}$ and $\{\bar{\tau}_N\}_{N\in\mathbb{N}}$ are q.s. increasing. Now we claim that

$$\mathbb{V}\left(\bigcup_{N=1}^{\infty}\left\{\omega : \tau_N(\omega) = T\right\}\right) = 1 \tag{29}$$

and

$$\mathbb{V}\left(\bigcup_{N=1}^{\infty}\left\{\omega : \bar{\tau}_N(\omega) = T\right\}\right) = 1. \tag{30}$$

By the definition of τ_N we know that $|X_\cdot^{\epsilon,N+1}| \le N$ on $[0,\tau_n]$, hence we have for $0 \le y \le T$

$$f\left(\frac{t}{\epsilon}, X_t^{\epsilon,N}\right)I_{[0,\tau_n]}(t) = f^N\left(\frac{t}{\epsilon}, X_t^{\epsilon,N}\right)I_{[0,\tau_n]}(t)$$

and

$$\bar{f}(\bar{X}_t^N)I_{[0,\bar{\tau}_n]}(t) = \bar{f}^N(\bar{X}_t^N)I_{[0,\bar{\tau}_n]}(t),$$

where $f = b, h, \sigma$ and $\bar{f} = \bar{b}, \bar{h}, \bar{\sigma}$. It follows from Lemma 4.2 in Li and Peng [14] that both $f^N(\frac{t}{\epsilon}, X_t^{\epsilon,N})I_{[0,\tau_n]}(t)$ and $\bar{f}^N(\bar{X}_t^N)I_{[0,\bar{\tau}_n]}(t)$ are $M_G^p(0,T;\mathbb{R})$-processes for any $p \ge 2$. Therefore, for any $t \in [0,T]$, we have

$$\begin{aligned}
X_{t\wedge\tau_N}^{\epsilon,N} &= X_0 + \int_0^t b^N\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,ds + \int_0^t h^N\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \sigma^N\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,dB_s \\
&= \int_0^t b\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,ds + \int_0^t h\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \sigma\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right)I_{[0,\tau_N]}\,dB_s,
\end{aligned}$$

and, similarly, we have

$$\begin{aligned}
\bar{X}_{t\wedge\bar{\tau}_N}^N &= \int_0^t \bar{b}(\bar{X}_s^N)I_{[0,\bar{\tau}_N]}\,ds + \int_0^t \bar{h}(\bar{X}_s^N)I_{[0,\bar{\tau}_N]}\,d\langle B\rangle_s \\
&\quad + \int_0^t \bar{\sigma}(\bar{X}_s^N)I_{[0,\bar{\tau}_N]}\,dB_s.
\end{aligned}$$

Applying G-Itô's formula to

$$\Phi\left(t\wedge\tau_N, X_{t\wedge\tau_N}^{\epsilon,N}\right) := e^{-C_L(t\wedge\tau_N)}V\left(t\wedge\tau_N, X_{t\wedge\tau_N}^{\epsilon,N}\right)$$

and

$$\bar{\Phi}\big(t \wedge \bar{\tau}_N, \bar{X}^N_{t\wedge\bar{\tau}_N}\big) := e^{-C_L(t\wedge\bar{\tau}_N)} V\big(t \wedge \tau_N, \bar{X}^N_{t\wedge\bar{\tau}_N}\big),$$

respectively, where we can take $V(t,x) = 1 + |x|^2$, we obtain

$$\begin{aligned}
&\Phi\big(t \wedge \tau_N, X^{\epsilon,N}_{t\wedge\tau_N}\big) - \Phi(0,0) \\
&= \int_0^{t\wedge\tau_N} \Big[\partial_t \Phi\big(s, X^{\epsilon,N}_s\big) + \partial_x \Phi\big(s, X^{\epsilon,N}_s\big) b\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big)\Big]\, ds \\
&\quad + \int_0^{t\wedge\tau_N} \partial_x \Phi\big(s, X^{\epsilon,N}_s\big)\sigma\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big)\, dB_s \\
&\quad + \int_0^{t\wedge\tau_N} \Big(\partial_x \Phi\big(s, X^{\epsilon,N}_s\big) h\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big) \\
&\quad + \frac{1}{2}\partial^2_{xx}\Phi\big(s, X^{\epsilon,N}_s\big)\Big[\sigma\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big)\Big]^2\Big)\, d\langle B\rangle_s
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
&\bar{\Phi}\big(t \wedge \bar{\tau}_N, \bar{X}^N_{t\wedge\bar{\tau}_N}\big) - \bar{\Phi}(0,0) \\
&= \int_0^{t\wedge\bar{\tau}_N} \big[\partial_t \bar{\Phi}\big(s, \bar{X}^N_s\big) + \partial_x \bar{\Phi}\big(s, X^{\epsilon,N}_s\big)\bar{b}\big(\bar{X}^N_s\big)\big]\, ds \\
&\quad + \int_0^{t\wedge\bar{\tau}_N} \partial_x \bar{\Phi}\big(s, \bar{X}^N_s\big)\bar{\sigma}\big(\bar{X}^N_s\big)\, dB_s \\
&\quad + \int_0^{t\wedge\bar{\tau}_N} \Big(\partial_x \bar{\Phi}\big(s, \bar{X}^N_s\big)\bar{h}\big(\bar{X}^N_s\big) + \frac{1}{2}\partial^2_{xx}\bar{\Phi}\big(s, \bar{X}^N_s\big)\big[\bar{\sigma}\big(\bar{X}^N_s\big)\big]^2\Big)\, d\langle B\rangle_s.
\end{aligned} \tag{32}$$

Letting

$$\eta_s\big(\Phi, X^{\epsilon,N}\big) := \partial_x \Phi\big(s, X^{\epsilon,N}_s\big) h\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big) + \frac{1}{2}\partial^2_{xx}\Phi\big(s, X^{\epsilon,N}_s\big)\sigma^2\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big), \tag{33}$$

$$\bar{\eta}_s\big(\bar{\Phi}, \bar{X}^N\big) := \partial_x \bar{\Phi}\big(s, \bar{X}^N_s\big)\bar{h}\big(\bar{X}^N_s\big) + \frac{1}{2}\partial^2_{xx}\bar{\Phi}\big(s, \bar{X}^N_s\big)\bar{\sigma}^2\big(\bar{X}^N_s\big), \tag{34}$$

we have $\eta_s(\Phi, X^{\epsilon,N}), \bar{\eta}_s(\bar{\Phi}, \bar{X}^N) \in M^2_w(0,T;\mathbb{R})$. Hence, substituting (33) in (31), we arrive at

$$\begin{aligned}
&\Phi\big(t \wedge \tau_N, X^{\epsilon,N}_{t\wedge\tau_N}\big) - \Phi(0,0) \\
&= \int_0^{t\wedge\tau_N} \Big(\partial_t \Phi\big(s, X^{\epsilon,N}_s\big) + \partial_x \Phi\big(s, X^{\epsilon,N}_s\big) b\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big) + 2G\big(\eta_s\big(\Phi, X^N\big)\big)\Big)\, ds \\
&\quad + \int_0^{t\wedge\tau_N} \partial_x \Phi\big(s, X^{\epsilon,N}_s\big)\sigma\Big(\frac{s}{\epsilon}, X^{\epsilon,N}_s\Big)\, dB_s \\
&\quad + \int_0^{t\wedge\tau_N} \eta_s\big(\Phi, X^N\big)\, d\langle B\rangle_s - \int_0^{t\wedge\tau_N} 2G\big(\eta_s\big(\Phi, X^N\big)\big)\, ds \\
&= \int_0^{t\wedge\tau_N} \mathcal{L}\Phi\big(s, X^{\epsilon,N}_s\big)\, ds + \int_0^{t\wedge\tau_N} \eta_s\big(\Phi, X^N\big)\, d\langle B\rangle_s - \int_0^{t\wedge\tau_N} 2G\big(\eta_s\big(\Phi, X^N\big)\big)\, ds
\end{aligned}$$

$$+ \int_0^{t\wedge\tau_N} \partial_x \Phi\big(s, X_s^{\epsilon,N}\big)\sigma\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right) dB_s. \tag{35}$$

By assumption (A2), $\mathcal{L}V \le C_L V$ implies $\mathcal{L}\Phi \le 0$. It follows from Proposition 1.4 in IV-1 of Peng [16] that, for $\eta \in M_w^1(0,T;\mathbb{R})$,

$$\int_0^{t\wedge\tau_N} \eta_s\big(\Phi, X^N\big)\, d\langle B\rangle_s - \int_0^{t\wedge\tau_N} 2G\big(\eta_s\big(\Phi, X^N\big)\big)\, ds \le 0, \quad \text{q.s.}$$

Taking expectation on (35), we obtain

$$\hat{\mathbb{E}}\big[\Phi\big(t\wedge\tau_N, X_{t\wedge\tau_N}^{\epsilon,N} - \bar{X}_{t\wedge\tau_N}^N\big)\big] \le \Phi(0,0).$$

In particular, we have

$$\hat{\mathbb{E}}\big[V\big(T\wedge\tau_N, X_{T\wedge\tau_N}^{\epsilon,N}\big)I_{\tau_N\le T}\big] \le V(0,0)e^{C_L T}.$$

Since $\tau_N < T$ implies $|X_{T\wedge\tau_N}^{\epsilon,N}| = N$, q.s., from which we deduce

$$\mathbb{V}(\tau_N < T)\cdot \inf_{|x|\ge N}\inf_{t\in[0,T]} V(t,x) \le V(0,0)e^{C_L T}.$$

Letting $N\to\infty$, by (A2), we obtain

$$1 \ge \lim_{N\to\infty}\mathbb{V}(\tau_N = T) \ge 1 - \lim_{N\to\infty}\mathbb{V}(\tau_N < T) = 1.$$

Since $\{\omega : \tau_N(\omega) = T\}$ is increasing, the upwards convergence theorem yields (29). Similarly, substituting (34) in (32), after same discussion as above, we get that (30) holds.

Therefore, there exists a polar set A such that for all $\omega \in A^c$ the following assertion holds: one can find an $N_0(\omega)$ that depends on ω such that for all $N \ge N_0(\omega), N \in \mathbb{N}, \tau_N(\omega) = T$. Then we define, for $t \in [0,T]$,

$$X_t^\epsilon(\omega) = \begin{cases} X_t^{\epsilon,N_0(\omega)}(\omega), & \omega \in A^c; \\ 0, & \omega \in A. \end{cases}$$

Similarly, one can find an $N_1(\omega)$ that depends on ω such that for all $N \ge N_1(\omega), N \in \mathbb{N}, \bar{\tau}_N(\omega) = T$. Then we define, for $t \in [0,T]$,

$$\bar{X}_t(\omega) = \begin{cases} \bar{X}_t^{N_1(\omega)}(\omega), & \omega \in A^c; \\ 0, & \omega \in A. \end{cases}$$

From the argument above, we have $X^\epsilon I_{[0,\tau_N]} = X^{\epsilon,N} I_{[0,\tau_N]} \in M_G^2(0,T;\mathbb{R})$, and thus $X^\epsilon \in M_w^2(0,T;\mathbb{R})$. Also, we have $\bar{X} \in M_w^2(0,T;\mathbb{R})$. Moreover,

$$\begin{aligned} X_{t\wedge\tau_N}^\epsilon &= X_{t\wedge\tau_N}^{\epsilon,N} \\ &= X_0 + \int_0^{t\wedge\tau_N} b^N\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right) ds + \int_0^{t\wedge\tau_N} h^N\left(\frac{s}{\epsilon}, X_s^{\epsilon,N}\right) d\langle B\rangle_s \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\tau_N}\sigma^N\left(\frac{s}{\epsilon},X_s^{\epsilon,N}\right)dB_s\\
&=\int_0^{t\wedge\tau_N}b\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)ds+\int_0^{t\wedge\tau_N}h\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)d\langle B\rangle_s+\int_0^{t\wedge\tau_N}\sigma\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)dB_s
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{X}_{t\wedge\bar{\tau}_N}&=\bar{X}^N_{t\wedge\bar{\tau}_N}\\
&=X_0+\int_0^{t\wedge\bar{\tau}_N}-\bar{b}(\bar{X}_s)\,ds+\int_0^{t\wedge\bar{\tau}_N}\bar{h}(\bar{X}_s)\,d\langle B\rangle_s+\int_0^{t\wedge\bar{\tau}_N}\bar{\sigma}(\bar{X}_s)\,dB_s,
\end{aligned}
$$

Hence, the two equations imply that $X^\epsilon,\bar{X}$ satisfy (4), (5), respectively. Furthermore, we obtain that $X^\epsilon-\bar{X}$ satisfies

$$
\begin{aligned}
X_t^\epsilon-\bar{X}_t=&\int_0^t\left[b\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)-\bar{b}(\bar{X}_s)\right]ds+\int_0^t\left[h\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)-\bar{h}(\bar{X}_s)\right]d\langle B\rangle_s\\
&+\int_0^t\left[\sigma\left(\frac{s}{\epsilon},X_s^{\epsilon}\right)-\bar{\sigma}(\bar{X}_s)\right]dB_s.
\end{aligned}
$$

Therefore, letting $N\to\infty$ in (28), we can deduce

$$
\lim_{\epsilon\to 0}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2\Big]=0,
$$

which completes the proof of this theorem. □

As a consequence of Theorem 3.6 and Markov-type inequality of capacity, the convergence also holds in the sense of capacity.

Corollary 3.7 *Under the same assumptions as in Theorem 3.6, for all $\delta>0$, we have*

$$
\lim_{\varepsilon\to 0}v\Big(\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2\le\delta\Big)=1.
$$

Proof By Proposition 2.3 and Theorem 3.6, for any number $\delta>0$, one can find

$$
\begin{aligned}
&\mathbb{V}\Big(\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2>\delta\Big)\\
&\quad\le\frac{1}{\delta}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2\Big].
\end{aligned}
$$

Noting that $v(A)=1-\mathbb{V}(A^c)$, we have

$$
v\Big(\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2\le\delta\Big)\ge 1-\frac{1}{\delta}\hat{\mathbb{E}}\Big[\sup_{0\le t\le T}\left|X_t^\epsilon-\bar{X}_t\right|^2\Big].
$$

Let $\epsilon\to 0$ and the required result follows from Theorem 3.6. □

4 Numerical simulation

In this section, we present numerical simulation of GSDE obtained using PYTHON and give three examples to demonstrate the averaging principle for the GSDE driven by G-Brownian motion.

Example 1 (Without quadratic variation term) Consider the following standard GSDE:

$$dX^\epsilon = -2\lambda X^\epsilon \sin^2\left(\frac{t}{\epsilon}\right) dt + dB_t \tag{36}$$

and the averaged GSDE

$$dZ = -\lambda Z\, dt + dB_t \tag{37}$$

with the same initial condition $X_0^\epsilon = Z_0 = X_0$, where B_t is a G-Brownian motion and satisfies

$$\underline{\sigma}^2 t \leq \langle B\rangle_t \leq \bar{\sigma}^2 t.$$

Obviously, $\frac{1}{\pi}\int_0^\pi \lambda X^\epsilon \sin^2(t)\, dt = \frac{1}{2}\lambda X^\epsilon$, all coefficients of the standard GSDE and averaged GSDE satisfy conditions (A1)–(A2) and (B1)–(B3) for the functions $b, h, \sigma, \bar{b}, \bar{h}, \bar{\sigma}$. Thus Theorem 3.5 and Theorem 3.6 hold, that is,

$$\lim_{\epsilon\to 0} \hat{\mathbb{E}}\left[\sup_{0\leq t\leq T} \left|X_t^\epsilon - Z_t\right|^2\right] = 0$$

and

$$\lim_{\epsilon\to 0} \nu\left(\sup_{0\leq t\leq T} \left|X_t^\epsilon - Z_t\right|^2 \leq \delta\right) = 1.$$

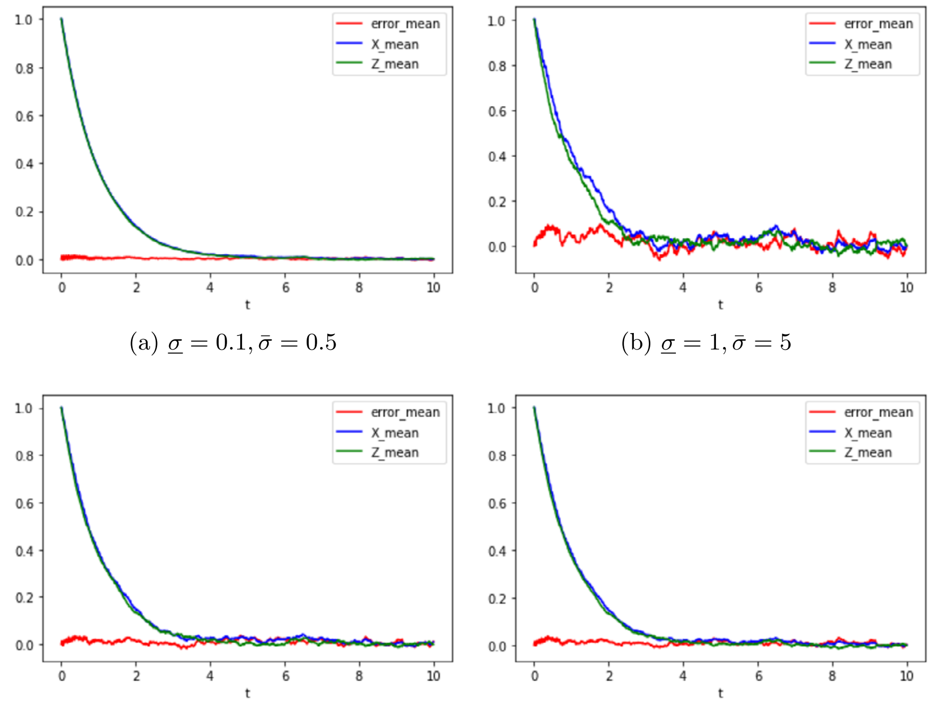
Now we carry out the numerical simulation to get the solutions of GSDE (36) and averaged GSDE (37) under conditions $X_0 = 1, \lambda = 1.0, \varepsilon = 0.01$, (a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$; (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$; (c) $\underline{\sigma} = 0.5, \bar{\sigma} = 2$, and (d) Brownian motion $\sigma = 1$, respectively. Figure 1 depicts a sample average of 5000 trajectories of the SDE X^ϵ, a sample average of 5000 trajectories of the averaged SDE Z, and a sample average of 5000 trajectories of the error $X^\epsilon - Z$. Not only do we see a good agreement between solutions of the equation and the averaged equation, but we are also aware of the fact: Larger volatility can cause greater fluctuation!

Example 2 (With quadratic variation term) Consider the following standard GSDE:

$$dX^\epsilon = \left[-\lambda X^\epsilon + \left(1 + \frac{t}{\epsilon}\right)^{-1} \sin\left(\frac{t}{\epsilon} + X^\epsilon\right)\right] d\langle B\rangle_t + dB_t \tag{38}$$

and the averaged GSDE

$$dZ = -\lambda Z\, d\langle B\rangle_t + dB_t, \tag{39}$$



(a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$ (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$

(c) $\underline{\sigma} = \frac{1}{2}, \bar{\sigma} = 2$ (d) $\underline{\sigma} = \bar{\sigma} = 1$

Figure 1 Sample averages of 5000 trajectories of X^ϵ and Z without quadratic variation terms

with the same initial condition $X_0^\epsilon = Z_0 = X_0$, where B_t is a G-Brownian motion and satisfies

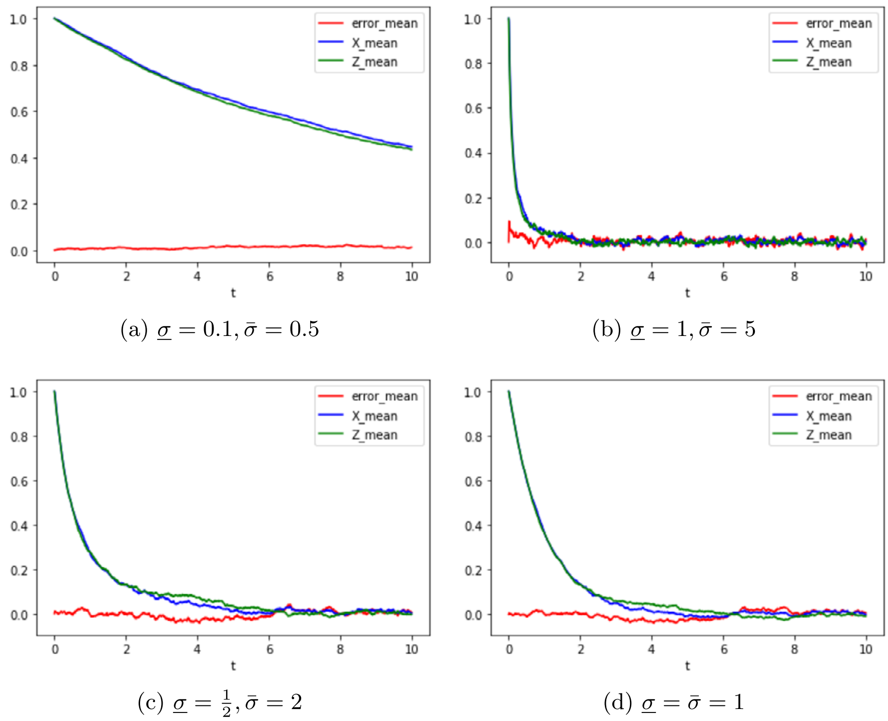
$$\underline{\sigma}^2 t \le \langle B \rangle_t \le \bar{\sigma}^2 t.$$

We can easily calculate the following claim for any $T_1 \in [0, T]$:

$$\sup_{t \ge 0} \frac{1}{T_1} \int_t^{t+T_1} \left| h(s,x) - \bar{h}(x) \right|^2 ds \le \varphi(T_1)\left(1 + |x|^2\right),$$

where φ is a positive bounded function with $\lim_{T_1 \to \infty} \varphi(T_1) = 0$. In fact, due to $|\sin(x)| \le 1$, for any $T_1 \in [0, T], \epsilon \in (0, 1)$, we have

$$\begin{aligned}
&\sup_{t \ge 0} \frac{1}{T_1} \int_t^{t+T_1} \left| \left[\lambda x + \frac{1}{1 + s/\epsilon} \sin(s/\epsilon + x) \right] - \lambda x \right|^2 ds \\
&\quad = \sup_{t \ge 0} \frac{1}{T_1} \int_t^{t+T_1} \left| \frac{1}{1 + s/\epsilon} \sin(s/\epsilon + x) \right|^2 ds \\
&\quad \le \sup_{t \ge 0} \frac{1}{T_1} \int_t^{t+T_1} \frac{1}{(1 + s/\epsilon)^2} ds \\
&\quad \le \sup_{t \ge 0} \epsilon^2 \frac{1}{T_1} \frac{T_1 + t}{T_1 + t + \epsilon} \\
&\quad \le \frac{1}{T_1} =: \varphi(T_1).
\end{aligned}$$



(a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$ (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$ (c) $\underline{\sigma} = \frac{1}{2}, \bar{\sigma} = 2$ (d) $\underline{\sigma} = \bar{\sigma} = 1$

Figure 2 Sample averages of 5000 trajectories of X^ϵ and Z with quadratic variation terms

Hence all the coefficients of GSDE and averaged GSDE satisfy conditions (A1)–(A2) and (B) for the functions $b, h, \sigma, \bar{b}, \bar{h}, \bar{\sigma}$. Thus we can use the solution Z of GSDE (39) to approximate the original solution X^ϵ of GSDE (38), and the convergence will be assured.

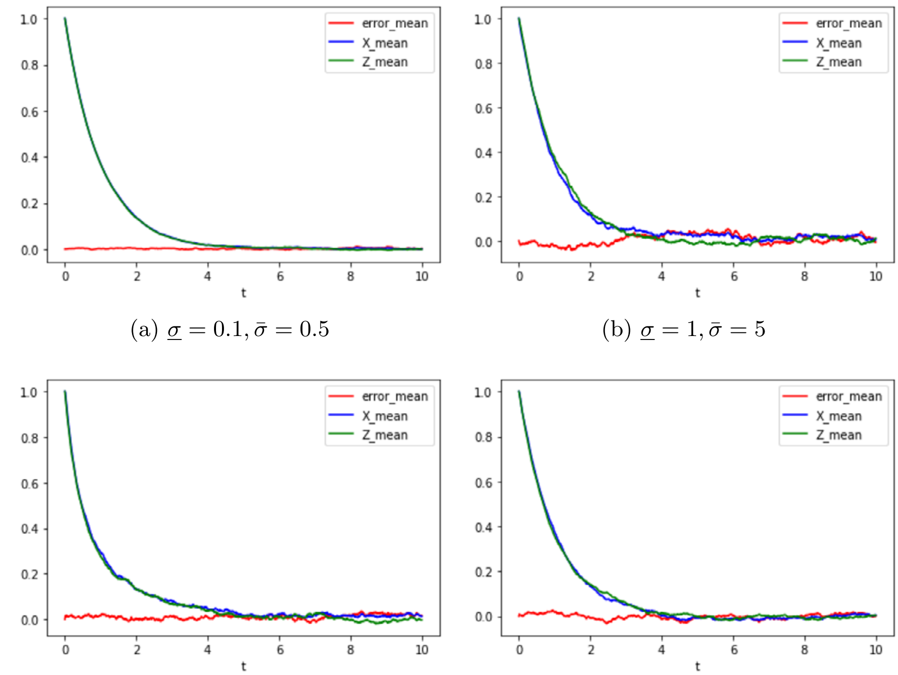
Now we carry out the numerical simulation to get the solutions of GSDE (38) and averaged GSDE (39) under conditions $X_0 = 1, \lambda = 1.0, \varepsilon = 0.01$, (a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$; (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$; (c) $\underline{\sigma} = 0.5, \bar{\sigma} = 2$, and (d) $\underline{\sigma} = \bar{\sigma} = 1$, respectively. Figure 2 depicts a sample average of 5000 trajectories of the SDE X^ϵ, a sample average of 5000 trajectories of the averaged SDE Z, and a sample average of 5000 trajectories of the error $X^\epsilon - Z$. We can see a good agreement between solutions of the equation and the averaged equation. Comparing (b)(c) with (d), we can observe the following fact: because GSDE contains quadratic variation term, the solution of the equation can better reflect the change of trend: The solution of this equation decays faster as the uncertainty of the volatility increases.

Example 3 (With diffusion coefficient term) Consider the following standard GSDE, where the diffusion coefficient is not a constant:

$$dX^\epsilon = -X^\epsilon\, dt + \left[\lambda \cos\left(X^\epsilon\right) + \left(1 + \frac{t}{\epsilon}\right)^{-1} \sin\left(\frac{t}{\epsilon} + X^\epsilon\right)\right] dB_t \tag{40}$$

and the averaged GSDE

$$dZ = -Z\, dt + \lambda \cos(Z)\, dB_t, \tag{41}$$



(a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$ (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$

(c) $\underline{\sigma} = \frac{1}{2}, \bar{\sigma} = 2$ (d) $\underline{\sigma} = \bar{\sigma} = 1$

Figure 3 Sample averages of 5000 trajectories of X^{ϵ} and Z with diffusion coefficient terms

with the same initial condition $X_0^{\epsilon} = Z_0 = X_0$, where B_t is a G-Brownian motion and satisfies

$$\underline{\sigma}^2 t \leq \langle B \rangle_t \leq \bar{\sigma}^2 t.$$

Here,

$$\sigma(t,x) = \lambda \cos(x) + \frac{1}{1+t/\epsilon} \sin(t/\epsilon + x), \ \text{and}\, \bar{\sigma}(x) = \lambda \cos(x).$$

Due to $|\sin(x)| \leq 1$, for any $\epsilon \in (0,1)$, we have

$$\begin{aligned}
&\sup_{t\geq 0} \frac{1}{T_1} \int_t^{t+T_1} \left| \left[\lambda \cos(x) + \frac{1}{1+s/\epsilon} \sin(s/\epsilon + x) \right] - \lambda \cos(x) \right|^2 ds \\
&\quad \leq \sup_{t\geq 0} \frac{1}{T_1} \int_t^{t+T_1} \frac{1}{(1+s/\epsilon)^2} ds \\
&\quad \leq \sup_{t\geq 0} \frac{1}{T_1} \frac{\epsilon^2 (T_1 + t)}{T_1 + t + \epsilon} \leq \frac{1}{T_1} := \varphi(T_1).
\end{aligned}$$

Hence, all the coefficients of GSDE and averaged GSDE satisfy conditions (A1)–(A2) and (B) for the functions $b, h, \sigma, \bar{b}, \bar{h}, \bar{\sigma}$. Thus we can use the solution Z of GSDE (41) to approximate the original solution X^{ϵ} of GSDE (40), and the convergence will be assured.

Now we carry out the numerical simulation to get the solutions of GSDE (40) and averaged GSDE (41) under conditions $X_0 = 1, \lambda = 1.0, \varepsilon = 0.01$, (a) $\underline{\sigma} = 0.1, \bar{\sigma} = 0.5$; (b) $\underline{\sigma} = 1, \bar{\sigma} = 5$; (c) $\underline{\sigma} = 0.5, \bar{\sigma} = 2$, and (d) $\underline{\sigma} = \bar{\sigma} = 1$, respectively. Figure 3 depicts a sample average of 5000 trajectories of the SDE X^ϵ, a sample average of 5000 trajectories of the averaged SDE Z, and a sample average of 5000 trajectories of the error $X^\epsilon - Z$. We can see a good agreement between solutions of the equation and the averaged equation, and the error is approximately zero. The numerical verification is consistent with the theoretical results.

5 Conclusion

In this paper we have studied the time-averaging principle for stochastic differential equations based on the Lyapunov condition in the presence of a family of probability measures, each corresponding to a different scenario for the volatility. To overcome the difficulty from the locally Lipschitz coefficients of G-SDEs, the G-stochastic calculus and the localization technique have been used. We show that the solution of a standard equation converges to the solution of the corresponding averaging equation in the sense of sublinear expectation with the help of some properties of G-stochastic calculus. We present numerical simulations of G-SDEs and give three examples to demonstrate the averaging method. The numerical results exhibit that there is a good agreement between solutions of the equation and the averaged equation, and the error is approximately zero. The numerical verification is consistent with the theoretical results.

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Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that he has no competing interests.

Author contributions
GZ is the unique author of the manuscript. GZ prepared the manuscript initially and performed all the steps of the proofs in this research. GZ read and approved the final manuscript.

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