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Mean-field optimal control in a multi-agent interaction model for prevention of maritime crime

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Abstract

We study a multi-agent system for the modeling maritime crime. The model involves three interacting populations of ships: commercial ships, pirate ships, and coast guard ships. Commercial ships follow commercial routes, are subject to traffic congestion, and are repelled by pirate ships. Pirate ships travel stochastically, are attracted by commercial ships and repelled by coast guard ships. Coast guard ships are controlled. We prove well-posedness of the model and existence of optimal controls that minimize dangerous contacts. Then we study, in a two-step procedure, the mean-field limit as the number of commercial ships and pirate ships is large, deriving a mean-field PDE/PDE/ODE model. Via Γ-convergence, we study the limit of the corresponding optimal control problems.

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Keywords: Multi-agent system; Mean-field limit; Stochastic differential equations; Optimal control; Γ-convergence

1 Introduction

Systems featuring interactions among multi-agents have attracted much attention of the scientific community in recent years as they find applications in various fields. They are a proper tool to study, *e.g.*, biological aggregation as in flocks, swarms, or fish schools [14, 19, 37], crowd dynamics [2], emergent economic behaviors [16, 21], consensus in collective decision-making [13, 29], coordination and cooperation in robotics [17, 34]. In this framework, mathematical analysis has played a role in the proof of well-posedness of the models, in the derivation of mean-field limit, and in the analysis of optimal control problems for this kind of models [1, 3–7, 12, 24, 25, 30].

In this paper, we exploit the tools developed for the analysis of multi-agent systems to study optimal control in a model for the prediction of maritime crime. The majority of world's goods is carried by sea [22], but the freedom of navigation is affected by the presence of modern maritime piracy, which poses serious threats to international traffic and individual safety. It is a priority to prevent crimes and suppress them [23].

To face this problem, we devise a model featuring three populations of agents, representing the types of ships. Our model is inspired by the macroscopic model (*i.e.*, with a large number of ships) introduced in [18], but it differs from it in that our derivation starts from a microscopic model (*i.e.*, with a finite number of ships). We briefly outline it in this Introduction, referring to Sect. 3 for the precise description of all the features and assumptions on the model.

We consider three populations: N commercial ships with trajectories $X_1,\dots,X_N$, M pirate (criminal) ships with trajectories $Y_1,\dots,Y_M$, and L coast guard (patrol) ships with trajectories $Z_1,\dots,Z_L$. The trajectory of each ship evolves in a time interval $[0,T]$ according to a specific dynamical law based on its type and on the presence of other surrounding ships, as we illustrate now.

Commercial ships tend to follow commercial routes, but their motion is affected by traffic congestion: a commercial ship obstructed by a high density of commercial ships travels slower than the one with free space. Moreover, in the presence of pirate ships, commercial ships are repelled by them and adjust their trajectory to travel far from danger. Hence, the nth commercial ship evolves according to

$$\frac{\mathrm{d}X_n}{\mathrm{d}t}(t) = v_n^N\big(X(t)\big)\left(\mathbf{r}\big(X_n(t)\big) + \frac{1}{M}\sum_{m=1}^{M} K^{\mathrm{cp}}\big(X_n(t) - Y_m(t)\big)\right), \tag{1.1}$$

where v_n^N is a suitable function depending on all the other commercial ships needed for the congestion phenomenon, $\mathbf{r}$ is the vector field indicating the commercial route, and K^{cp} is the term due to the repulsion from pirate ships that adjusts the direction of the trajectory.

Pirate ships are attracted by commercial ships and are repelled by coast guard ships. Moreover, in the absence of other ships, they travel randomly in search of targets. Hence, the mth pirate ship evolves stochastically according to

$$\begin{aligned}\mathrm{d}Y_m(t) = {}&\left(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(Y_m(t) - Z_\ell(t)\big) - \frac{1}{N}\sum_{\ell=1}^{N} K^{\mathrm{pc}}\big(Y_m(t) - X_n(t)\big)\right)\mathrm{d}t\\ &+ \sqrt{2\kappa}\,\mathrm{d}W_m(t),\end{aligned} \tag{1.2}$$

where K^{pg} and K^{pc} are the repulsion and attraction terms with coast guard ships and commercial ships, respectively. The term $(W_m(t))_{t\in[0,T]}$ is a Brownian motion accounting for the stochastic behavior mentioned above. Its effect is a white noise with coefficient $\sqrt{2\kappa}$ added to the velocity of Y_m.

Finally, for coast guard ships, we only impose that they are repelled by each other and that their trajectory is controllable, at a cost. Hence, the ℓth coast guard ship evolves according to

$$\frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}\big(Z_\ell(t) - Z_{\ell'}(t)\big) + u_\ell(t),$$

where K^{gg} is the repulsion term among coast guard ships and the u_ℓs are the control.

The search of coast guard ships for dangerous contacts between commercial and pirate ships will be driven by the optimal control of the system based on the cost defined as follows. The cost of a control $u = (u_1,\dots,u_L)$ takes into account the effort in modifying the trajectories of coast guard ships (it can be thought as the cost of fuel) and the total number of dangerous contacts among commercial and pirate ships

$$\mathcal{J}_{N,M}(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M} H^{\mathrm{d}}\big(X_n(t) - Y_m(t)\big)\,\mathrm{d}t\right), \tag{1.3}$$

where H^{d} is a compactly supported convolution kernel used for counting dangerous contacts and $\mathbb{E}$ denotes the expected value. We study the problem of finding a control that minimizes $\mathcal{J}_{N,M}$.

In Sect. 4 we prove well-posedness of the model that describes the evolution and we prove the existence of an optimal control.

Next, we proceed with the derivation of the mean-field limit of the optimal control problem. We carry out this analysis in two steps: first, we let $M \to +\infty$ (a large number of pirate ships), and then $N \to +\infty$ (a large number of commercial ships). The reason thereof is that the limit as $M \to +\infty$ is interesting *per se*, as we explain forthwith.

Under suitable conditions, in Sect. 7 (see Theorem 7.1 and Proposition 7.2) we show that, as $M \to +\infty$, the mean-field behavior of pirate ships is described by a probability distribution $\bar{\mu}^{\mathrm{p}}$. The trajectories of commercial ships $\bar{X}_n$ in this mean-field model satisfy

$$\frac{\mathrm{d}\bar{X}_n}{\mathrm{d}t}(t) = v_n^N\big(\bar{X}(t)\big)\Big(\mathbf{r}\big(\bar{X}_n(t)\big) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\Big), \tag{1.4}$$

which corresponds to (1.1) with the trajectories of pirate ships replaced by their mean-field behavior. The probability distribution $\bar{\mu}^{\mathrm{p}}$ of pirate ships solves the diffusive PDE

$$\partial_t \bar{\mu}^{\mathrm{p}} - \kappa\,\Delta_y \bar{\mu}^{\mathrm{p}} + \mathrm{div}_y\left(\left(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(\cdot - Z_\ell(t)\big) - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}\big(\cdot - \bar{X}_n(t)\big)\right)\bar{\mu}^{\mathrm{p}}\right) = 0. \tag{1.5}$$

This mean-field model is interesting *per se* when the precise location of pirate ships is not known, but one can only predict the probability of finding them in certain regions of the sea. Proving convergence of solutions of the original model to the mean-field model as $M \to +\infty$ requires some technical steps, mainly done following the guidelines in [9]. First, in Sect. 5 we introduce an auxiliary averaged model where the evolution of pirate ships is replaced by a single stochastic process evolving according to the same dynamics of (1.2), *i.e.*,

$$\mathrm{d}\bar{Y}(t) = \left(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(\bar{Y}(t) - Z_\ell(t)\big) - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}\big(\bar{Y}(t) - \bar{X}_n(t)\big)\right)\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t),$$

where $\bar{X}_n$ evolves according to (1.4), $\bar{\mu}^{\mathrm{p}}$ being the law of the stochastic process $(\bar{Y}(t))_{t\in[0,T]}$. In Sect. 5 we prove well-posedness for this averaged model using a fixed point argument. Solutions to the original model converge, as $M \to +\infty$, to solutions of this auxiliary averaged model. To see this, in Propositions 6.1–6.2 we rely on a propagation of the chaos principle [15], from which we deduce that the solutions to (1.2) are independent and identically distributed stochastic processes if so are the initial conditions. Then, a Glivenko–Cantelli-type result allows us to deduce convergence of the empirical measures of the Y_ms to their common law $\bar{\mu}^{\mathrm{p}}$. The parabolic PDE (1.5) is then the Fokker–Planck equation for pirate ships, as shown in Proposition 7.2.

After deriving the mean-field limit as $M \to +\infty$, in Theorem 7.3 we show that the costs $\mathcal{J}_{N,M}$ defined in (1.3) Γ-converge, as $M \to +\infty$, to the cost for the limit problem

$$\mathcal{J}_N(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\, \mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\, \mathrm{d}t. \tag{1.6}$$

As a consequence, optimal controls for the original problem converge as $M \to +\infty$ to optimal controls for the limit problem, see Proposition 7.4. This concludes the analysis as $M \to +\infty$.

The next step is to study the mean-field limit as the number of commercial ships is large, *i.e.*, when $N \to +\infty$. In Theorem 8.1 and Proposition 8.3, we show that the mean-field limit of commercial ships is described in terms of their distribution μ^{c}, which solves a scalar conservation law with a nonlocal flux, apt to describe traffic flow in sea. More precisely, μ^{c} is a solution to the PDE

$$\partial_t \mu^{\mathrm{c}} + \mathrm{div}_x\Big(v\big(\eta *_2 \mu^{\mathrm{c}}\big)\big(\mathbf{r} + K^{\mathrm{cp}} * \mu^{\mathrm{p}}\big)\mu^{\mathrm{c}}\Big) = 0,$$

where $v(\eta *_2 \mu^{\mathrm{c}})$ arises from the limit of the congestion velocities and μ^{p} is the probability distribution of pirate ships, evolving according to the parabolic PDE

$$\partial_t \mu^{\mathrm{p}} - \kappa\,\Delta_y \mu^{\mathrm{p}} + \mathrm{div}_y\left(\left(\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}\big(\cdot - Z_\ell(t)\big) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}\right)\mu^{\mathrm{p}}\right) = 0.$$

Under suitable assumptions, in Theorem 8.4 we prove the uniqueness of solutions to this PDE system and, as observed in Remark 8.5, that the measures are absolutely continuous, *i.e.*, $\mu^{\mathrm{c}} = \rho^{\mathrm{c}}\,\mathrm{d}x$ and $\mu^{\mathrm{p}} = \rho^{\mathrm{p}}\,\mathrm{d}y$.

We conclude the paper by finding in Theorem 8.6 the Γ-limit of the costs $\mathcal{J}_N$ defined in (1.6) as $N \to +\infty$. It is given by the cost for the following mean-field system:

$$\mathcal{J}(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\, \mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\, \mathrm{d}t.$$

Also in this case, we deduce convergence of optimal controls as $N \to +\infty$, see Proposition 8.7. The limit problem is an optimal control problem with a finite number of coast guard ships driving the densities of commercial and criminal ships.

2 Notation and preliminary results

2.1 Basic notation and preliminary results

Given a matrix A, we let $|A|$ its Frobenius norm. We shall often consider matrices of the form $A \in \mathbb{R}^{2\times d}$. By writing $A = (A_1,\dots,A_d)$, we make explicit its columns $A_i \in \mathbb{R}^2$.

If Ω, Ω' are measurable spaces, μ is a measure on Ω, and $\psi\colon \Omega \to \Omega'$ is a measurable map, then the push-forward $\psi_\#\mu$ is the measure on Ω' satisfying $\int_{\Omega'} \phi(\omega')\,\mathrm{d}\psi_\#\mu(\omega') := \int_\Omega \phi(\psi(\omega))\,\mathrm{d}\mu(\omega)$ for every measurable function ϕ.

Throughout the paper, we systematically apply Grönwall's inequality. We recall that if $u, v, w\colon [0,T] \to \mathbb{R}$ are continuous and nonnegative functions satisfying

$$u(t) \le w(t) + \int_0^t v(s)u(s)\,\mathrm{d}s \quad \text{for every } t \in [0,T],$$

then

$$u(t) \le w(t) + \int_0^t v(s)w(s)e^{\int_s^t v(r)\,\mathrm{d}r}\,\mathrm{d}s \quad \text{for every } t \in [0,T],$$

cf. [32, Theorem 1.3.2]. If, in addition, $w\colon [0,T] \to \mathbb{R}$ is continuous, positive, and nondecreasing, then

$$u(t) \le w(t)e^{\int_0^t v(s)\,\mathrm{d}s} \quad \text{for every } t \in [0,T],$$

cf. [32, Theorem 1.3.1]

If not specified otherwise, we let C denote a constant that might change from line to line. We make precise the dependence of C on other constants when it is relevant for the discussion.

2.2 Stochastic processes and Brownian motion

For the theory of stochastic processes and stochastic differential equations, we refer to the monographs [27, 28, 31]. Here we recall some basic facts and definitions used in the paper.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ used throughout the paper. By a.s. (almost surely) we mean $\mathbb{P}$-almost everywhere. We let $\mathbb{E}$ denote the expectation.

A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]}$ increasing in t, *i.e.*, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$. When $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ is said to satisfy the usual conditions if it is right-continuous (*i.e.*, $\mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t$ for all s) and if $\mathcal{N}_\mathbb{P} \subset \mathcal{F}_0$, where $\mathcal{N}_\mathbb{P} = \{A \subset \Omega \text{ s.t. } A \subset B \text{ with } B \in \mathcal{F} \text{ and } \mathbb{P}(B) = 0\}$ (if $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, this means that $\mathcal{F}_0$ contains $\mathbb{P}$-null sets).

A stochastic process is a parametrized collection of random variables $(S(t))_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}^d$ (equipped with σ-algebra of Borel sets). Given $t \in [0,T]$ and $\omega \in \Omega$, we will write $S(t,\omega) = S(t)(\omega)$ for the realization of the random variable $S(t)$ at ω. A path of the stochastic process is a curve in $\mathbb{R}^d$ obtained as the realization $t \mapsto S(t,\omega)$ for some $\omega \in \Omega$. A stochastic process $(S(t))_{t\in[0,T]}$ is adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if $S(t)$ is $\mathcal{F}_t$-measurable for every $t \in [0,T]$.

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. A d-dimensional Brownian motion (or Wiener process) is an $\mathbb{R}^d$-valued stochastic process $(W(t))_{t\in[0,T]}$, adapted to $(\mathcal{F}_t)_{t\in[0,T]}$, a.s. with continuous paths such that: $W(0) = 0$ a.s.; $W(t) - W(s) \sim \mathcal{N}(0,(t-s)\mathrm{Id}_d)$; $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for $t \ge s$.[1] Equivalently, it has components $W(t) = (W_1(t), \dots, W_d(t))$ with $(W_1(t))_{t\in[0,T]}, \dots, (W_d(t))_{t\in[0,T]}$ independent 1-dimensional Brownian motions.

[1] One can speak of a Brownian motion without introducing the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ by replacing the condition that $W(t) - W(s)$ is independent of $\mathcal{F}_s$ with the requirement that it has independent increments. In this case, one implicitly considers a filtration constructed from $(W(t))_{t\in[0,T]}$ by letting $\mathcal{F}_t^W$ be σ-algebra generated by $\{W(s) | s \le t\}$. If the filtration needs to satisfy the usual conditions, then $\mathcal{F}_t^W$ is modified with the augmentation $\mathcal{F}_t$ defined as the σ-algebra generated by $\mathcal{F}_t^W$ and $\mathcal{N}_\mathbb{P}$, see [28, p. 16] or [27, Proposition 2.7.7].

2.3 Stochastic differential equation

For the general theory about SDEs, we refer to [27, 28, 31]. We recall here some basic facts. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration satisfying the usual conditions, let $(W(t))_{t\in[0,T]}$ be a d-dimensional Brownian motion, and let us consider an initial datum S^0 given by an $\mathcal{F}_0$-measurable random variable.[2]

However, in this paper we are only interested in a specific class of SDEs, *i.e.*, those with a constant dispersion matrix of the form

$$\begin{cases} \mathrm{d}S(t) = b(t,S(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ S(0) = S^0 \quad \text{a.s.} \end{cases} \tag{2.1}$$

A stochastic process $(S(t))_{t\in[0,T]}$ is a strong solution to (2.1) if $(S(t))_{t\in[0,T]}$ has a.s. continuous paths, it is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, satisfies a.s. $\int_0^T |b(t,S(t))|\,\mathrm{d}t < \infty$, and for every $t \in [0,T]$

$$S(t) = S^0 + \int_0^t b\bigl(s,S(s)\bigr)\,\mathrm{d}s + \sigma\,\mathrm{d}W(s).$$

For this class of SDEs, it is well known that the well-posedness theory is simpler [27, Equation (2.34)] and requires weaker assumptions on the initial datum S^0 than those usually stated in general theorems. For the reader's convenience, we state and prove the result in the form needed in this paper, as we did not find a precise reference in the literature. Besides, some of the tools used in the proof will be exploited later in the paper. The result is stated with the Euclidean norm $|\cdot|$ on $\mathbb{R}^d$, but we remark that it holds true when replacing it with any equivalent norm, *e.g.*, also $\max_h |S_h|$, as long as the assumptions on b are satisfied with that norm.

Proposition 2.1 *Let $b\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ be a Carathéodory function satisfying:*

- *$|b(t,S)| \le C_b(1+|S|)$ for every $t\in[0,T]$ and $S\in\mathbb{R}^d$;*
- *For every $R>0$, there exists C_R such that $|b(t,S)-b(t,S')| \le \mathrm{Lip}_R(b)|S-S'|$ for all $t\in[0,T]$ and $S,S'\in\mathbb{R}^d$ such that $|S|,|S'|\le R$.*

Let $\sigma\in\mathbb{R}^{d\times d}$. Let $(W(t))_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued Brownian motion, and let S^0 be a random variable such that a.s. $|S^0| < +\infty$. Then there exists a unique strong solution $(S(t))_{t\in[0,T]}$ to (2.1). Moreover, if $\mathbb{E}(|S^0|) < +\infty$, then $\mathbb{E}(\|S\|_\infty) \le C(1+\mathbb{E}(|S^0|))$, where the constant C depends on C_b, T, and W.

Proof The scheme of the proof is the classical one, see [28, Theorem 3.3].

Let us fix $\omega\in\Omega$ such that $|S^0(\omega)| < +\infty$ and $t\mapsto W(t,\omega)$ is continuous, which occurs almost surely. We consider the Picard iterations

$$\widetilde{S}^0(t,\omega) := S^0(\omega) \quad \text{for } t\in[0,T], \tag{2.2}$$

$$\widetilde{S}^{j+1}(t,\omega) := S^0(\omega) + \int_0^t b\bigl(s,\widetilde{S}^j(s,\omega)\bigr)\,\mathrm{d}s + W(t,\omega) \quad \text{for } t\in[0,T], j\ge 0. \tag{2.3}$$

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[2] If this is not the case, then the construction explained in Footnote 1 is modified by considering the σ-algebra generated by S^0, $\{W(s)|s\le t\}$, and $\mathcal{N}_{\mathbb{P}}$.

Note that the curve $t \mapsto \widetilde{S}^j(t,\omega)$ is continuous. First of all, let us prove that for all j and for all $t \in [0,T]$

$$\left|\widetilde{S}^j(t,\omega)\right| \le \left(e^{C_b t} - 1\right) + \left(\left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty\right)e^{C_b t}, \tag{2.4}$$

where C_b is the constant appearing in $|b(t,S)| \le C_b(1+|S|)$. For $j=0$, (2.4) is trivially satisfied. Assume that (2.4) is true for j. Then, by (2.3) and by the linear growth of b,

$$\begin{aligned}
&\left|\widetilde{S}^{j+1}(t,\omega)\right| \\
&\quad\le \left|S^0(\omega)\right| + \int_0^t C_b\left(1 + \left|\widetilde{S}^j(s,\omega)\right|\right)\mathrm{d}s + \left|W(t,\omega)\right| \\
&\quad\le C_b t + \left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty \\
&\qquad + \int_0^t C_b\left(e^{C_b s} - 1\right) + C_b\left(\left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty\right)e^{C_b s}\,\mathrm{d}s \\
&\quad\le C_b t + \left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty + \left(e^{C_b t} - 1\right) - C_b t \\
&\qquad + \left(\left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty\right)\left(e^{C_b t} - 1\right) \\
&\quad= \left(e^{C_b t} - 1\right) + \left(\left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty\right)e^{C_b t},
\end{aligned}$$

which proves (2.4). In particular,

$$\left\|\widetilde{S}^j(\cdot,\omega)\right\|_\infty \le \left(1 + \left|S^0(\omega)\right| + \left\|W(\cdot,\omega)\right\|_\infty\right)e^{C_b T} =: R(\omega). \tag{2.5}$$

Since b is locally Lipschitz, there exists a constant $\mathrm{Lip}_{R(\omega)}(b)$ such that $|b(t,S) - b(t,S')| \le \mathrm{Lip}_{R(\omega)}(b)|S - S'|$ for all $t \in [0,T]$ and $S, S' \in \mathbb{R}^d$ such that $|S|, |S'| \le R(\omega)$. Thanks to this, we show that

$$\sup_{0\le s\le t}\left|\widetilde{S}^{j+1}(s,\omega) - \widetilde{S}^j(s,\omega)\right| \le C(\omega)\frac{(\mathrm{Lip}_{R(\omega)}(b)t)^j}{j!} \tag{2.6}$$

for a suitable constant $C(\omega)$ depending on ω. Indeed, for $j=0$, by the linear growth of b, we have that for every $s \in [0,T]$

$$\left|\widetilde{S}^1(s,\omega) - \widetilde{S}^0(s,\omega)\right| \le \int_0^s \left|b\left(r, S^0(\omega)\right)\right|\mathrm{d}r + \left|W(s,\omega)\right| \le C_b s\left(1 + \left|S^0(\omega)\right|\right) + \left|W(s,\omega)\right|,$$

hence

$$\sup_{0\le s\le t}\left|\widetilde{S}^1(s,\omega) - \widetilde{S}^0(s,\omega)\right| \le C_b T\left(1 + \left|S^0(\omega)\right|\right) + \left\|W(\cdot,\omega)\right\|_\infty =: C(\omega). \tag{2.7}$$

Moreover, by the local Lipschitz continuity of b, we have that for every $s \in [0,T]$

$$\begin{aligned}
\left|\widetilde{S}^{j+1}(s,\omega) - \widetilde{S}^j(s,\omega)\right| &\le \int_0^s \left|b\left(r,\widetilde{S}^j(r,\omega)\right) - b\left(r,\widetilde{S}^{j-1}(r,\omega)\right)\right|\mathrm{d}r \\
&\le \mathrm{Lip}_{R(\omega)}(b)\int_0^s \left|\widetilde{S}^j(r,\omega) - \widetilde{S}^{j-1}(r,\omega)\right|\mathrm{d}r.
\end{aligned}$$

Assuming (2.6) true for $j-1$, we have that

$$\sup_{0\le s\le t}\left|\widetilde{S}^{j+1}(s,\omega)-\widetilde{S}^{j}(s,\omega)\right|\le \mathrm{Lip}_{R(\omega)}(b)\int_0^t \sup_{0\le r\le s}\left|\widetilde{S}^{j}(r,\omega)-\widetilde{S}^{j-1}(r,\omega)\right|\,\mathrm{d}s$$

$$\le \mathrm{Lip}_{R(\omega)}(b)\int_0^t C(\omega)\frac{(\mathrm{Lip}_{R(\omega)}(b)s)^{j-1}}{(j-1)!}\,\mathrm{d}s = C(\omega)\frac{(\mathrm{Lip}_{R(\omega)}(b)t)^{j}}{j!}.$$

This implies that $\widetilde{S}^{j}(\cdot,\omega)$ is a Cauchy sequence in the uniform norm since for $j\ge i$

$$\left\|\widetilde{S}^{j}(\cdot,\omega)-\widetilde{S}^{i}(\cdot,\omega)\right\|_\infty \le C(\omega)\sum_{h=i}^{+\infty}\frac{(\mathrm{Lip}_{R(\omega)}(b)T)^{h}}{h!}\to 0 \quad \text{as } i\to+\infty. \tag{2.8}$$

Thus there exists a continuous curve $S(\cdot,\omega)$ such that

$$\left\|\widetilde{S}^{j}(\cdot,\omega)-S(\cdot,\omega)\right\|_\infty \to 0.$$

We have constructed $S(\cdot,\omega)$ for a.e. $\omega\in\Omega$. The stochastic processes $(\widetilde{S}^{j}(t))_{t\in[0,T]}$ are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and have a.s. continuous paths. This implies that the limit $(S(t))_{t\in[0,T]}$ is a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and has a.s. continuous paths. Moreover, passing to the limit in (2.2) for a.e. $\omega\in\Omega$, it is a strong solution to (2.1).

Uniqueness is proven in a more general setting in [27, Theorem 2.5] via a stopping time argument.

Assume now $\mathbb{E}(|S^0|)<+\infty$ and let us prove the estimate on $\mathbb{E}(\|S\|_\infty)$. Passing to the limit in (2.5), we get that for a.e. $\omega\in\Omega$

$$\left\|S(\cdot,\omega)\right\|_\infty \le \left(1+\left|S^0(\omega)\right|+\left\|W(\cdot,\omega)\right\|_\infty\right)e^{C_bT}. \tag{2.9}$$

By Doob's maximal inequality [27, Chap. 1, Theorem 3.8-(iv)] we have that

$$\mathbb{E}\left(\|W\|_\infty^2\right)\le 4\mathbb{E}\left(\left|W(T)\right|^2\right),$$

and thus, by Hölder's inequality,

$$\mathbb{E}\left(\|W\|_\infty\right)\le \left(\mathbb{E}\left(\|W\|_\infty^2\right)\right)^{\frac{1}{2}}\le 2\mathbb{E}\left(\left|W(T)\right|^2\right)^{\frac{1}{2}}. \tag{2.10}$$

Hence, taking the expectation in (2.9),

$$\mathbb{E}\left(\|S\|_\infty\right)\le\left(1+\mathbb{E}\left(\left|S^0\right|\right)+2\mathbb{E}\left(\left|W(T)\right|^2\right)^{\frac{1}{2}}\right)e^{C_bT},$$

which concludes the proof. □

Remark 2.2 A comment about the Picard iterations used in the proof of Proposition 2.1 is in order. If b is globally Lipschitz, *i.e.*, $|b(t,S)-b(t,S')|\le \mathrm{Lip}(b)|S-S'|$ and $\mathbb{E}(|S^0|)<+\infty$, then the convergence of the Picard iterations can be improved. Indeed, $\mathbb{E}(|S^0|)<+\infty$ and (2.10) yield $\mathbb{E}(C(\omega))<+\infty$, where $C(\omega)$ is the constant defined in (2.7). Then, taking the expectation in (2.8) and replacing $\mathrm{Lip}_{R(\omega)}(b)$ with the global Lipschitz constant $\mathrm{Lip}(b)$, we deduce that $\mathbb{E}(\|\widetilde{S}^{j}-S\|_\infty)\to 0$.

2.4 Wasserstein space

Given a complete metric space (B,d), we let $\mathcal{P}_1(B)$ denote the 1-Wasserstein space, *i.e.*, the space of Borel probability measures $\mu \in \mathcal{P}(B)$ such that

$$\int_B d(x,x_0)\,\mathrm{d}\mu(x) < +\infty,$$

where $x_0 \in B$ is fixed. The 1-Wasserstein space is equipped with the 1-Wasserstein distance defined for every $\mu_1,\mu_2 \in \mathcal{P}_1(B)$ by (see [39, Definition 6.1])

$$\mathcal{W}_1(\mu_1,\mu_2) := \inf_\gamma \int_{B\times B} d(x,x')\,\mathrm{d}\gamma(x,x'),$$

where the infimum is taken over all transport plans $\gamma \in \mathcal{P}(B\times B)$ with marginals $\pi^1_\# \gamma = \mu_1$ and $\pi^2_\# \gamma = \mu_2$, where π^i is the projection on the ith component.

We shall often exploit the dual formulation of the 1-Wasserstein distance. By Kantorovich's duality [39, Theorem 5.10], we have that

$$\mathcal{W}_1(\mu_1,\mu_2) = \sup_{\substack{\psi\in L^1(\mu_1)\\ \psi\ d\text{-convex}}} \left(\int_B \psi^d(x')\,\mathrm{d}\mu_2(x') - \int_B \psi(x)\,\mathrm{d}\mu_1(x) \right),$$

where ψ^d is the d-transform $\psi^d(x') = \inf_{x\in B}(\psi(x) + d(x,x'))$. Since d is a distance on a metric space, a d-convex function ψ is a Lipschitz function with Lipschitz constant 1 and it coincides with its d-transform, *cf.* [39, Particular Case 5.4]. Hence, if ψ is a Lipschitz function with Lipschitz constant $\mathrm{Lip}(\psi)$, then we have that

$$\left| \int_B \psi(x)\,\mathrm{d}(\mu_2 - \mu_1)(x) \right| \le \mathrm{Lip}(\psi)\mathcal{W}_1(\mu_1,\mu_2).$$

When in this paper we refer to Kantorovich's duality, we apply this inequality. Note that the condition $\psi \in L^1(\mu_1)\cap L^1(\mu_2)$ is satisfied since $|\psi(x)| \le |\psi(0)| + \mathrm{Lip}(\psi)|x|$ and $\mu_1,\mu_2 \in \mathcal{P}_1(B)$.

2.5 Wiener space

Given an interval $[0,T]$, we shall consider the so-called Wiener space of $\mathbb{R}^d$-valued continuous functions $C^0([0,T];\mathbb{R}^d)$ equipped with the uniform norm. Given $t \in [0,T]$, we consider the evaluation function $\mathrm{ev}_t\colon C^0([0,T];\mathbb{R}^d) \to \mathbb{R}^d$ defined by $\mathrm{ev}_t(\varphi) := \varphi(t)$ for every $\varphi \in C^0([0,T];\mathbb{R}^d)$. The family of evaluation functions $\{\mathrm{ev}_t\}_{t\in[0,T]}$ generates a σ-algebra on $C^0([0,T];\mathbb{R}^d)$, which coincides with the Borel σ-algebra with respect to the uniform norm in $C^0([0,T];\mathbb{R}^d)$.[3] This is generated by cylindrical sets of the form $\{\varphi \in C^0([0,T];\mathbb{R}^d) : \varphi(t_1) \in A_1,\dots,\varphi(t_k) \in A_k\}$, where $A_1,\dots,A_k \subset \mathbb{R}^d$ are Borel sets.

Let $(S(t))_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued stochastic process a.s. with continuous paths. This means that there exists an event $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 1$ and $t \mapsto S(t,\omega)$ is continuous for all $\omega \in E$. We can redefine $S(t,\omega) = 0$ for all $t \in [0,T]$ when $\omega \in \Omega \setminus E$. This new stochastic process is indistinguishable from the previous one and satisfies $S(\cdot,\omega) \in C^0([0,T];\mathbb{R}^d)$

[3] The reason for this is that the evaluation maps ev_t are continuous with respect to the uniform norm, thus Borel measurable; conversely, open balls in the Wiener space (which is separable) are measurable with respect to the σ-algebra generated by $\{\mathrm{ev}_t\}_{t\in[0,T]}$ since $\|\varphi\|_\infty = \sup_{t\in[0,T]\cap\mathbb{Q}} |\mathrm{ev}_t(\varphi)|$.

for all $\omega \in \Omega$. The stochastic process $(S(t))_{t\in[0,T]}$ can be regarded as the random variable $S\colon \Omega \to C^0([0,T];\mathbb{R}^d)$ such that $\omega \mapsto S(\cdot,\omega)$.

The σ-algebra generated by this random variable is the σ-algebra generated by sets of the form $S^{-1}(A)$ where $A \subset C^0([0,T];\mathbb{R}^d)$ is a cylindrical Borel set. This means that $A = \{\varphi \in C^0([0,T];\mathbb{R}^d) : \varphi(t_1) \in A_1, \dots, \varphi(t_k) \in A_k\}$, where $A_1,\dots,A_k \subset \mathbb{R}^d$ are Borel sets. For these sets, we have that

$$S^{-1}(A) = \big\{\omega \in \Omega : S(\cdot,\omega) \in A\big\} = \big\{\omega \in \Omega : S(t_1,\omega) \in A_1, \dots, S(t_k,\omega) \in A_k\big\},$$

and thus the σ-algebra generated by $S\colon \Omega \to C^0([0,T];\mathbb{R}^d)$ coincides with the σ-algebra generated by the family $\{S(t)\}_{t\in[0,T]}$ of random variables $S(t,\cdot)\colon \Omega \to \mathbb{R}^d$, *i.e.*, the σ-algebra generated by the stochastic process.

In particular, if $(S_1(t))_{t\in[0,T]}, \dots, (S_K(t))_{t\in[0,T]}$ are stochastic processes a.s. with continuous paths, then they are independent as stochastic processes if and only if they are independent as random variables $S_1,\dots,S_K\colon \Omega \to C^0([0,T];\mathbb{R}^d)$.

Finally, we remark that a random variable $S\colon \Omega \to C^0([0,T];\mathbb{R}^d)$ induces the probability measure $S_\#\mathbb{P}$ on the space $C^0([0,T];\mathbb{R}^d)$. We let $\mathrm{Law}(S) := S_\#\mathbb{P} \in \mathcal{P}(C^0([0,T];\mathbb{R}^d))$.[4]

If $\mu \in \mathcal{P}(C^0([0,T];\mathbb{R}^d))$, then we let $\mu(t) := (\mathrm{ev}_t)_\#\mu \in \mathcal{P}(\mathbb{R}^d)$.

2.6 Empirical measures

Given random variables $X_1,\dots,X_K\colon \Omega \to \mathbb{R}^d$ with $\mathbb{E}(|X_k|) < +\infty$, we define their empirical measure as the random measure[5] $\mu_K\colon \Omega \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$\mu_K(\omega) := \frac{1}{K}\sum_{k=1}^K \delta_{X_k(\omega)}$$

for a.e. $\omega \in \Omega$. Note that indeed $\mu_K \in \mathcal{P}_1(\mathbb{R}^d)$ a.s. since

$$\mathbb{E}\bigg(\int_{\mathbb{R}^d} |x|\,\mathrm{d}\mu_K(x)\bigg) = \frac{1}{K}\sum_{k=1}^K \mathbb{E}\big(|X_k|\big) < +\infty.$$

Empirical measures of independent samples from a law approximate the law itself. More precisely, let us fix a law $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ and $(X_k)_{k\in\mathbb{N}}$ a sequence of i.i.d. random variables with law μ (which thus satisfy $\mathbb{E}(|X_k|) < +\infty$). Let μ_K be the empirical measure of $X_1,\dots,X_K$. Then $\mathbb{E}(\mathcal{W}_1(\mu_K,\mu)) \to 0$ as $K \to +\infty$, see, *e.g.*, [33, Lemma 4.7.1]. In fact, also precise rates of convergence are available in the literature, see [26, Theorem 1].

2.7 Γ-convergence

For the theory of Γ-convergence, we refer to the monograph [20]. In this paper it will be used to find the limits of optimal control problems.

[4] This discussion applies, in particular, to a Brownian motion $(W(t))_{t\in[0,T]}$. The probability measure $\mathrm{Law}(W)$ is known as Wiener measure on $C^0([0,T];\mathbb{R}^d)$.

[5] The map $\mu_K\colon \Omega \to \mathcal{P}_1(\mathbb{R}^d)$ is indeed measurable with respect to the Borel σ-algebra on the 1-Wasserstein space $\mathcal{P}_1(\mathbb{R}^d)$. To see this, we observe that $\mathcal{P}_1(\mathbb{R}^d)$ endowed with the 1-Wasserstein distance is separable, see, *e.g.*, [39, Theorem 6.18], hence the Borel σ-algebra is generated by balls $\{\mu \in \mathcal{P}_1(\mathbb{R}^d) : \mathcal{W}_1(\mu,\mu_0) < r\}$. The pre-image of such a ball through μ_K is the event $\{\omega \in \Omega : \mathcal{W}_1(\frac{1}{K}\sum_{k=1}^K \delta_{X_k(\omega)}, \mu_0) < r\}$. This is measurable since the function $(x_1,\dots,x_K) \mapsto \mathcal{W}_1(\frac{1}{K}\sum_{k=1}^K \delta_{x_k}, \mu_0)$ is Lipschitz continuous.

Table 1 Summary of the notation for ships

Item	*Meaning*	*Comment*
$[0,T]$	time interval	fixed
$X=(X_1,\dots,X_N)$	trajectories of N commercial ships	unknown of the system
$Y=(Y_1,\dots,Y_M)$	trajectories of M pirate ships	unknown of the system
$Z=(Z_1,\dots,Z_L)$	trajectories of L guard ships	unknown of the system
$.^{\mathrm{c}}$	related to commercial ships	
$.^{\mathrm{p}}$	related to pirate ships	
$.^{\mathrm{g}}$	related to guard ships	

3 Description of the model

To better describe the phenomena that we aim to capture, we introduce all the ingredients that enter in the model step by step. For the reader's convenience, all the unknowns, the parameters, and the initial data of the model are summarized in Tables 1–5.

The model is an evolutionary system analyzed in a fixed time interval $[0,T]$.

Ships. The system describes the evolution of N commercial ships, M pirate (criminal) ships, and L coast guard (patrol) ships, whose trajectories are curves $X_n\colon [0,T]\to\mathbb{R}^2$ for $n\in\{1,\dots,N\}$, $Y_m\colon [0,T]\to\mathbb{R}^2$ for $m\in\{1,\dots,M\}$, and $Z_\ell\colon [0,T]\to\mathbb{R}^2$ for $\ell\in\{1,\dots,L\}$, respectively.

We shall often collect the trajectories based on their type by considering the matrix-valued curves $X=(X_1,\dots,X_N)\colon [0,T]\to\mathbb{R}^{2\times N}$, $Y=(Y_1,\dots,Y_M)\colon [0,T]\to\mathbb{R}^{2\times M}$, and $Z=(Z_1,\dots,Z_M)\colon [0,T]\to\mathbb{R}^{2\times L}$. The letters X, Y, Z will unambiguously indicate the type of ship, even when decorated, *e.g.*, as $\bar{X}$, $\widetilde{X}$, or with superscripts and subscripts.

Hereafter, whenever a variable is related to commercial, pirate, or guard ships, it is indexed with the superscript c, p, or g, respectively.

Evolution of commercial ships.

Step 1. We start by describing the evolution of commercial ships in safe waters (absence of pirate ships) and in the absence of congestion in the traffic. We assume that there is a vector field $\mathbf{r}\colon\mathbb{R}^2\to\mathbb{R}^2$ indicating safe commercial routes. In this ideal setting, commercial ships evolve according to the ODEs

$$\begin{cases} \frac{\mathrm{d}X_n}{\mathrm{d}t}(t)=\mathbf{r}(X_n(t)),\\ X_n(0)=X_n^0, \quad n=1,\dots,N, \end{cases}$$

where $X^0=(X_1^0,\dots,X_N^0)\in\mathbb{R}^{2\times N}$ is the initial position of commercial ships.

We shall assume that $\mathbf{r}$ is globally Lipschitz continuous.

Step 2. To include congestion in the model, we introduce $v^N=(v_1^N,\dots,v_N^N)\colon\mathbb{R}^{2\times N}\to[0,v_{\max}]^N$. The component v_n^N weighs the speed of the trajectory of the nth commercial ship according to the presence of all the other commercial ships:

$$\begin{cases} \frac{\mathrm{d}X_n}{\mathrm{d}t}(t)=v_n^N(X(t))\mathbf{r}(X_n(t)),\\ X_n(0)=X_n^0, \quad n=1,\dots,N. \end{cases}$$

The assumptions on v^N needed throughout the paper are the following: v^N is Lipschitz continuous with respect to the max norm with a Lipschitz constant independent of N, *i.e.*, $|v^N(X)-v^N(X')|\le C\max_n|X_n-X_n'|$.

For v^N, we have in mind a precise expression that will be used in Sect. 8. We consider a globally Lipschitz smooth convolution kernel $\eta\colon \mathbb{R}^2\times\mathbb{R}^2 \to [0,1]$ satisfying $\eta(X,0)=0$. The quantity

$$\sum_{n'=1}^{N} \eta\big(X_n(t), X_n(t)-X_{n'}(t)\big)$$

suitably counts[6] the number of commercial ships around the nth commercial ship at time t. Hence, the quantity

$$\frac{1}{N-1}\sum_{n'=1}^{N} \eta\big(X_n(t), X_n(t)-X_{n'}(t)\big)$$

can be regarded as the density of commercial ships around the nth commercial ship at time t. The precise expression of the scaling factor $\frac{1}{N-1}$ is relevant only to interpret the previous expression as a density and can, in fact, be replaced by a sequence converging to zero with the same rate of $\frac{1}{N}$. Given a Lipschitz function $v\colon [0,1]\to[0,v_{\max}]$, the corrected speed of the nth commercial ship depends on the density of its surrounding ships as follows:

$$v_n^N\big(X(t)\big) = v\left(\frac{1}{N-1}\sum_{n'=1}^{N} \eta\big(X_n(t), X_n(t)-X_{n'}(t)\big)\right).$$

To model congestion, v must be assumed to be nonincreasing in the density.

Step 3. Eventually, let us modify the dynamics of commercial ships in the presence of pirate ships. We consider a globally Lipschitz vector-valued interaction kernel $K^{\mathrm{cp}}\colon \mathbb{R}^2\to\mathbb{R}^2$ (here cp stands for commercial-pirate). To model repulsion of the nth commercial ship from the pirate ships, we modify the direction of the trajectory $X_n(t)$ by averaging the vectors $K^{\mathrm{cp}}(X_n(t)-Y_m(t))$, *i.e.*,

$$\begin{cases} \frac{\mathrm{d}X_n}{\mathrm{d}t}(t) = v_n^N(X(t))(\mathbf{r}(X_n(t)) + \frac{1}{M}\sum_{m=1}^{M} K^{\mathrm{cp}}(X_n(t)-Y_m(t))), \\ X_n(0) = X_n^0, \quad n=1,\dots,N. \end{cases}$$

For K^{cp}, we have in mind the following expression:

$$K^{\mathrm{cp}}\big(X_n(t)-Y_m(t)\big) = H^{\mathrm{cp}}\big(X_n(t)-Y_m(t)\big)\big(X_n(t)-Y_m(t)\big), \tag{3.1}$$

where H^{cp} has compact support with a radius given by the length for which the presence of a pirate ship at $Y_m(t)$ affects the trajectory $X_n(t)$. An example of H^{cp} is $H^{\mathrm{cp}}(w)=\frac{h(|w|)}{|w|}$, where h is compactly supported in $(0,+\infty)$ so that $K^{\mathrm{cp}}(X_n(t)-Y_m(t)) = h(|X_n(t)-Y_m(t)|)\frac{X_n(t)-Y_m(t)}{|X_n(t)-Y_m(t)|}$ and $\frac{X_n(t)-Y_m(t)}{|X_n(t)-Y_m(t)|}$ is, for $X_n(t)$, the direction pointing opposite to $Y_m(t)$.

[6] For example, let $\hat\eta \in C_c^\infty(\mathbb{R}^2)$ be supported in a ball $B_{2\delta}$ of radius 2δ with $\hat\eta = 1$ on B_δ. If $\eta(X,X') = \hat\eta(X-X')$, then $\sum_{n'=1}^{N}\hat\eta(X_n(t)-X_{n'}(t))$ (approximately) counts the number of ships in a δ-neighborhood of $X_n(t)$ (around all directions). Instead, if $\eta(X,X')=\hat\eta(X-X'-\delta\mathbf{r}(X))$, then $\sum_{n'=1}^{N}\hat\eta(X_n(t)-X_{n'}(t)-\delta\mathbf{r}(X_n(t)))$ (approximately) counts the number of commercial ships obstructing the commercial route in front of $X_n(t)$.

Table 2 Summary of functions used in the model for evolution of commercial ships

Item	*Meaning*	*Comment*
η	kernel to compute the density of commercial ships	smooth and globally Lipschitz
v	velocity as a function of the density	Lipschitz continuous
$v^N = (v_1^N, \dots, v_N^N)$	obtained from η and v	Lipschitz continuous, with Lipschitz constant independent of N

Table 3 Interaction kernels used in the model

Item	*Meaning*	*Comment*
K^{cp}	effect of pirate ships on commercial ships	Lipschitz continuous
K^{pg}	effect of guard ships on pirate ships	Lipschitz continuous
K^{pc}	effect of commercial ships on pirate ships	Lipschitz continuous
K^{gg}	effect of guard ships on guard ships	Lipschitz continuous

Evolution of pirate ships.

Step 1. Pirate ships are repelled by guard ships and are attracted by commercial ships. To model this, we consider globally Lipschitz vector-valued interaction kernels $K^{\mathrm{pg}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $K^{\mathrm{pc}}\colon \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$\begin{cases} \frac{\mathrm{d}Y_m}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(Y_m(t) - Z_\ell(t)) - \frac{1}{N}\sum_{\ell=1}^{N} K^{\mathrm{pc}}(Y_m(t) - X_n(t)), \\ Y_m(0) = Y_m^0, \quad m = 1,\dots,M, \end{cases}$$

where $Y^0 = (Y_1^0,\dots,Y_M^0) \in \mathbb{R}^{2\times M}$ is the initial position of pirate ships.

For the precise form of $K^{\mathrm{pg}}, K^{\mathrm{pc}}$, see the analogous discussion for commercial ships done after (3.1).

Step 2. In the absence of commercial and guard ships, pirate ships explore the environment in search of targets by navigating randomly. To model this, we add a stochastic term in the evolution of pirate ships by considering M Brownian motions $(W_1(t))_{t\in[0,T]},\dots,(W_M(t))_{t\in[0,T]}$. The pirate ships then evolve according to the following SDEs:

$$\begin{cases} \mathrm{d}Y_m(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(Y_m(t) - Z_\ell(t)) \\ \quad - \frac{1}{N}\sum_{\ell=1}^{N} K^{\mathrm{pc}}(Y_m(t) - X_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_m(t), \\ Y_m(0) = Y_m^0 \quad \text{a.s.}, m = 1,\dots,M, \end{cases}$$

where $\kappa > 0$.

Evolution of guard ships. The last part of the system describes guard ships. In the absence of other ships, guard ships tend to repel each other. To model this, we consider globally Lipschitz vector-valued interaction kernel $K^{\mathrm{gg}}\colon \mathbb{R}^2 \to \mathbb{R}^2$. In this setting, the guard ships evolve according to

$$\begin{cases} \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L, \end{cases}$$

where $Z^0 = (Z_1^0,\dots,Z_L^0) \in \mathbb{R}^{2\times L}$ is the initial position of guard ships. We do not require more on the dynamics of guard ships as we want the global dynamics of the system to be governed by the optimal control policy for guard ships.

Table 4 Summary of initial data

Item	*Meaning*	*Comment*
$X^0 = (X_1^0,\dots,X_N^0)$	initial positions of commercial ships	points in $\mathbb{R}^2$, $\lvert X_n^0\rvert \le R_0$
$Y_1^0,\dots,Y_M^0$	initial positions of pirate ships	random variables in $\mathbb{R}^2$
$Z^0 = (Z_1^0,\dots,Z_L^0)$	initial positions of guard ships	points in $\mathbb{R}^2$

Controls. We consider a set of admissible controls $\mathcal{U} \subset \mathbb{R}^{2\times L}$. We assume $\mathcal{U}$ to be compact. A fixed control $u = (u_1,\dots,u_L) \in L^\infty([0,T];\mathcal{U})$ drives the evolution of guard ships as follows:

$$\begin{cases} \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L. \end{cases}$$

Full model. In conclusion, we are interested in the following ODE/SDE/ODE model:

$$\begin{cases} \mathrm{d}X_n(t) = v_n^N(X(t))(\mathbf{r}(X_n(t)) + \frac{1}{M}\sum_{m=1}^{M} K^{\mathrm{cp}}(X_n(t) - Y_m(t)))\,\mathrm{d}t, \\ \mathrm{d}Y_m(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(Y_m(t) - Z_\ell(t)) - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}(Y_m(t) - X_n(t)))\,\mathrm{d}t \\ \qquad\qquad + \sqrt{2\kappa}\,\mathrm{d}W_m(t), \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ X_n(0) = X_n^0 \quad \text{a.s.}, \qquad Y_m(0) = Y_m^0 \quad \text{a.s.}, \qquad Z_\ell(0) = Z_\ell^0, \\ n = 1,\dots,N, m = 1,\dots,M, \ell = 1,\dots,L. \end{cases} \tag{3.2}$$

(The first equation is expressed as an SDE to stress that the solution X is a stochastic process. However, given a trajectory Y, the first equation is, in fact, an ODE.)

We prove well-posedness for (3.2) in Sect. 4.1.

Initial data. The initial data in (3.2) will be given by $X^0 = (X_1^0,\dots,X_N^0) \in \mathbb{R}^{2\times N}$ with $|X_n^0| \le R_0$ for some $R_0 > 0$; $\mathbb{R}^2$-valued i.i.d. random variables $Y_1^0,\dots,Y_M^0$; $Z^0 = (Z_1^0,\dots,Z_L^0) \in \mathbb{R}^{2\times L}$.

Optimal control. As previously mentioned, the dynamics of guard ships will be driven by an optimal control. To define the cost, we consider a bounded and globally Lipschitz function $H^{\mathrm{d}}\colon \mathbb{R}^2 \to \mathbb{R}$. If the quantity $H^{\mathrm{d}}(X_n(t) - Y_m(t))$ is significantly different from zero when $Y_m(t)$ is close to $X_n(t)$ and is small when $Y_m(t)$ is far from $X_n(t)$ (*e.g.*, when H^{d} is compactly supported), then this function can be used to count contacts between commercial and pirate ships (the superscript d stands for "danger"). Hence we consider the cost functional $\mathcal{J}_{N,M}\colon L^\infty([0,T];\mathcal{U}) \to \mathbb{R}$ defined for every control $u \in L^\infty([0,T];\mathcal{U})$ by

$$\mathcal{J}_{N,M}(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M} H^{\mathrm{d}}\big(X_n(t) - Y_m(t)\big)\,\mathrm{d}t\right), \tag{3.3}$$

where the stochastic processes $(X(t))_{t\in[0,T]} = (X_1(t),\dots,X_N(t))_{t\in[0,T]}$ and $(Y(t))_{t\in[0,T]} = (Y_1(t),\dots,Y_M(t))_{t\in[0,T]}$ are given by the unique strong solutions to (3.2) corresponding to the control u obtained in Proposition 4.1.

The objective is to minimize the cost $\mathcal{J}_{N,M}$.

4 Well-posedness of the ODE/SDE/ODE model

4.1 Well-posedness of the ODE/SDE/ODE model for a fixed control

In this section we prove well-posedness for the model presented in (3.2).

Table 5 Summary of the items regarding control

Item	*Meaning*	*Comment*
$\mathcal{U} \subset \mathbb{R}^{2\times L}$	set of admissible controls	compact
H^{d}	kernel for dangerous contacts in cost functional	bounded and Lipschitz continuous
$\mathcal{J}_{N,M}$	cost functional associated to (3.2), for fixed N, M	defined in (3.3)
$\mathcal{J}_N$	cost functional defined in (7.16), for fixed N	obtained as the Γ-limit of $\mathcal{J}_{N,M}$ as $M \to +\infty$ in Theorem 7.3
$\mathcal{J}$	cost functional defined in (8.58)	obtained as the Γ-limit of $\mathcal{J}_N$ as $N \to +\infty$ in Theorem 8.6

We remark that the solutions depend on N and M. Not to overburden the notation, in this section we drop the dependence on N and M as we will not consider limits as $N \to +\infty$ or $M \to +\infty$.

Proposition 4.1 *Assume the following*:

- *Let* $(W_1(t))_{t\in[0,T]}, \dots, (W_M(t))_{t\in[0,T]}$ *be independent Brownian motions*;
- *Let* $X^0 = (X_1^0, \dots, X_N^0) \in \mathbb{R}^{2\times N}$;
- *Let* $Y_1^0, \dots, Y_M^0$ *be* $\mathbb{R}^2$*-valued random variables with* $|Y_m^0| < +\infty$ *a.s. for* $m = 1, \dots, M$;
- *Let* $Z^0 = (Z_1^0, \dots, Z_L^0) \in \mathbb{R}^{2\times L}$;
- *Let* $u \in L^\infty([0,T];\mathcal{U})$.

Then there exists a unique strong solution to (3.2), $(X(t))_{t\in[0,T]} = (X_1(t), \dots, X_N(t))_{t\in[0,T]}$, $(Y(t))_{t\in[0,T]} = (Y_1(t), \dots, Y_M(t))_{t\in[0,T]}$, *and* $Z = (Z_1, \dots, Z_L)$. *Moreover, if* $\mathbb{E}(|Y_m^0|) < +\infty$ *for* $m = 1, \dots, M$, *then* $\mathbb{E}(\max_m \|Y_m\|_\infty) < +\infty$.

Proof We start by noticing that the ODEs involving the variables Z_ℓ are decoupled from the equations involving X_n and Y_m. Given a control $u = (u_1, \dots, u_L) \in L^\infty([0,T];\mathcal{U})$, we solve

$$\begin{cases} \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1, \dots, L. \end{cases}$$

We observe that there exists a unique solution for all times $t \in [0,T]$ to the previous ODE system. Too see this, we introduce the function $f = f_u = (f_{u,1}, \dots, f_{u,L})\colon [0,T]\times\mathbb{R}^{2\times L} \to \mathbb{R}^{2\times L}$ (we drop the dependence on u for ease of notation) defined by

$$f_\ell(t,Z) := \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell - Z_{\ell'}) + u_\ell(t) \quad \text{for } \ell = 1, \dots, L,$$

and we notice that the system reads

$$\begin{cases} \frac{\mathrm{d}Z}{\mathrm{d}t}(t) = f(t, Z(t)), \\ Z(0) = Z^0, \end{cases} \tag{4.1}$$

where $Z = (Z_1, \dots, Z_L)$. The right-hand side $f(t,Z)$ is a Carathéodory function, globally Lipschitz continuous in the Z variable (with Lipschitz constant independent of t). These properties are sufficient for the well-posedness of the ODE.[7] We remark that solutions to (4.1) are bounded. Indeed,

$$\left|f(t,Z)\right| \le \left|f(t,0)\right| + \left|f(t,Z)-f(t,0)\right| \le \left\|K^{\mathrm{gg}}\right\|_\infty + \|u\|_\infty + C|Z| \le C\big(1+|Z|\big), \tag{4.2}$$

hence

$$\left|Z(t)\right| \le \left|Z^0\right| + \int_0^t C\big(1+\left|Z(s)\right|\big)\,\mathrm{d}s \le \left|Z^0\right| + CT + \int_0^t \left|Z(s)\right|\,\mathrm{d}s$$

and, by Grönwall's inequality, for $t\in[0,T]$,

$$\left|Z(t)\right| \le \big(\left|Z^0\right| + CT\big)e^{Ct} \le \big(\left|Z^0\right| + CT\big)e^{CT}, \tag{4.3}$$

where the constant C depends on K^{gg} and $\mathcal{U}$ (compact).

We exploit the solution $Z(t)$ to solve the ODE/SDE/ODE system, which now we write in a more compact way. Let us introduce the $\mathbb{R}^{2\times(M+N)}$-valued stochastic process $(S(t))_{t\in[0,T]}$ defined by

$$S(t) := \big(Y_1(t),\dots,Y_M(t),X_1(t),\dots,X_N(t)\big)$$

(we put the components $Y_1(t),\dots,Y_M(t)$ in the first block for consistency later). We consider the drift vector $b_Z = b = (b_1,\dots,b_{M+N})\colon [0,T]\times\mathbb{R}^{2\times(M+N)} \to \mathbb{R}^{2\times(M+N)}$ (we drop the dependence on Z for the ease of notation) defined for every $S=(S_1,\dots,S_{M+N})\in\mathbb{R}^{2\times(M+N)}$ by

$$b_i(t,S) := \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(S_i - Z_\ell(t)\big) - \frac{1}{N}\sum_{j=M+1}^{M+N} K^{\mathrm{pc}}(S_i - S_j) \tag{4.4}$$

for $i=1,\dots,M$ and

$$b_i(t,S) := \nu^N_{i-M}\big((S_j)_{j=M+1}^{M+N}\big)\left(\mathbf{r}(S_i) + \frac{1}{M}\sum_{j=1}^{M} K^{\mathrm{cp}}(S_i - S_j)\right) \tag{4.5}$$

for $i=M+1,\dots,M+N$. Moreover, let $\sigma\in\mathbb{R}^{(2\times2)\times(M+N)}$ be the constant dispersion tensor given by the collection $\sigma=(\sigma_1,\dots,\sigma_{M+N})$ of the matrices $\sigma_i\colon\mathbb{R}^{2\times(M+N)}\to\mathbb{R}^{2\times2}$ defined by

$$\sigma_i := \sqrt{2\kappa}\,\mathrm{Id}_2 \quad \text{for } i=1,\dots,M$$

and $\sigma_i := 0$ for $i=M+1,\dots,M+N$. For $W=(W_1,\dots,W_{M+N})\in\mathbb{R}^{2\times(M+N)}$, we adopt the short-hand notation σW to denote the element in $\mathbb{R}^{2\times(M+N)}$ with columns $(\sigma W)_1,\dots,(\sigma W)_{M+N}\in\mathbb{R}^2$ given by $(\sigma W)_i = \sigma_i W_i$.

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[7]The result is classical: one considers the Picard operator $\mathcal{S}\colon C^0([0,T];\mathbb{R}^{2\times L})\to C^0([0,T];\mathbb{R}^{2\times L})$ defined by $\mathcal{S}(Z)(t) := Z^0 + \int_0^t f(s,Z(s))\,\mathrm{d}s$, which is a contraction with respect to the norm (equivalent to the uniform norm) $|||\varphi|||_\alpha := \sup_{t\in[0,T]}(e^{-\alpha t}|\varphi(t)|)$ for suitable $\alpha>0$ (depending on the Lipschitz constant of f).

By setting $S^0 := (Y_1^0, \dots, Y_M^0, X_1^0, \dots, X_N^0)$, the system reads

$$\begin{cases} \mathrm{d}S(t) = b(t, S(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ S(0) = S^0 \quad \text{a.s.}, \end{cases} \tag{4.6}$$

where $(W(t))_{t\in[0,T]}$ is an $\mathbb{R}^{2\times(M+N)}$-valued Brownian motion. Note that $W_1(t), \dots, W_M(t)$ correspond to the M independent $\mathbb{R}^2$-valued Brownian motions already introduced for (3.2). This is the reason why we chose to put the Y_ms in place of the X_ns in the first block of S.

We are now left to check that the conditions for the existence and uniqueness stated in Proposition 2.1 are satisfied by (4.6). By the continuity of $Z(t)$, the function $t \mapsto b(t, S)$ is continuous for every S. Let $i \in \{1, \dots, M\}$, so that b_i is given by (4.4). By the Lipschitz continuity of K^{pg}, we have that

$$\begin{aligned} \left|K^{\mathrm{pg}}(z_1 - z_2)\right| &\le \left|K^{\mathrm{pg}}(z_1 - z_2) - K^{\mathrm{pg}}(0)\right| + \left|K^{\mathrm{pg}}(0)\right| \le C|z_1 - z_2| + \left|K^{\mathrm{pg}}(0)\right| \\ &\le C\big(1 + |z_1| + |z_2|\big). \end{aligned} \tag{4.7}$$

Reasoning analogously for K^{pc}, it follows that

$$\begin{aligned} \left|b_i(t,S)\right| &\le \frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\big(S_i - Z_\ell(t)\big)\right| + \frac{1}{N}\sum_{j=M+1}^{M+N}\left|K^{\mathrm{pc}}(S_i - S_j)\right| \\ &\le \frac{1}{L}\sum_{\ell=1}^{L} C\big(1 + |S_i| + \left|Z_\ell(t)\right|\big) + \frac{1}{N}\sum_{j=M+1}^{M+N} C\big(1 + |S_i| + |S_j|\big) \\ &\le C\Big(1 + \max_h |S_h|\Big), \end{aligned} \tag{4.8}$$

where we used the continuity, and thus boundedness, of $Z_\ell(t)$ for $t \in [0, T]$. Let us check the Lipschitz continuity condition. By the Lipschitz continuity of K^{pg} and K^{pc}, we have that

$$\begin{aligned} \left|b_i(t,S) - b_i\big(t,S'\big)\right| &\le \frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\big(S_i - Z_\ell(t)\big) - K^{\mathrm{pg}}\big(S_i' - Z_\ell(t)\big)\right| \\ &\quad + \frac{1}{N}\sum_{j=M+1}^{M+N}\left|K^{\mathrm{pc}}(S_i - S_j) - K^{\mathrm{pc}}\big(S_i' - S_j'\big)\right| \\ &\le \frac{1}{L}\sum_{\ell=1}^{L} C\left|S_i - S_i'\right| + \frac{1}{N}\sum_{j=M+1}^{M+N} C\left|S_i - S_j - S_i' + S_j'\right| \\ &\le \frac{1}{L}\sum_{\ell=1}^{L} C\left|S_i - S_i'\right| + \frac{1}{N}\sum_{j=M+1}^{M+N} C\big(\left|S_i - S_i'\right| + \left|S_j - S_j'\right|\big) \\ &\le C\max_h\left|S_h - S_h'\right|, \end{aligned} \tag{4.9}$$

where the constant C depends on $K^{\mathrm{pg}}, K^{\mathrm{pc}}$. (In fact, b_i is even globally Lipschitz continuous for $i \in \{1, \dots, M\}$).

Let now $i \in \{M+1,\dots,M+N\}$, so that b_i is given by (4.5). By the boundedness of v^N, by the bound $\mathbf{r}(x) \leq C(1+|x|)$, and reasoning for $K^{\rm cp}$ as in (4.7), we have that

$$\left|b_i(t,S)\right| \leq \left\|v^N\right\|_\infty \left(C\left(1+|S_i|\right) + \frac{1}{M}\sum_{j=1}^M \left|K^{\rm cp}(S_i-S_j)\right| \right) \leq C\left(1+\max_h |S_h|\right). \tag{4.10}$$

To check the local Lipschitz continuity of b_i, let us fix $R>0$. For $t \in [0,T]$ and $\max_h |S_h| \leq R$, $\max_h |S'_h| \leq R$. By the boundedness and the Lipschitz property of v^N (recall that it has a Lipschitz constant independent of N) and by the Lipschitz continuity of $\mathbf{r}$ and $K^{\rm cp}$, we have that

$$\begin{aligned}
&\left|b_i(t,S) - b_i\left(t,S'\right)\right| \\
&\quad \leq \left|v^N_{i-M}\left((S_j)_{j=M+1}^{M+N}\right) - v^N_{i-M}\left((S'_j)_{j=M+1}^{M+N}\right)\right| \cdot \left|\mathbf{r}(S_i) + \frac{1}{M}\sum_{j=1}^M K^{\rm cp}(S_i-S_j)\right| \\
&\qquad + \left|v^N_{i-M}\left((S'_j)_{j=M+1}^{M+N}\right)\right| \\
&\qquad \times \left|\mathbf{r}(S_i) + \frac{1}{M}\sum_{j=1}^M K^{\rm cp}(S_i-S_j) - \mathbf{r}\left(S'_i\right) - \frac{1}{M}\sum_{j=1}^M K^{\rm cp}\left(S'_i-S'_j\right)\right| \\
&\quad \leq \max_h \left(C\left|S_h-S'_h\right|\left(1+|S_h|\right) + C\left|S_h-S'_h\right|\right) \\
&\quad \leq \max_h \left(C\left|S_h-S'_h\right|\left(1+|S_h|\right)\right) \leq C\max_h\left|S_h-S'_h\right|(1+R),
\end{aligned} \tag{4.11}$$

where the constant C depends on v^N, $\mathbf{r}$, and $K^{\rm cp}$ (independent of N). Choosing $C_R = C(1+R)$, we get the desired inequality.

Applying Proposition 2.1, we conclude the proof of existence and uniqueness. Moreover, we also get $\mathbb{E}(\max_h \|S_h\|_\infty) < +\infty$ and, in particular, $\mathbb{E}(\max_m \|Y_m\|_\infty) < +\infty$. □

4.2 Existence of an optimal control for the ODE/SDE/ODE model

Let $\mathcal{J}_{N,M}$ be the cost defined in (3.3). We have the following result concerning the existence of optimal controls.

Proposition 4.2 *Under the assumptions of Proposition 4.1, there exists an optimal control $u^* \in L^\infty([0,T];\mathcal{U})$, i.e.,*

$$\mathcal{J}_{N,M}\left(u^*\right) = \min_{u\in L^\infty([0,T];\mathcal{U})} \mathcal{J}_{N,M}(u).$$

Proof The result is obtained via the direct method in the calculus of variations. We divide the proof into steps for the sake of presentation.

<u>*Step 1*</u>. (Preliminary steps) Let $u^j \in L^\infty([0,T];\mathcal{U})$ be a minimizing sequence, *i.e.*, $\mathcal{J}_{N,M}(u^j) \to \min \mathcal{J}_{N,M}$ as $j \to +\infty$. Since u^j is bounded in $L^\infty([0,T];\mathcal{U})$, there exist $u^* \in L^\infty([0,T];\mathcal{U})$ and a subsequence (not relabeled) such that $u^j \overset{*}{\rightharpoonup} u^*$ weakly-* in $L^\infty([0,T];\mathcal{U})$. We claim that u^* is an optimal control.

To prove the claim, let us fix $(X^j(t))_{t\in[0,T]} = (X^j_1(t),\dots,X^j_N(t))_{t\in[0,T]}$, $(Y^j(t))_{t\in[0,T]} = (Y^j_1(t),\dots,Y^j_M(t))_{t\in[0,T]}$, and $Z^j = (Z^j_1,\dots,Z^j_L)$, the strong solutions to (3.2) corresponding to the controls u^j obtained in Proposition 4.1. We adopt the notation of the proof of Proposition 4.1 and let $S = (Y_1, \dots, Y_M, X_1, \dots, X_N)$. In this way, for every j, we have that

$$\begin{cases} \frac{\mathrm{d}Z^j}{\mathrm{d}t}(t) = f_{u^j}(t, Z^j(t)), \\ Z^j(0) = Z^0, \end{cases}$$

(we stress the dependence of f_{u^j} on the controls u^j) and

$$\begin{cases} \mathrm{d}S^j(t) = b_{Z^j}(t, S^j(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ S^j(0) = S^0 \quad \text{a.s.}, \end{cases}$$

(we stress the dependence of the drift vector $\mathbb{R}^{2\times L}$ on the trajectories Z^j).

Step 2. (Identifying the limit of Z^j) We remark that (4.3) yields $\|Z^j\|_\infty \le C$ for every j, where C depends on Z^0, T, K^{gg}, and $\mathcal{U}$. Let us check that the Z^js are also equicontinuous. By (4.2), for every j and for $s \le t$, we have that

$$\begin{aligned} \left|Z^j(t) - Z^j(s)\right| &\le \int_s^t \left|f_{u^j}(r, Z^j(r))\right| \mathrm{d}r \le \int_s^t \left(\left\|K^{\mathrm{gg}}\right\|_\infty + \left\|u^j\right\|_\infty + C\left\|Z^j\right\|_\infty\right) \mathrm{d}r \\ &\le \left(\left\|K^{\mathrm{gg}}\right\|_\infty + \left\|u^j\right\|_\infty + C\left\|Z^j\right\|_\infty\right)|t-s| \le C|t-s|, \end{aligned}$$

where C depends on Z^0, T, K^{gg}, and $\mathcal{U}$ (compact). By Arzelà–Ascoli's theorem, we obtain $Z^* \in C^0([0,T];\mathbb{R}^2)$ such that $\|Z^j - Z^*\|_\infty \to 0$ up to a subsequence that we do not relabel. This together with the convergence $u^j \overset{*}{\rightharpoonup} u^*$ and

$$Z^j_\ell(t) = Z^0 + \int_0^t \left(\frac{1}{L} \sum_{\ell'=1}^L K^{\mathrm{gg}}\big(Z^j_\ell(s) - Z^j_{\ell'}(s)\big) + u^j_\ell(s) \right) \mathrm{d}s$$

yields, letting $j \to +\infty$,

$$Z^*_\ell(t) = Z^0 + \int_0^t \left(\frac{1}{L} \sum_{\ell'=1}^L K^{\mathrm{gg}}\big(Z^*_\ell(s) - Z^*_{\ell'}(s)\big) + u^*_\ell(s) \right) \mathrm{d}s,$$

i.e., Z^* is the solution to

$$\begin{cases} \frac{\mathrm{d}Z^*}{\mathrm{d}t}(t) = f_{u^*}(t, Z^*(t)), \\ Z^*(0) = Z^0. \end{cases}$$

Step 3. (Identifying the limit of S^j) We let $(S^*(t))_{t\in[0,T]}$ be the $\mathbb{R}^2$-valued stochastic process obtained as the strong solution to

$$\begin{cases} \mathrm{d}S^*(t) = b_{Z^*}(t, S^*(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ S^*(0) = S^0 \quad \text{a.s.} \end{cases}$$

We claim that a.s. $\max_h \|S^j_h - S^*_h\|_\infty \to 0$ as $j \to +\infty$. We start by observing that a.s. for $0 \le s \le t \le T$ and $i = 1, \dots, M+N$

$$\begin{aligned} \left|S^j_i(s) - S^*_i(s)\right| &\le \int_0^s \left|b_{i,Z^j}\big(r, S^j(r)\big) - b_{i,Z^*}\big(r, S^*(r)\big)\right| \mathrm{d}r \\ &\le \int_0^s \left|b_{Z^j,i}\big(r, S^j(r)\big) - b_{Z^j,i}\big(r, S^*(r)\big)\right| \mathrm{d}r \\ &\quad + \int_0^s \left|b_{Z^j,i}\big(r, S^*(r)\big) - b_{Z^*,i}\big(r, S^*(r)\big)\right| \mathrm{d}r. \end{aligned} \tag{4.12}$$

We estimate the former integrand by exploiting the Lipschitz property of $b_{Z^j,i}$ obtained in (4.9) and (4.11)

$$\begin{aligned} &\left|b_{Z^j,i}\big(r, S^j(r)\big) - b_{Z^j,i}\big(r, S^*(r)\big)\right| \\ &\quad \le \max_h \Big(C\big(1 + \left|S^*_h(r)\right|\big)\left|S^j_h(r) - S^*_h(r)\right|\Big) \\ &\quad \le \max_h \Big(C\big(1 + \left\|S^*_h\right\|_\infty\big) \sup_{0 \le r \le s} \left|S^j_h(r) - S^*_h(r)\right|\Big) \quad \text{a.s.,} \end{aligned} \tag{4.13}$$

where the constant C depends on K^{pg}, K^{pc}, K^{cp}, ν^N, and $\mathbf{r}$ (independent of N). To estimate the latter integrand in (4.12), we resort to the definition of b_Z. By (4.4), for $i = 1, \dots, M$, we get that

$$\begin{aligned} \left|b_{Z^j,i}\big(r, S^*(r)\big) - b_{Z^*,i}\big(r, S^*(r)\big)\right| &\le \frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\big(S^*_i - Z^j_\ell(r)\big) - K^{\mathrm{pg}}\big(S^*_i - Z^*_\ell(r)\big)\right| \\ &\le \frac{1}{L}\sum_{\ell=1}^{L} C\left|Z^j_\ell(r) - Z^*_\ell(r)\right| \le C\left\|Z^j - Z^*\right\|_\infty \quad \text{a.s.,} \end{aligned} \tag{4.14}$$

where the constant C depends on K^{pg}. For $i = M+1, \dots, M+N$, by (4.5) we have instead that $|b_{Z^j,i}(r, S^*(r)) - b_{Z^*,i}(r, S^*(r))| = 0$. We observe that Proposition 2.1 also gives us that $\mathbb{E}(\max_h \|S^*_h\|_\infty) < C(1 + \mathbb{E}(\max_h |S^0_h|))$, thus a.s. $\max_h \|S^*_h\|_\infty < +\infty$.

We are now in a position to prove that a.s. $\max_h \|S^j_h - S^*_h\|_\infty \to 0$. For $k \ge 1$, let us consider the events

$$A_k := \Big\{\omega \in \Omega : \max_h \left\|S^*_h(\cdot, \omega)\right\|_\infty \le k\Big\}.$$

We remark that $\mathbb{P}(\bigcup_k A_k) = 1$ since a.s. $\max_h \|S^*_h\|_\infty < +\infty$. Let us fix $\omega \in A_k$ and such that (4.12)–(4.14) hold true. Then we have that

$$\begin{aligned} &\max_h \sup_{0 \le s \le t} \left|S^j_h(s, \omega) - S^*_h(s, \omega)\right| \\ &\quad \le C\Big(1 + \max_k \left\|S^*_k(\cdot, \omega)\right\|_\infty\Big) \int_0^t \max_h \sup_{0 \le r \le s} \left|S^j_h(r, \omega) - S^*_h(r, \omega)\right| \mathrm{d}s + CT\left\|Z^j - Z^*\right\|_\infty. \end{aligned}$$

Integrating on A_k, we get that

$$\int_{A_k} \max_h \sup_{0\le s\le t} \left|S_h^j(s,\omega) - S_h^*(s,\omega)\right| \,\mathrm{d}\mathbb{P}(\omega)$$
$$\le CT\left\|Z^j - Z^*\right\|_\infty + C(1+k)\int_0^t \int_{A_k} \max_h \sup_{0\le r\le s} \left|S_h^j(r,\omega) - S_h^*(r,\omega)\right| \,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}s.$$

By Grönwall's inequality, we deduce that

$$\int_{A_k} \max_h \sup_{0\le s\le t} \left|S_h^j(s,\omega) - S_h^*(s,\omega)\right| \,\mathrm{d}\mathbb{P}(\omega) \le CT\left\|Z^j - Z^*\right\|_\infty e^{C(1+k)t}$$

and, in particular,

$$\int_{A_k} \max_h \left\|S_h^j(\cdot,\omega) - S_h^*(\cdot,\omega)\right\|_\infty \,\mathrm{d}\mathbb{P}(\omega) \le CT\left\|Z^j - Z^*\right\|_\infty e^{C(1+k)T}.$$

By *Step 2* we have that $\|Z^j - Z^*\|_\infty \to 0$ as $j \to +\infty$, and thus $\max_h \|S_h^j(\cdot,\omega) - S_h^*(\cdot,\omega)\|_\infty \to 0$ for a.e. $\omega \in A_k$. Since $\mathbb{P}(\bigcup_k A_k) = 1$, we conclude that a.s. $\max_h \|S_h^j - S_h^*\|_\infty \to 0$.

Step 4. (Limit of the cost) Let us show that

$$\mathcal{J}_{N,M}(u^*) \le \liminf_{j\to+\infty} \mathcal{J}_{N,M}(u^j).$$

Since u^j is a minimizing sequence, this will be sufficient to conclude that $\mathcal{J}_{N,M}(u^*) = \min_u \mathcal{J}_{N,M}(u)$.

By sequential weak semicontinuity of the L^2-norm, we get that

$$\frac{1}{2}\int_0^T \left|u^*(t)\right| \mathrm{d}t \le \liminf_{j\to+\infty} \frac{1}{2}\int_0^T \left|u^j(t)\right| \mathrm{d}t.$$

From *Step 3* we have that a.s. $\max_h \|S_h^j - S_h^*\|_\infty \to 0$, thus a.s. $\max_m \|Y_m^j - Y_m^*\|_\infty \to 0$ and $\max_n \|X_n^j - X_n^*\|_\infty \to 0$ (recall that $S = (Y_1,\dots,Y_M,X_1,\dots,X_N)$). Then, using the fact that H^{d} is bounded, by the dominated convergence theorem

$$\mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n,m} H^{\mathrm{d}}\left(X_n^j(t) - Y_m^j(t)\right)\mathrm{d}t\right) \to \mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n,m} H^{\mathrm{d}}\left(X_n^*(t) - Y_m^*(t)\right)\mathrm{d}t\right)$$

as $j \to +\infty$. By the superadditivity of the lim inf, we conclude the proof. □

5 An averaged ODE/SDE/ODE system

5.1 Introducing the averaged ODE/SDE/ODE system

To study the mean-field limit of (3.2) as $M \to +\infty$, we consider an averaged ODE/SDE/ODE system, where the trajectories $Y_m(t)$ are replaced by a single trajectory $\bar{Y}(t)$ interacting with the other agents via its probability distribution. More precisely, let $(W(t))_{t\in[0,T]}$

be an $\mathbb{R}^2$-valued Brownian motion and consider the problem

$$\begin{cases} \frac{\mathrm{d}\bar{X}_n}{\mathrm{d}t}(t) = v_n^N(\bar{X}(t))(\mathbf{r}(\bar{X}_n(t)) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(t)(\bar{X}_n(t))), \\ \mathrm{d}\bar{Y}(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\bar{Y}(t) - Z_\ell(t)) \\ \quad - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}(\bar{Y}(t) - \bar{X}_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \bar{X}_n(0) = X_n^0, \qquad Z_\ell(0) = Z_\ell^0, \quad n = 1, \dots, N, \ell = 1, \dots, L, \\ \bar{Y}(0) = \bar{Y}^0 \quad \text{a.s.}, \qquad \bar{\mu}^{\mathrm{p}} = \mathrm{Law}(\bar{Y}). \end{cases} \tag{5.1}$$

We start by giving a precise definition for the notion of solutions to the previous system.

Definition 5.1 A *strong solution* to (5.1) is given by a curve $\bar{X} = (\bar{X}_1, \dots, \bar{X}_N) \in C^0([0,T]; \mathbb{R}^{2\times N})$, an $\mathbb{R}^2$-valued stochastic process $(\bar{Y}(t))_{t\in[0,T]}$ a.s. with continuous paths, and a curve $Z = (Z_1, \dots, Z_L) \in C^0([0,T]; \mathbb{R}^{2\times L})$ such that

(1) a.s. for all $t \in [0,T]$,

$$\bar{Y}(t) = \bar{Y}^0 + \int_0^t \left(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(\bar{Y}(s) - Z_\ell(s)\big) - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}\big(\bar{Y}(s) - \bar{X}_n(s)\big) \right) \mathrm{d}t \\ + \sqrt{2\kappa}\, W(t);$$

(2) setting $\bar{\mu}^{\mathrm{p}} := \mathrm{Law}(\bar{Y}) \in \mathcal{P}(C^0([0,T];\mathbb{R}^2))$, the curves $\bar{X}$ and Z satisfy

$$\bar{X}_n(t) = X_n^0 + \int_0^t v_n^N\big(\bar{X}(s)\big)\big(\mathbf{r}\big(\bar{X}_n(s)\big) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(t)\big(\bar{X}_n(s)\big)\big)\,\mathrm{d}s$$

and

$$Z_\ell(t) = Z_\ell^0 + \int_0^t \left(\frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}\big(Z_\ell(s) - Z_{\ell'}(s)\big) + u_\ell(s) \right) \mathrm{d}s$$

for all $t \in [0,T]$.

5.2 Well-posedness of the averaged ODE/SDE/ODE system

Let us prove the following well-posedness result.

Proposition 5.2 *Assume the following*:

- *Let* $(W(t))_{t\in[0,T]}$ *be a Brownian motion*;
- *Let* $X^0 = (X_1^0, \dots, X_N^0) \in \mathbb{R}^{2\times N}$;
- *Let* $\bar{Y}^0$ *be a random variable with* $\mathbb{E}(|\bar{Y}^0|) < +\infty$;
- *Let* $Z^0 = (Z_1^0, \dots, Z_L^0) \in \mathbb{R}^{2\times L}$;
- *Let* $u \in L^\infty([0,T];\mathcal{U})$.

Then there exists a unique strong solution to (5.1). *Moreover,* $\mathbb{E}(\|\bar{Y}\|_\infty) < +\infty$ *and* $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$.

Proof As recalled in the proof of Proposition 4.1, for every control $u = (u_1,\dots,u_L) \in L^\infty([0,T];\mathcal{U})$, there exists a unique continuous solution to

$$Z_\ell(t) = Z_\ell^0 + \int_0^t \left(\frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}\big(Z_\ell(s) - Z_{\ell'}(s)\big) + u_\ell(s) \right) \mathrm{d}s, \quad t \in [0,T], \tag{5.2}$$

hence $Z_\ell(t)$ will be treated as fixed in the following.

The proof now mainly follows the lines of [9, Theorem 3.1]. For the sake of brevity, we let $C^0 = C^0([0,T];\mathbb{R}^2)$.

<u>*Step 1*</u>. (Decoupling the system) Let us fix $\mu \in \mathcal{P}_1(C^0)$ (μ plays the role of $\bar{\mu}^{\mathrm{p}}$ in the equation and is used to apply a fixed point argument). Let us consider the decoupled system

$$\begin{cases} \frac{\mathrm{d}\widetilde{X}_n}{\mathrm{d}t}(t) = v_n^N(\widetilde{X}(t))(\mathbf{r}(\widetilde{X}_n(t)) + K^{\mathrm{cp}} * \mu(t)(\widetilde{X}_n(t))), \\ \widetilde{X}_n(0) = X_n^0, \quad n = 1,\dots,N, \end{cases} \tag{5.3}$$

$$\begin{cases} \mathrm{d}\widetilde{Y}(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\widetilde{Y}(t) - Z_\ell(t)) \\ \qquad - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}(\widetilde{Y}(t) - \widetilde{X}_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \widetilde{Y}(0) = \bar{Y}^0 \quad \text{a.s.}, \end{cases} \tag{5.4}$$

where the $Z_\ell(t)$ are obtained in (5.2).

Substep 1.1. We start by commenting about the existence (and uniqueness) of continuous curves $\widetilde{X} = (\widetilde{X}_1,\dots,\widetilde{X}_N) \in C^0([0,T];\mathbb{R}^{2\times N})$ solutions to (5.3). For this, we need to check the conditions for well-posedness of ODE systems. Let us consider the function $g_\mu = (g_{\mu,1},\dots,g_{\mu,N})\colon [0,T]\times\mathbb{R}^{2\times N} \to \mathbb{R}^{2\times N}$ defined by

$$g_{\mu,n}(t,X) := v_n^N(X)\big(\mathbf{r}(X_n) + K^{\mathrm{cp}} * \mu(t)(X_n)\big) \tag{5.5}$$

for $n = 1,\dots,N$. The system then reads

$$\begin{cases} \frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t}(t) = g_\mu(t,\widetilde{X}(t)), \\ \widetilde{X}_n(0) = X_n^0, \quad n = 1,\dots,N. \end{cases} \tag{5.6}$$

The dependence of g_μ on the time variable t is only due to the terms

$$\begin{aligned} K^{\mathrm{cp}} * \mu(t)(X_n) &= \int_{\mathbb{R}^2} K^{\mathrm{cp}}(X_n - x)\,\mathrm{d}\mu(t)(x) = \int_{\mathbb{R}^2} K^{\mathrm{cp}}(X_n - x)\,\mathrm{d}\big((\mathrm{ev}_t)_\#\mu\big)(x) \\ &= \int_{C^0} K^{\mathrm{cp}}\big(X_n - \mathrm{ev}_t(\varphi)\big)\,\mathrm{d}\mu(\varphi) = \int_{C^0} K^{\mathrm{cp}}\big(X_n - \varphi(t)\big)\,\mathrm{d}\mu(\varphi), \end{aligned}$$

which are continuous in t. This follows from, *e.g.*, the dominated convergence theorem by observing that the Lipschitz continuity of K^{cp} yields

$$\big|K^{\mathrm{cp}}\big(X_n - \varphi(t)\big)\big| \le \big|K^{\mathrm{cp}}(0)\big| + C\big|X_n - \varphi(t)\big| \le C\big(1 + |X_n| + \|\varphi\|_\infty\big)$$

and $\int_{C^0} \|\varphi\|_\infty\,\mathrm{d}\mu(\varphi) < +\infty$ since $\mu \in \mathcal{P}_1(C^0)$. The functions $g_{\mu,n}$ are locally Lipschitz in X, *i.e.*, given $R > 0$, there exists $C_R > 0$ such that for $t \in [0,T]$ and $\max_n |X_n| \le R$, $\max_n |X'_n| \le R$

it holds that

$$\max_n \bigl|g_{\mu,n}(t,X) - g_{\mu,n}\bigl(t,X'\bigr)\bigr| \le C_R \max_n \bigl|X_n - X'_n\bigr|. \tag{5.7}$$

The computations are analogous to those in (4.11), the only difference being in the term

$$\begin{aligned}\bigl|K^{\mathrm{cp}} * \mu(t)(X_n) - K^{\mathrm{cp}} * \mu(t)\bigl(X'_n\bigr)\bigr| &\le \int_{\mathbb{R}^2} \bigl|K^{\mathrm{cp}}(X_n - x) - K^{\mathrm{cp}}\bigl(X'_n - x\bigr)\bigr|\,\mathrm{d}\mu(t)(x) \\ &\le \int_{\mathbb{R}^2} C\bigl|X_n - X'_n\bigr|\,\mathrm{d}\mu(t)(x) \le C\bigl|X_n - X'_n\bigr|.\end{aligned}$$

In conclusion, $g_\mu(t,X)$ is continuous in t and locally Lipschitz in X with respect to the max norm. By Picard–Lindelhöf's theorem, the ODE system (5.6) admits a unique solution for small times. For the existence for all times, with computation analogous to those in (4.10), we observe that we have linear growth for g_μ, *i.e.*,

$$\max_n \bigl|g_{\mu,n}(t,X)\bigr| \le C\Bigl(1 + \max_n |X_n|\Bigr),$$

the constant C above depending on $\|\nu^N\|_\infty$, $\mathbf{r}$, and K^{cp}. This upper bound allows for a Grönwall inequality. Indeed,

$$\begin{aligned}\bigl|\widetilde{X}_n(t)\bigr| &\le \bigl|X_n^0\bigr| + \int_0^t \biggl|\frac{\mathrm{d}\widetilde{X}_n}{\mathrm{d}t}(s)\biggr|\,\mathrm{d}s = \bigl|X_n^0\bigr| + \int_0^t \bigl|g_{\mu,n}\bigl(s,\widetilde{X}(s)\bigr)\bigr|\,\mathrm{d}s \\ &\le \max_{n'}\bigl|X_{n'}^0\bigr| + \int_0^t C\Bigl(1 + \max_{n'}|X_{n'}|\Bigr)\,\mathrm{d}s = \max_{n'}\bigl|X_{n'}^0\bigr| + CT + C\int_0^t \max_{n'}|X_{n'}|\,\mathrm{d}s,\end{aligned}$$

which yields

$$\max_n \bigl|\widetilde{X}_n(t)\bigr| \le C\Bigl(\max_n \bigl|X_n^0\bigr| + T\Bigr)e^{Ct} \quad \text{for all } t \in [0,T], \tag{5.8}$$

and, in particular, the boundedness of solutions in terms of the initial datum X^0 and final time T (in addition to $\|\nu^N\|_\infty$, $\mathbf{r}$, and K^{cp}). This is enough to deduce global existence in time.

Substep 1.2. Given the continuous curves $\widetilde{X}$ and Z obtained previously, we consider SDE (5.4). We rewrite this SDE by introducing the drift vector $b_{\widetilde{X}}\colon [0,T]\times\mathbb{R}^2 \to \mathbb{R}^2$ (depending on $\widetilde{X}$)

$$b_{\widetilde{X}}(t,Y) := \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\bigl(Y - Z_\ell(t)\bigr) - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}\bigl(Y - \widetilde{X}_n(t)\bigr) \tag{5.9}$$

and by considering the constant dispersion matrix $\sigma = \sqrt{2\kappa}\,\mathrm{Id}_2$, so that the SDE reads

$$\begin{cases} \mathrm{d}\widetilde{Y}(t) = b_{\widetilde{X}}(t,\widetilde{Y}(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ \widetilde{Y}(0) = \bar{Y}^0 \quad \text{a.s.} \end{cases} \tag{5.10}$$

For the existence and uniqueness of a strong solution to this SDE, we check that the assumptions of Proposition 2.1 are satisfied. The drift $b_{\widetilde{X}}$ is continuous in t: it follows from the continuity of the curves $\widetilde{X}_n$ and Z_ℓ. The drift $b_{\widetilde{X}}$ is globally Lipschitz continuous in Y. Indeed, we have that

$$\begin{aligned} &\left|b_{\widetilde{X}}(t,Y)-b_{\widetilde{X}}\left(t,Y'\right)\right| \\ &\quad\le \frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\left(Y-Z_\ell(t)\right)-K^{\mathrm{pg}}\left(Y'-Z_\ell(t)\right)\right| \\ &\qquad+\frac{1}{N}\sum_{n=1}^{N}\left|K^{\mathrm{pc}}\left(Y-\widetilde{X}_n(t)\right)-K^{\mathrm{pc}}\left(Y'-\widetilde{X}_n(t)\right)\right|\le C\left|Y-Y'\right|, \end{aligned} \tag{5.11}$$

the constant C only depending on the Lipschitz constants of K^{pg} and K^{pc}. Finally, $b_{\widetilde{X}}$ satisfies the linear growth condition. This follows from (4.7) and the analogous condition for K^{pc}, which yield

$$\begin{aligned} \left|b_{\widetilde{X}}(t,Y)\right| &\le \frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\left(Y-Z_\ell(t)\right)\right|+\frac{1}{N}\sum_{n=1}^{N}\left|K^{\mathrm{pc}}\left(Y-\widetilde{X}_n(t)\right)\right| \\ &\le \left|K^{\mathrm{pg}}(0)\right|+\frac{1}{L}\sum_{\ell=1}^{L}C\left|Y-Z_\ell(t)\right|+\left|K^{\mathrm{pc}}(0)\right|+\frac{1}{N}\sum_{n=1}^{N}C\left|Y-\widetilde{X}_n(t)\right| \\ &\le \left|K^{\mathrm{pg}}(0)\right|+\left|K^{\mathrm{pc}}(0)\right|+\|Z\|_\infty+\max_n\|\widetilde{X}_n\|_\infty+|Y|\le C\left(1+|Y|\right), \end{aligned}$$

where the constant C depends on K^{pg}, K^{pc}, $\|Z\|_\infty$, and $\max_n\|\widetilde{X}_n\|_\infty$ and we used the boundedness of $\widetilde{X}$ obtained in (5.8).

We are in a position to apply Proposition 2.1, which also gives us that

$$\mathbb{E}\left(\|\widetilde{Y}\|_\infty\right)\le C\left(1+\mathbb{E}\left(\left|\bar{Y}^0\right|\right)\right). \tag{5.12}$$

This implies that $\mathrm{Law}(\widetilde{Y})\in\mathcal{P}_1(C^0)$. Indeed,

$$\begin{aligned} \int_{C^0}\|\varphi\|_\infty\,\mathrm{dLaw}(\widetilde{Y})(\varphi) &= \int_{C^0}\|\varphi\|_\infty\,\mathrm{d}(\widetilde{Y}_\#\mathbb{P})(\varphi)=\int_\Omega\left\|\widetilde{Y}(\cdot,\omega)\right\|_\infty\mathrm{d}\mathbb{P}(\omega) \\ &= \mathbb{E}\left(\|\widetilde{Y}\|_\infty\right)<+\infty. \end{aligned}$$

Step 2. (Fixed-point argument) Let us implement the machinery to carry out a fixed point argument.

Substep 2.1. (Definition of Picard operator) We consider the functional $\mathcal{L}\colon\mathcal{P}_1(C^0)\to\mathcal{P}_1(C^0)$ defined as follows: given $\mu\in\mathcal{P}_1(C^0)$, we let $\widetilde{X}=(\widetilde{X}_1,\dots,\widetilde{X}_N)$ and $(\widetilde{Y}(t))_{t\in[0,T]}$ be the unique solution to (5.3)–(5.4) obtained as explained in the previous step. Then we set $\mathcal{L}(\mu):=\mathrm{Law}(\widetilde{Y})$, which belongs to $\mathcal{P}_1(C^0)$, as explained in the previous step. We shall show that $\mathcal{L}$ is a contraction with respect to a suitable auxiliary distance on $\mathcal{P}_1(C^0)$ to deduce the existence of a fixed point.

Substep 2.2. (Definition of equivalent Wasserstein distance) The auxiliary distance we consider on $\mathcal{P}_1(C^0)$ is defined as follows. We let $\alpha>0$ (its choice is made precise later in (5.22)) and we define on C^0 the norm

$$|||\varphi|||_\alpha := \sup_{t\in[0,T]}\left(e^{-\alpha t}\left|\varphi(t)\right|\right). \tag{5.13}$$

Then we define the auxiliary distance on $\mathcal{P}_1(C^0)$ by

$$\mathcal{W}_{1,\alpha}(\mu_1,\mu_2) := \inf_\gamma \int_{C^0\times C^0} |||\varphi-\psi|||_\alpha \, \mathrm{d}\gamma(\varphi,\psi),$$

where the infimum is taken over all transport plans $\gamma \in \mathcal{P}(C^0\times C^0)$ with marginals $\pi^1_\#\gamma = \mu_1$ and $\pi^2_\#\gamma = \mu_2$, where π^i is the projection on the ith component. Since the norm $|||\cdot|||_\alpha$ is equivalent to the usual uniform norm $\|\cdot\|_\infty$ on C^0, the distance $\mathcal{W}_{1,\alpha}$ is equivalent to the usual 1-Wasserstein distance $\mathcal{W}_1$ on $\mathcal{P}_1(C^0)$.

Substep 2.3. (Start of the proof of the contraction property) Given $\mu,\mu' \in \mathcal{P}_1(C^0)$, let us estimate $\mathcal{W}_{1,\alpha}(\mathcal{L}(\mu),\mathcal{L}(\mu'))$. Let $\widetilde{X} = (\widetilde{X}_1,\dots,\widetilde{X}_N)$, $\widetilde{Y}$ and $\widetilde{X}' = (\widetilde{X}'_1,\dots,\widetilde{X}'_N)$, $\widetilde{Y}'$ be solutions obtained in *Step 1* corresponding to μ and μ', respectively. By Kantorovich's duality, there exists a functional $\Psi\colon C^0 \to C^0$ Lipschitz continuous with respect to $|||\cdot|||_\alpha$ with Lipschitz constant 1 such that, using the fact that $\mathcal{L}(\mu) = \mathrm{Law}(\widetilde{Y})$ and $\mathcal{L}(\mu') = \mathrm{Law}(\widetilde{Y}')$,

$$\begin{aligned}\mathcal{W}_{1,\alpha}\bigl(\mathcal{L}(\mu),\mathcal{L}(\mu')\bigr) &= \int_{C^0}\Psi(\varphi)\mathcal{L}(\mu)(\varphi) - \int_{C^0}\Psi(\varphi')\mathcal{L}(\mu')(\varphi') \\ &= \mathbb{E}\bigl(\Psi(\widetilde{Y}) - \Psi(\widetilde{Y}')\bigr) \le \mathbb{E}\bigl(|||\widetilde{Y}-\widetilde{Y}'|||_\alpha\bigr).\end{aligned} \tag{5.14}$$

The following substeps show how to estimate the term $\mathbb{E}(|||\widetilde{Y}-\widetilde{Y}'|||_\alpha)$.

Substep 2.4. (Estimate of $|\widetilde{Y}(t)-\widetilde{Y}'(t)|$) We start by observing that from (5.10), from the definition of $b_{\widetilde{X}}$ in (5.9), by the Lipschitz continuity of K^{pc}, and by the Lipschitz continuity of $b_{\widetilde{X}'}$ obtained in (5.11), we have that a.s.

$$\begin{aligned}&\bigl|\widetilde{Y}(t)-\widetilde{Y}'(t)\bigr| \\ &\quad= \left|\int_0^t b_{\widetilde{X}}\bigl(s,\widetilde{Y}(s)\bigr)\,\mathrm{d}s - \int_0^t \widetilde{b}_{\widetilde{X}'}\bigl(s,\widetilde{Y}'(s)\bigr)\,\mathrm{d}s\right| \\ &\quad\le \int_0^t \Bigl(\bigl|b_{\widetilde{X}}\bigl(s,\widetilde{Y}(s)\bigr) - b_{\widetilde{X}'}\bigl(s,\widetilde{Y}(s)\bigr)\bigr| + \bigl|b_{\widetilde{X}'}\bigl(s,\widetilde{Y}(s)\bigr) - b_{\widetilde{X}'}\bigl(s,\widetilde{Y}'(s)\bigr)\bigr|\Bigr)\,\mathrm{d}s \\ &\quad\le \int_0^t \left(\frac{1}{N}\sum_{n=1}^N \bigl|K^{\mathrm{pc}}\bigl(\widetilde{Y}(s)-\widetilde{X}_n(s)\bigr) - K^{\mathrm{pc}}\bigl(\widetilde{Y}(s)-\widetilde{X}'_n(s)\bigr)\bigr| + C\bigl|\widetilde{Y}(s)-\widetilde{Y}'(s)\bigr|\right)\mathrm{d}s \\ &\quad\le \int_0^t C\Bigl(\max_n\bigl|\widetilde{X}_n(s)-\widetilde{X}'_n(s)\bigr| + \bigl|\widetilde{Y}(s)-\widetilde{Y}'(s)\bigr|\Bigr)\,\mathrm{d}s,\end{aligned} \tag{5.15}$$

the constant C depending only on the Lipschitz constants of K^{pg} and K^{pc}.

Substep 2.5. (Estimate of $|\widetilde{X}_n(s)-\widetilde{X}'_n(s)|$) The curves $\widetilde{X}$ and $\widetilde{X}'$ are solutions to (5.6). As obtained in (5.8), they are bounded by a constant $R>0$ depending on the initial datum X^0, the final time T, and the parameters of the problem ($\|v\|_\infty$, $\mathbf{r}$, and K^{cp}), *i.e.*, $\max_n\|\widetilde{X}_n\|_\infty \le R$, $\max_n\|\widetilde{X}'_n\|_\infty \le R$. We recall that g_μ and $g_{\mu'}$ are locally Lipschitz, hence there exists $C>0$ (depending on R) such that (5.7) is satisfied. It follows that for $n=1,\dots,N$

$$\begin{aligned}&\bigl|\widetilde{X}_n(s)-\widetilde{X}'_n(s)\bigr| \\ &\quad\le \int_0^s \bigl|g_{\mu,n}\bigl(r,\widetilde{X}(r)\bigr) - g_{\mu',n}\bigl(r,\widetilde{X}'(r)\bigr)\bigr|\,\mathrm{d}r \\ &\quad\le \int_0^s \Bigl(\bigl|g_{\mu,n}\bigl(r,\widetilde{X}(r)\bigr) - g_{\mu,n}\bigl(r,\widetilde{X}'(r)\bigr)\bigr| + \bigl|g_{\mu,n}\bigl(r,\widetilde{X}'(r)\bigr) - g_{\mu',n}\bigl(r,\widetilde{X}'(r)\bigr)\bigr|\Bigr)\,\mathrm{d}r\end{aligned} \tag{5.16}$$

$$\leq \int_0^s \Big(C \max_{n'} \big| \widetilde{X}_{n'}(r) - \widetilde{X}'_{n'}(r) \big| + \big| g_{\mu,n}\big(r, \widetilde{X}'(r)\big) - g_{\mu',n}\big(r, \widetilde{X}'(r)\big) \big| \Big) \, \mathrm{d}r.$$

Let us now apply the definition of g_μ and $g_{\mu'}$ in (5.5) to estimate for $n = 1, \dots, N$ and $r \in [0, s]$

$$\begin{aligned} \big| g_{\mu,n}\big(r, \widetilde{X}'(r)\big) - g_{\mu',n}\big(r, \widetilde{X}'(r)\big) \big| &\leq \|\nu\|_\infty \big| K^{\mathrm{cp}} * \mu(r)\big(\widetilde{X}'_n(r)\big) - K^{\mathrm{cp}} * \mu'(r)\big(\widetilde{X}'_n(r)\big) \big| \\ &\leq C \left| \int_{\mathbb{R}^2} K^{\mathrm{cp}}\big(\widetilde{X}'_n(r) - x\big) \, \mathrm{d}\big(\mu(r) - \mu'(r)\big)(x) \right|, \end{aligned} \tag{5.17}$$

where the constant C depends on $\|\nu\|_\infty$. We observe that by the Lipschitz continuity of $x \mapsto K^{\mathrm{cp}}(\widetilde{X}'_n(r) - x)$ and by Kantorovich's duality,

$$\left| \int_{\mathbb{R}^2} K^{\mathrm{cp}}\big(\widetilde{X}'_n(r) - x\big) \, \mathrm{d}\big(\mu(r) - \mu'(r)\big)(x) \right| \leq C \mathcal{W}_1\big(\mu(r), \mu'(r)\big), \tag{5.18}$$

where C is the Lipschitz constant of K^{cp}. To bound this term, let us fix an optimal plan $\gamma \in \mathcal{P}(C^0 \times C^0)$ with marginals $\pi^1_\# \gamma = \mu$, $\pi^2_\# \gamma = \mu'$ and satisfying

$$\mathcal{W}_{1,\alpha}\big(\mu, \mu'\big) = \int_{C^0 \times C^0} |\!|\!| \varphi - \psi |\!|\!|_\alpha \, \mathrm{d}\gamma(\varphi, \psi).$$

We remark that $\gamma(r) = (\mathrm{ev}_r)_\# \gamma \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ has marginals $\pi^1_\# (\mathrm{ev}_r)_\# \gamma = (\mathrm{ev}_r)_\# \pi^1_\# \gamma = \mu(r)$ and $\pi^2_\# (\mathrm{ev}_r)_\# \gamma = (\mathrm{ev}_r)_\# \pi^2_\# \gamma = \mu'(r)$, hence, by the optimality of $\mathcal{W}_1$ and by the definition of $|\!|\!| \cdot |\!|\!|_\alpha$ in (5.13), we obtain for $r \in [0, s]$

$$\begin{aligned} &\mathcal{W}_1\big(\mu(r), \mu'(r)\big) \\ &\quad \leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \big| x - x' \big| \, \mathrm{d}\gamma(r)\big(x, x'\big) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \big| x - x' \big| \, \mathrm{d}(\mathrm{ev}_r)_\# \gamma\big(x, x'\big) \\ &\quad = \int_{C^0 \times C^0} \big| \varphi(r) - \psi(r) \big| \, \mathrm{d}\gamma(\varphi, \psi) \leq e^{\alpha r} \int_{C^0 \times C^0} e^{-\alpha r} \big| \varphi(r) - \psi(r) \big| \, \mathrm{d}\gamma(\varphi, \psi) \\ &\quad \leq e^{\alpha r} \int_{C^0 \times C^0} |\!|\!| \varphi - \psi |\!|\!|_\alpha \, \mathrm{d}\gamma(\varphi, \psi) = e^{\alpha r} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big). \end{aligned}$$

Integrating in r, we get that

$$\int_0^s \mathcal{W}_1\big(\mu(r), \mu'(r)\big) \, \mathrm{d}r \leq \frac{e^{\alpha s} - 1}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big) \leq \frac{e^{\alpha s}}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big). \tag{5.19}$$

Putting together (5.16)–(5.19), we conclude that

$$\max_n \big| \widetilde{X}_n(s) - \widetilde{X}'_n(s) \big| \leq C \left(\int_0^s \max_n \big| \widetilde{X}_n(r) - \widetilde{X}'_n(r) \big| \, \mathrm{d}r + \frac{e^{\alpha s}}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big) \right) \mathrm{d}r.$$

By Grönwall's inequality, we conclude that

$$\begin{aligned} \max_n \big| \widetilde{X}_n(s) - \widetilde{X}'_n(s) \big| &\leq C \frac{e^{\alpha s}}{\alpha} e^{Cs} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big) \leq C e^{CT} \frac{e^{\alpha s}}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big) \\ &\leq C \frac{e^{\alpha s}}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu, \mu'\big). \end{aligned} \tag{5.20}$$

To sum up, the constant C in the previous formula depends on X^0, T, $\|\nu\|_\infty$, $\mathbf{r}$, and K^{cp}.

Substep 2.6. (Concluding the estimate of $|\widetilde{Y}(t) - \widetilde{Y}'(t)|$) Substituting (5.20) in (5.15), we obtain that

$$\begin{aligned}\big|\widetilde{Y}(s) - \widetilde{Y}'(s)\big| &\leq C \int_0^s \bigg(\frac{e^{\alpha r}}{\alpha} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \bigg) \,\mathrm{d}r \\ &\leq C \bigg(\frac{e^{\alpha s} - 1}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \int_0^s \big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \,\mathrm{d}r \bigg) \\ &\leq C \bigg(\frac{e^{\alpha s}}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \int_0^s \big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \,\mathrm{d}r \bigg).\end{aligned}$$

Multiplying both sides by $e^{-\alpha s}$ and using that $e^{-\alpha s} \leq e^{-\alpha r}$, we get that a.s. for $s \in [0,t]$

$$\begin{aligned}e^{-\alpha s}\big|\widetilde{Y}(s) - \widetilde{Y}'(s)\big| &\leq C \bigg(\frac{1}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \int_0^s e^{-\alpha r}\big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \,\mathrm{d}r \bigg) \\ &\leq C \bigg(\frac{1}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \int_0^t \sup_{0\leq r\leq s} e^{-\alpha r}\big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \,\mathrm{d}s \bigg).\end{aligned}$$

Taking the supremum for $s \in [0,t]$ and the expectation, we deduce that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leq s\leq t} e^{-\alpha s}\big|\widetilde{Y}(s) - \widetilde{Y}'(s)\big| \Big) \\ &\quad\leq C \bigg(\frac{1}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) + \int_0^t \mathbb{E}\Big(\sup_{0\leq r\leq s} e^{-\alpha r}\big|\widetilde{Y}(r) - \widetilde{Y}'(r)\big| \Big) \,\mathrm{d}s \bigg),\end{aligned}$$

and thus, by Grönwall's inequality,

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} e^{-\alpha s}\big|\widetilde{Y}(s) - \widetilde{Y}'(s)\big| \Big) \leq \frac{C}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) e^{Ct},$$

which for $t = T$ yields

$$\mathbb{E}\big(\big|\big|\big|\widetilde{Y} - \widetilde{Y}'\big|\big|\big|_\alpha\big) \leq \frac{C}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big) e^{CT} \leq \frac{C}{\alpha^2} \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big). \tag{5.21}$$

Keeping track of the constant C, it depends on X^0, T, $\|\nu\|_\infty$, $\mathbf{r}$, K^{cp}, K^{pg}, and K^{pc}.

Substep 2.7. (Choice of α and end of the proof of the contraction property) We choose $\alpha > 0$ in such a way that

$$C_\alpha := \frac{C}{\alpha^2} < 1, \tag{5.22}$$

where C is the constant obtained in (5.21). In this way, by (5.14) and (5.21) we conclude that

$$\mathcal{W}_{1,\alpha}\big(\mathcal{L}(\mu), \mathcal{L}\big(\mu'\big)\big) \leq C_\alpha \mathcal{W}_{1,\alpha}\big(\mu,\mu'\big),$$

i.e., $\mathcal{L}\colon \mathcal{P}_1(C^0) \to \mathcal{P}_1(C^0)$ is a contraction with respect to the equivalent Wasserstein distance $\mathcal{W}_{1,\alpha}$. As such, it admits a unique fixed point $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0)$.

Step 3. Given the fixed point $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0)$ of $\mathcal{L}$, we define $\bar{X} = (\bar{X}_1, \dots, \bar{X}_N)$ as the solution to (5.3) corresponding to $\bar{\mu}^{\mathrm{p}}$, and then we let $\bar{Y}$ be the solution to (5.4) corresponding to $\bar{X}$. Since $\bar{\mu}^{\mathrm{p}}$ is a fixed point, we have that $\mathcal{L}(\bar{\mu}^{\mathrm{p}}) = \bar{\mu}^{\mathrm{p}}$, *i.e.*, $\mathrm{Law}(\bar{Y}) = \bar{\mu}^{\mathrm{p}}$. Hence we found the unique strong solution to the coupled system. This concludes the proof. □

Remark 5.3 By (5.8), it follows that $\max_n \|\bar{X}_n\|_\infty$ is bounded by a constant depending on the initial datum X^0, the final time T, $\|\nu^N\|_\infty$, $\mathbf{r}$, and K^{cp}.

By (5.12), it follows that $\mathbb{E}(\|\bar{Y}\|_\infty) \le C(1 + \mathbb{E}(|\bar{Y}^0|))$, where the constant C depends on K^{pg}, K^{pc}, $\|Z\|_\infty$, $\max_n \|\bar{X}_n\|_\infty$, T, and W.

6 Propagation of chaos

Proposition 6.1 *Assume the following*:

- *Let* $(W_1(t))_{t\in[0,T]}$ *and* $(W_2(t))_{t\in[0,T]}$ *be two* $\mathbb{R}^2$*-valued Brownian motions.*
- *Let* $X^0 = (X_1^0, \dots, X_N^0) \in \mathbb{R}^{2\times N}$;
- *Let* Y_1^0, Y_2^0 *be identically distributed* $\mathbb{R}^2$*-valued random variables with* $\mathbb{E}(|Y_m^0|) < +\infty$;
- *Let* $Z^0 = (Z_1^0, \dots, Z_L^0) \in \mathbb{R}^{2\times L}$;
- *Let* $u \in L^\infty([0,T]; \mathcal{U})$.

For $m = 1, 2$, *let* $\bar{X}_m = (\bar{X}_{m,1}, \dots, \bar{X}_{m,N})$, $(\bar{Y}_m(t))_{t\in[0,T]}$, $Z = (Z_1, \dots, Z_L)$ *be the unique strong solution to*[8]

$$\begin{cases} \frac{\mathrm{d}\bar{X}_{m,n}}{\mathrm{d}t}(t) = \nu_n^N(\bar{X}_m(t))(\mathbf{r}(\bar{X}_{m,n}(t)) + K^{\mathrm{cp}} * \bar{\mu}_m^{\mathrm{p}}(t)(\bar{X}_{m,n}(t))), \\ \mathrm{d}\bar{Y}_m(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_m(t) - Z_\ell(t)) \\ \qquad\qquad - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\bar{Y}_m(t) - \bar{X}_{m,n}(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_m(t), \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \bar{X}_{m,n}(0) = X_n^0, \qquad Z_\ell(0) = Z_\ell^0, \quad n = 1, \dots, N, \ell = 1, \dots, L, \\ \bar{Y}_m(0) = Y_m^0 \quad a.s., \qquad \bar{\mu}_m^{\mathrm{p}} = \mathrm{Law}(\bar{Y}_m). \end{cases} \tag{6.1}$$

Then the stochastic processes $(\bar{Y}_1(t))_{t\in[0,T]}$ *and* $(\bar{Y}_2(t))_{t\in[0,T]}$ *are identically distributed and* $\bar{X}_1(t) = \bar{X}_2(t)$ *for* $t \in [0,T]$.

Proof We fix $Z = (Z_1, \dots, Z_L)$ as the solution to

$$\begin{cases} \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1, \dots, L, \end{cases}$$

which is independent of m since it is decoupled from the first two sets of equations.

Let $m \in \{1, 2\}$. We resort to some tools already considered in *Step 2* in the proof of Proposition 5.2. As in that proof, we set $C^0 := C^0([0,T]; \mathbb{R}^2)$.

Step 1. (Exploiting the decoupled system) Given $\mu \in \mathcal{P}_1(C^0)$, we let $\widetilde{X} = (\widetilde{X}_1, \dots, \widetilde{X}_N)$ and $(\widetilde{Y}_m(t))_{t\in[0,T]}$ be the unique solution to the decoupled system

$$\begin{cases} \frac{\mathrm{d}\widetilde{X}_n}{\mathrm{d}t}(t) = \nu_n^N(\widetilde{X}(t))(\mathbf{r}(\widetilde{X}_n(t)) + K^{\mathrm{cp}} * \mu(t)(\widetilde{X}_n(t))), \\ \widetilde{X}_n(0) = X_n^0, \quad n = 1, \dots, N, \end{cases} \tag{6.2}$$

[8] This corresponds to the averaged ODE/SDE/ODE system (5.1) with initial data X^0, Y_m^0, Z^0, with Brownian motion W_m, and with control u provided by Proposition 5.2.

$$\begin{cases} \mathrm{d}\widetilde{Y}_m(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\widetilde{Y}_m(t) - Z_\ell(t)) \\ \qquad\qquad - \frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}(\widetilde{Y}_m(t) - \widetilde{X}_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_m(t), \\ \widetilde{Y}_m(0) = Y_m^0 \quad \text{a.s.}, \end{cases} \tag{6.3}$$

obtained as explained in *Step 1* in the proof of Proposition 5.2. We claim that

$$\mathrm{Law}(\widetilde{Y}_1) = \mathrm{Law}(\widetilde{Y}_2). \tag{6.4}$$

Using the short-hand notation introduced in (5.9), we have a.s. for $t \in [0,T]$

$$\widetilde{Y}_m(t) = Y_m^0 + \int_0^t b_{\widetilde{X}}\big(s, \widetilde{Y}_m(s)\big)\,\mathrm{d}s + \sqrt{2\kappa}\,W_m(t).$$

Substep 1.1. (Proof of claim (6.4) for Picard iterations $\widetilde{Y}_m^j$) We consider the Picard iterations (used in the proof of Proposition 2.1) constructed as follows for $m = 1,2$: for $\omega \in \Omega$,

$$\widetilde{Y}_m^0(t,\omega) := Y_m^0(\omega) \quad \text{for } t \in [0,T], \tag{6.5}$$

$$\widetilde{Y}_m^j(t,\omega) := Y_m^0(\omega) + \int_0^t b_{\widetilde{X}}\big(s, \widetilde{Y}_m^{j-1}(s,\omega)\big)\,\mathrm{d}s + \sqrt{2\kappa}\,W_m(t,\omega) \quad \text{for } t \in [0,T], j \geq 1. \tag{6.6}$$

We observe that $\mathrm{Law}(\widetilde{Y}_1^0) = \mathrm{Law}(\widetilde{Y}_2^0)$, as by (6.5) they coincide with the common law of the identically distributed random variables given by the initial data Y_1^0, Y_2^0. This is the base step of an induction argument. Let $j \geq 1$ and assume $\mathrm{Law}(\widetilde{Y}_1^{j-1}) = \mathrm{Law}(\widetilde{Y}_2^{j-1})$. Let $\Psi\colon \mathbb{R}^2 \times C^0 \times C^0 \to C^0$ be the continuous map defined by

$$\Psi(\xi,\varphi,w)(t) := \xi + \int_0^t b_{\widetilde{X}}\big(s, \varphi(s)\big)\,\mathrm{d}s + \sqrt{2\kappa}\,w(t).$$

With this notation, (6.6) reads $Y_m^j(\cdot,\omega) = \Psi(Y_m^0(\omega), \widetilde{Y}_m^{j-1}(\cdot,\omega), W_m(\cdot,\omega))$ for $\omega \in \Omega$ such that $W_m(\cdot,\omega)$ is a continuous path (this occurs a.s.). Then we have that

$$\mathrm{Law}\big(\widetilde{Y}_m^j\big) = \big(\widetilde{Y}_m^j\big)_\# \mathbb{P} = \Psi_\#\big(Y_m^0, \widetilde{Y}_m^{j-1}, W_m\big)_\# \mathbb{P}.$$

Since Y_1^0, Y_2^0 are identically distributed, by the induction assumption, and since $\mathrm{Law}(W_1) = \mathrm{Law}(W_2)$ (it is the Wiener measure), we have that $(Y_1^0, \widetilde{Y}_1^{j-1}, W_1)_\#\mathbb{P} = (Y_2^0, \widetilde{Y}_2^{j-1}, W_2)_\#\mathbb{P}$. Thus, repeating backward the same computations for $\widetilde{Y}_2^j$, we conclude that $\mathrm{Law}(\widetilde{Y}_1^j) = \mathrm{Law}(\widetilde{Y}_2^j)$.

Substep 1.2. (Convergence of Picard iterations to $\widetilde{Y}_m$) By Remark 2.2 we have that $\mathbb{E}(\|\widetilde{Y}_m^j - \widetilde{Y}_m\|_\infty) \to 0$. (Note that $b_{\widetilde{X}}$ is globally Lipschitz continuous, as proven in (5.11).)

Substep 1.3. (Proof of claim (6.4)) The convergence $\mathbb{E}(\|\widetilde{Y}_m^j - \widetilde{Y}_m\|_\infty) \to 0$ implies that $\widetilde{Y}_m^j \to \widetilde{Y}_m$ in law, hence $\mathrm{Law}(\widetilde{Y}_1) = \mathrm{Law}(\widetilde{Y}_2)$, which is our claim (6.4).

Step 2. (Exploiting the fixed point) For $m = 1,2$, we consider the functionals $\mathcal{L}_m = \mathcal{L}_{Y_m^0,W_m}\colon \mathcal{P}_1(C^0) \to \mathcal{P}_1(C^0)$ defined as in *Step 2* in the proof of Proposition 5.2 (we stress here the dependence on m to keep track of the dependence on the initial datum Y_m^0 and the Brownian motion W_m). Given $\mu \in \mathcal{P}_1(C^0)$, we let $\widetilde{X} = (\widetilde{X}_1,\dots,\widetilde{X}_N)$ and $(\widetilde{Y}_m(t))_{t\in[0,T]}$ be the unique solution to the decoupled system (6.2)–(6.3). Then we set $\mathcal{L}_m(\mu) := \mathrm{Law}(\widetilde{Y}_m)$. By the discussion in *Step 1*, we have that $\mathcal{L}_1(\mu) = \mathcal{L}_2(\mu)$.

Let us now fix an initial guess for the law μ, *e.g.*, $\mu = \delta_0 \in \mathcal{P}_1(C^0)$ (it is enough that it satisfies $\mathcal{W}_1(\mu, \bar{\mu}^{\mathrm{p}}_m) < +\infty$). We apply iteratively $\mathcal{L}^0_m(\mu) = \mu$, $\mathcal{L}^j_m(\mu) = \mathcal{L}_m(\mathcal{L}^{j-1}_m(\mu))$. Since $\mathcal{L}_m$ is a contraction with respect to the modified 1-Wasserstein distance $\mathcal{W}_{1,\alpha}$, $\mathcal{L}^j_m(\mu) \to \bar{\mu}^{\mathrm{p}}_m$ as $j \to +\infty$, where $\bar{\mu}^{\mathrm{p}}_m$ is the unique fixed point $\bar{\mu}^{\mathrm{p}}_m = \mathcal{L}_m(\bar{\mu}^{\mathrm{p}}_m)$. Since $\mathcal{L}_1(\mu) = \mathcal{L}_2(\mu)$, we conclude that $\bar{\mu}^{\mathrm{p}}_1 = \bar{\mu}^{\mathrm{p}}_2$, *i.e.*, the law given by the solution $\bar{Y}_m$ to (6.1) does not depend on m. In conclusion, $\bar{Y}_1, \bar{Y}_2$ are identically distributed. We let $\bar{\mu}^{\mathrm{p}}$ denote their common law.

The solution $\bar{X}_m = (\bar{X}_{m,1}, \dots, \bar{X}_{m,N})$ is then obtained as the solution to (6.2) corresponding to $\bar{\mu}^{\mathrm{p}}$. Thus it does not depend on m, yielding $\bar{X}_1 = \bar{X}_2$. □

Proposition 6.2 *Assume the following*:

- *Let* $(W_m(t))_{t\in[0,T]}$, $m = 1, \dots, M$ *be* M *independent* $\mathbb{R}^2$*-valued Brownian motions*;
- *Let* $X^0 = (X^0_1, \dots, X^0_N) \in \mathbb{R}^{2\times N}$;
- *Let* $Y^0_1, \dots, Y^0_M$ *be i.i.d.* $\mathbb{R}^2$*-valued random variables with* $\mathbb{E}(|Y^0_m|) < +\infty$ *and independent of the Brownian motions* $(W_m(t))_{t\in[0,T]}$;
- *Let* $Z^0 = (Z^0_1, \dots, Z^0_L) \in \mathbb{R}^{2\times L}$;
- *Let* $u \in L^\infty([0,T];\mathcal{U})$.

For every $m = 1, \dots, M$, *let* $\bar{X} = (\bar{X}_1, \dots, \bar{X}_N)$, $(\bar{Y}_m(t))_{t\in[0,T]}$, $Z = (Z_1, \dots, Z_L)$ *be the unique strong solution to*[9]

$$\begin{cases} \frac{\mathrm{d}\bar{X}_n}{\mathrm{d}t}(t) = v^N_n(\bar{X}(t))(\mathbf{r}(\bar{X}_n(t)) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(t)(\bar{X}_n(t))), \\ \mathrm{d}\bar{Y}_m(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_m(t) - Z_\ell(t)) \\ \qquad\qquad - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\bar{Y}_m(t) - \bar{X}_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_m(t) \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \bar{X}_n(0) = X^0_n, \qquad Z_\ell(0) = Z^0_\ell, \quad n = 1, \dots, N, \ell = 1, \dots, L, \\ \bar{Y}_m(0) = Y^0_m \quad a.s., \qquad \bar{\mu}^{\mathrm{p}} = \mathrm{Law}(\bar{Y}_m). \end{cases} \tag{6.7}$$

Then the stochastic processes $(\bar{Y}_1(t))_{t\in[0,T]}, \dots, (\bar{Y}_M(t))_{t\in[0,T]}$ *are independent.*

Proof The leading idea of the proof is to write $\bar{Y}_m$ in terms of the initial datum Y^0_m and the Brownian motion W_m.

We consider the solution operator $\mathcal{S}\colon \mathbb{R}^2 \times C^0 \to C^0$ defined by $\mathcal{S}(\xi, w) := \varphi$, where φ is the unique solution to the integral equation

$$\varphi(t) = \xi + \int_0^t b_{\bar{X}}\big(s, \varphi(s)\big)\,\mathrm{d}s + \sqrt{2\kappa}\,w(t), \quad t \in [0,T].$$

The fact that there exists a unique solution to the previous problem follows from the fact that the operator $\Psi\colon \mathbb{R}^2 \times C^0 \times C^0 \to C^0$ defined by

$$\Psi(\xi, \varphi, w)(t) := \xi + \int_0^t b_{\bar{X}}\big(s, \varphi(s)\big)\,\mathrm{d}s + \sqrt{2\kappa}\,w(t) \quad \text{for } t \in [0,T]$$

[9] This corresponds to the averaged ODE/SDE/ODE system (5.1) with initial data X^0, Y^0_m, Z^0, with Brownian motion W_m, and with control u. The solution is provided by Proposition 5.2. Note that we applied Proposition 6.1 to deduce that $(\bar{Y}_1(t))_{t\in[0,T]}, \dots, (\bar{Y}_M(t))_{t\in[0,T]}$ are identically distributed with common law $\bar{\mu}^{\mathrm{p}}$ and the curve $\bar{X}$ is independent of m.

is such that $\Psi(\xi,\cdot,w)\colon C^0 \to C^0$ is a contraction with respect to the auxiliary norm $|||\varphi|||_\alpha := \sup_{t\in[0,T]}(e^{-\alpha t}|\varphi(t)|)$ for suitable $\alpha > 0$. Indeed, by the Lipschitz continuity of $b_{\bar{X}}$,

$$\begin{aligned} &e^{-\alpha t}\big|\Psi(\xi,\varphi_1,w)(t) - \Psi(\xi,\varphi_2,w)(t)\big| \\ &\quad\le e^{-\alpha t}\int_0^t \big|b_{\bar{X}}\big(s,\varphi_1(s)\big) - b_{\bar{X}}\big(s,\varphi_2(s)\big)\big|\,\mathrm{d}s \\ &\quad\le Ce^{-\alpha t}\int_0^t e^{\alpha s}e^{-\alpha s}\big|\varphi_1(s) - \varphi_2(s)\big|\,\mathrm{d}s \le Ce^{-\alpha t}|||\varphi_1 - \varphi_2|||_\alpha \frac{e^{\alpha t} - 1}{\alpha} \le \frac{C}{\alpha}|||\varphi_1 - \varphi_2|||_\alpha, \end{aligned}$$

hence, choosing $\alpha > 0$ such that $C_\alpha = \frac{C}{\alpha} < 1$,

$$|||\Psi(\xi,\varphi_1,w) - \Psi(\xi,\varphi_2,w)|||_\alpha \le C_\alpha |||\varphi_1 - \varphi_2|||_\alpha,$$

and thus it has a unique fixed point.

We now observe that the solution operator $\mathcal{S}\colon \mathbb{R}^2\times C^0 \to C^0$ is continuous. Indeed, it is Lipschitz with respect to both variables. Letting $\varphi_1 = \mathcal{S}(\xi_1,w)$ and $\varphi_2 = \mathcal{S}(\xi_2,w)$, by the Lipschitz continuity of $b_{\bar{X}}$, we have that

$$\begin{aligned} \big|\varphi_1(t) - \varphi_2(t)\big| &\le |\xi_1 - \xi_2| + \int_0^t \big|b_{\bar{X}}\big(s,\varphi_1(s)\big) - b_{\bar{X}}\big(s,\varphi_2(s)\big)\big|\,\mathrm{d}s \\ &\le |\xi_1 - \xi_2| + C\int_0^t \big|\varphi_1(s) - \varphi_2(s)\big|\,\mathrm{d}s, \end{aligned}$$

thus, by Grönwall's inequality,

$$\big|\varphi_1(t) - \varphi_2(t)\big| \le |\xi_1 - \xi_2|e^{Ct} \quad\Longrightarrow\quad \big\|\mathcal{S}(\xi_1,w) - \mathcal{S}(\xi_2,w)\big\|_\infty \le |\xi_1 - \xi_2|e^{CT}.$$

Analogously, letting $\varphi_1 = \mathcal{S}(\xi,w_1)$ and $\varphi_2 = \mathcal{S}(\xi,w_2)$, by the Lipschitz continuity of $b_{\bar{X}}$, we have that

$$\begin{aligned} \big|\varphi_1(t) - \varphi_2(t)\big| &\le \int_0^t \big|b_{\bar{X}}\big(s,\varphi_1(s)\big) - b_{\bar{X}}\big(s,\varphi_2(s)\big)\big|\,\mathrm{d}s + \big|w_1(t) - w_2(t)\big| \\ &\le C\int_0^t \big|\varphi_1(s) - \varphi_2(s)\big|\,\mathrm{d}s + \|w_1 - w_2\|_\infty, \end{aligned}$$

thus, by Grönwall's inequality,

$$\big|\varphi_1(t) - \varphi_2(t)\big| \le \|w_1 - w_2\|_\infty e^{Ct} \quad\Longrightarrow\quad \big\|\mathcal{S}(\xi,w_1) - \mathcal{S}(\xi,w_2)\big\|_\infty \le \|w_1 - w_2\|_\infty e^{CT}.$$

We are now in a position to write the stochastic processes $(\bar{Y}_m(t))_{t\in[0,T]}$ as $Y_m(\cdot,\omega) = \mathcal{S}(Y_m^0(\omega), W_m(\cdot,\omega))$ for a.e. $\omega \in \Omega$. Note that $Y_1^0,\dots,Y_M^0\colon \Omega \to \mathbb{R}^2$ and $W_1,\dots,W_M\colon \Omega \to C^0$ are independent random variables. It follows that $(\bar{Y}_1(t))_{t\in[0,T]},\dots,(\bar{Y}_M(t))_{t\in[0,T]}$ are independent stochastic processes. This concludes the proof. □

7 Mean-field limit for a large number of pirate ships

In this section we study the limit of the problem as $M \to +\infty$. For this reason we will stress the dependence of initial data and solutions on M. Still, we do not stress dependence on N, not to overburden the notation.

7.1 Mean-field ODE/SDE/ODE limit model as $M \to +\infty$

In the following theorem we shall describe convergence of solutions in terms of empirical measures. Given stochastic processes $(S_1(t))_{t\in[0,T]}, \dots, (S_M(t))_{t\in[0,T]}$ a.s. with continuous paths, we associate the empirical measure[10] $\nu_M \colon \Omega \to \mathcal{P}(C^0([0,T];\mathbb{R}^2))$ defined for a.e. $\omega \in \Omega$ by

$$\nu_M(\cdot,\omega) := \frac{1}{M}\sum_{m=1}^{M} \delta_{S_m(\cdot,\omega)}.$$

(The first placeholder is kept free for the time variable.) If $\max_m \mathbb{E}(\|S_m\|_\infty) < +\infty$, then a.s. $\nu_M \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$. Indeed,

$$\begin{aligned}\mathbb{E}\biggl(\int_{C^0([0,T];\mathbb{R}^2)} \|\varphi\|_\infty \, \mathrm{d}\nu_M(\cdot,\cdot)(\varphi)\biggr) &= \frac{1}{M}\sum_{m=1}^{M}\mathbb{E}\biggl(\int_{C^0([0,T];\mathbb{R}^2)} \|\varphi\|_\infty \, \mathrm{d}\delta_{S_m}\biggr) \\ &= \frac{1}{M}\sum_{m=1}^{M}\mathbb{E}\bigl(\|S_m\|_\infty\bigr) < +\infty.\end{aligned} \tag{7.1}$$

We set $\nu_M(t,\omega) := (\mathrm{ev}_t)_\#\nu_M(\cdot,\omega)$ for all $\omega \in \Omega$ and $t \in [0,T]$. With a slight abuse of notation, we let $\nu_M(t)$ denote the random measure $\nu_M(t)\colon \Omega \to \mathcal{P}(\mathbb{R}^2)$.

Theorem 7.1 *Assume the following*:

- *Let* $(W_m(t))_{t\in[0,T]}$, $m \geq 1$, *be a sequence of independent* $\mathbb{R}^2$*-valued Brownian motions*;
- *Let* $X^0 = (X^0_1,\dots,X^0_N) \in \mathbb{R}^{2\times N}$;
- *Let* $Y^0 = (Y^0_1,\dots,Y^0_M)$, *where* $Y^0_1,\dots,Y^0_M$ *are i.i.d.* $\mathbb{R}^2$*-valued random variables with* $\mathbb{E}(|Y^0_m|) < +\infty$ *and independent of the Brownian motions* $(W_m(t))_{t\in[0,T]}$;
- *Let* $Z^0 = (Z^0_1,\dots,Z^0_L) \in \mathbb{R}^{2\times L}$;
- *Let* $(W(t))_{t\in[0,T]}$ *be a Brownian motion*;
- *Let* $\bar{Y}^0$ *be an* $\mathbb{R}^2$*-valued random variable identically distributed to* $Y^0_1,\dots,Y^0_M$.

Let $u^M, u \in L^\infty([0,T];\mathcal{U})$ *be such that* $u^M \overset{*}{\rightharpoonup} u$ *weakly* in* $L^\infty([0,T];\mathcal{U})$.[11] *Let* $(X^M(t))_{t\in[0,T]} = (X^M_1(t),\dots,X^M_N(t))_{t\in[0,T]}$, $(Y^M(t))_{t\in[0,T]} = (Y^M_1(t),\dots,Y^M_M(t))_{t\in[0,T]}$, *and* $Z^M = (Z^M_1,\dots,Z^M_L)$ *be the unique strong solution to*[12]

$$\begin{cases} \mathrm{d}X^M_n(t) = \nu^N_n(X^M(t))(\mathbf{r}(X^M_n(t)) + \frac{1}{M}\sum_{m=1}^M K^{\mathrm{cp}}(X^M_n(t) - Y^M_m(t)))\,\mathrm{d}t, \\ \mathrm{d}Y^M_m(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(Y^M_m(t) - Z^M_\ell(t)) \\ \qquad\qquad - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(Y^M_m(t) - X^M_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W_m(t), \\ \frac{\mathrm{d}Z^M_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z^M_\ell(t) - Z^M_{\ell'}(t)) + u^M_\ell(t), \\ X^M_n(0) = X^0_n \quad a.s., \qquad Y^M_m(0) = Y^0_m \quad a.s., \qquad Z^M_\ell(0) = Z^0_\ell, \\ n = 1,\dots,N, m = 1,\dots,M, \ell = 1,\dots,L. \end{cases} \tag{7.2}$$

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[10] The measurability of these random variables is proven with an argument analogous to the one in Footnote 5, keeping in mind the separability of $C^0([0,T];\mathbb{R}^2)$.

[11] In fact, by the boundedness of $\mathcal{U}$, this is equivalent to requiring that $u^M \rightharpoonup u$ weakly in $L^1([0,T];\mathcal{U})$.

[12] This corresponds to the original ODE/SDE/ODE system (3.2) with initial data X^0, Y^0, Z^0 and with control u^M. The solution is provided by Proposition 4.1. We stressed the dependence on M since we are interested in the limit as $M \to +\infty$.

Let ν_M^{p} be the empirical measures associated with $(Y_1^M(t))_{t\in[0,T]},\dots,(Y_M^M(t))_{t\in[0,T]}$. Then there exist $\bar{X}=(\bar{X}_1,\dots,\bar{X}_N)$, $(\bar{Y}(t))_{t\in[0,T]}$, and $Z=(Z_1,\dots,Z_L)$ such that

$$\mathbb{E}\Big(\max_n \big\|X_n^M-\bar{X}_n\big\|_\infty\Big)+\int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\nu_M^{\mathrm{p}}(t),\bar{\mu}^{\mathrm{p}}(t)\big)\big)\,\mathrm{d}t+\big\|Z^M-Z\big\|_\infty\to 0 \quad \text{as } M\to+\infty. \tag{7.3}$$

Moreover, $\bar{X}=(\bar{X}_1,\dots,\bar{X}_N)$, $(\bar{Y}(t))_{t\in[0,T]}$, and $Z=(Z_1,\dots,Z_L)$ are the unique strong solution to (5.1).[13]

Proof To prove the result, we need to exploit an intermediate problem. For $m=1,\dots,M$, let $\bar{X}=(\bar{X}_1,\dots,\bar{X}_N)$, $(\bar{Y}_m^M(t))_{t\in[0,T]}$, and $Z=(Z_1,\dots,Z_L)$ be the unique strong solution to[14]

$$\begin{cases} \frac{\mathrm{d}\bar{X}_n}{\mathrm{d}t}(t)=\nu_n^N(\bar{X}(t))(\mathbf{r}(\bar{X}_n(t))+K^{\mathrm{cp}}*\bar{\mu}^{\mathrm{p}}(t)(\bar{X}_n(t))),\\ \mathrm{d}\bar{Y}_m^M(t)=(\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_m^M(t)-Z_\ell(t))\\ \qquad -\frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\bar{Y}_m^M(t)-\bar{X}_n(t)))\,\mathrm{d}t+\sqrt{2\kappa}\,\mathrm{d}W_m(t),\\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t)=\frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t)-Z_{\ell'}(t))+u_\ell(t),\\ \bar{X}_n(0)=X_n^0,\qquad Z_\ell(0)=Z_\ell^0,\quad n=1,\dots,N,\ell=1,\dots,L,\\ \bar{Y}_m^M(0)=Y_m^0\quad \text{a.s.},\qquad \bar{\mu}^{\mathrm{p}}=\mathrm{Law}(\bar{Y}_m^M). \end{cases} \tag{7.4}$$

Our first task is to prove that

$$\mathbb{E}\Big(\max_n\big\|X_n^M-\bar{X}_n\big\|_\infty\Big)+\mathbb{E}\Big(\max_m\big\|Y_m^M-\bar{Y}_m^M\big\|_\infty\Big)+\big\|Z^M-Z\big\|_\infty\to 0\quad \text{as } M\to+\infty, \tag{7.5}$$

from which (7.3) will follow as shown in *Step 5*.

As in the previous proofs, let $C^0:=C^0([0,T];\mathbb{R}^2)$. Let us also consider the empirical measures[15] $\bar{\nu}_M^{\mathrm{p}}:\Omega\to\mathcal{P}(C^0)$ associated with $(\bar{Y}_1^M(t))_{t\in[0,T]},\dots,(\bar{Y}_M^M(t))_{t\in[0,T]}$. To be precise, we have that for a.e. $\omega\in\Omega$

$$\nu_M^{\mathrm{p}}(\cdot,\omega):=\frac{1}{M}\sum_{m=1}^M\delta_{Y_m^M(\cdot,\omega)},\qquad \bar{\nu}_M^{\mathrm{p}}(\cdot,\omega):=\frac{1}{M}\sum_{m=1}^M\delta_{\bar{Y}_m^M(\cdot,\omega)}.$$

(The first placeholder is kept free for the time variable.) Notice that, in fact, a.s. $\nu_M^{\mathrm{p}}\in\mathcal{P}_1(C^0)$ and $\bar{\nu}_M^{\mathrm{p}}\in\mathcal{P}_1(C^0)$ by (7.1) and since by Proposition 4.1 and Proposition 5.2 we have that $\mathbb{E}(\max_m\|Y_m^M\|_\infty)<+\infty$ and $\mathbb{E}(\max_m\|\bar{Y}_m^M\|_\infty)<+\infty$, respectively.

<u>*Step 1*</u>. (Estimate of $|Y_m^M-\bar{Y}_m^M|$) Using the fact that Y_m^M and $\bar{Y}_m^M$ are strong solutions to (7.2) and (7.4), respectively, and by the Lipschitz continuity of K^{pg} and K^{pc}, we have

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[13] Corresponding to the initial data $X^0,\bar{Y}^0,Z^0$ with Brownian motion W and control u. We recall that the solution is provided by Proposition 5.2.

[14] This corresponds to the averaged ODE/SDE/ODE system (5.1) with initial data X^0, Y_m^0, Z^0, with Brownian motion W_m, and with control u. The solution is provided by Proposition 5.2. Note that we applied Proposition 6.1 to deduce that $(\bar{Y}_1^M(t))_{t\in[0,T]},\dots,(\bar{Y}_M^M(t))_{t\in[0,T]}$ are identically distributed with common law $\bar{\mu}^{\mathrm{p}}$ and the curve $\bar{X}$ does not depend on m and M.

[15] The random measure $\bar{\nu}_M^{\mathrm{p}}:\Omega\to\mathcal{P}(C^0)$ (empirical measure of $\bar{Y}_1^M,\dots,\bar{Y}_M^M$) must not be confused with $\nu_M^{\mathrm{p}}:\Omega\to\mathcal{P}(C^0)$ (empirical measure of $Y_1^M,\dots,Y_M^M$) or $\bar{\mu}^{\mathrm{p}}\in\mathcal{P}(C^0)$ (common law of the stochastic processes $(\bar{Y}_1^M(t))_{t\in[0,T]},\dots,(\bar{Y}_M^M(t))_{t\in[0,T]}$, and $(\bar{Y}(t))_{t\in[0,T]}$).

that a.s. for $0 \le s \le t$ and $m = 1, \dots, M$

$$\begin{aligned}
&\left|Y_m^M(s) - \bar{Y}_m^M(s)\right| \\
&\quad\le \int_0^s \left(\frac{1}{L}\sum_{\ell=1}^{L}\left|K^{\mathrm{pg}}\bigl(Y_m^M(r) - Z_\ell^M(r)\bigr) - K^{\mathrm{pg}}\bigl(\bar{Y}_m(r) - Z_\ell(r)\bigr)\right|\right. \\
&\qquad \left. + \frac{1}{N}\sum_{n=1}^{N}\left|K^{\mathrm{pc}}\bigl(Y_m^M(r) - X_n^M(r)\bigr) - K^{\mathrm{pc}}\bigl(\bar{Y}_m^M(r) - \bar{X}_n(r)\bigr)\right|\right) \mathrm{d}r \\
&\quad\le \int_0^s C\left(\left|Y_m^M(r) - \bar{Y}_m^M(r)\right| + \frac{1}{L}\sum_{\ell=1}^{L}\left|Z_\ell^M(r) - Z_\ell(r)\right| + \frac{1}{N}\sum_{n=1}^{N}\left|X_n^M(r) - \bar{X}_n(r)\right|\right) \mathrm{d}r \\
&\quad\le \int_0^t C\left(\max_n \sup_{0\le r\le s}\left|X_n^M(r) - \bar{X}_n(r)\right| + \max_{m'} \sup_{0\le r\le s}\left|Y_{m'}^M(r) - \bar{Y}_{m'}^M(r)\right|\right) \mathrm{d}s \\
&\qquad + CT\left\|Z^M - Z\right\|_\infty,
\end{aligned}$$

the constant C depending on K^{pg} and K^{pc}. Taking the supremum in $s \in [0, t]$, the maximum in m and then the expectation, we obtain that for every $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\left(\max_m \sup_{0\le s\le t}\left|Y_m^M(s) - \bar{Y}_m^M(s)\right|\right) \\
&\quad\le C\int_0^t \mathbb{E}\left(\max_m \sup_{0\le r\le s}\left|Y_m^M(r) - \bar{Y}_m^M(r)\right| + \max_n \sup_{0\le r\le s}\left|X_n^M(r) - \bar{X}_n(r)\right|\right) \mathrm{d}s \qquad (7.6)\\
&\qquad + CT\left\|Z^M - Z\right\|_\infty.
\end{aligned}$$

Step 2. (Estimate of $|X_n^M - \bar{X}_n|$) To estimate $|X_n^M(s) - \bar{X}_n(s)|$, we rewrite

$$\frac{1}{M}\sum_{m=1}^{M} K^{\mathrm{cp}}\bigl(X_n^M(t) - Y_m^M(t)\bigr) = \int_{\mathbb{R}^2} K^{\mathrm{cp}}\bigl(X_n^M(t) - y\bigr)\,\mathrm{d}\nu_M^{\mathrm{p}}(t)(y) = K^{\mathrm{cp}} * \nu_M^{\mathrm{p}}(t)\bigl(X_n^M(t)\bigr). \quad (7.7)$$

Then we exploit the properties of ν^N, $\mathbf{r}$, and K^{cp} and (7.7) to get from (7.2) that a.s. for $0 \le s \le t$ and $n = 1, \dots, N$

$$\begin{aligned}
&\left|X_n^M(s) - \bar{X}_n(s)\right| \\
&\quad\le \int_0^s \left|\nu_n^N\bigl(X^M(r)\bigr)\left(\mathbf{r}\bigl(X_n^M(r)\bigr) + \frac{1}{M}\sum_{m=1}^{M} K^{\mathrm{cp}}\bigl(X_n^M(t) - Y_m^M(r)\bigr)\right)\right. \\
&\qquad \left. - \nu_n^N\bigl(\bar{X}(r)\bigr)\bigl(\mathbf{r}\bigl(\bar{X}_n(r)\bigr) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\bigr)\right| \mathrm{d}r \qquad (7.8)\\
&\quad\le \int_0^s \left\|\nu^N\right\|_\infty \Bigl(\left|\mathbf{r}\bigl(X_n^M(r)\bigr) - \mathbf{r}\bigl(\bar{X}_n^M(r)\bigr)\right| \\
&\qquad + \left|K^{\mathrm{cp}} * \nu_M^{\mathrm{p}}(r)\bigl(X_n(r)\bigr) - K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\right|\Bigr) \\
&\qquad + \left|\nu_n^N\bigl(X^M(r)\bigr) - \nu_n^N\bigl(\bar{X}(r)\bigr)\right|\left|\mathbf{r}\bigl(\bar{X}_n(r)\bigr) - K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\right| \mathrm{d}r.
\end{aligned}$$

To estimate the term involving $|K^{\mathrm{cp}} * \nu_M^{\mathrm{p}}(r)(X_n^M(r)) - K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)(\bar{X}_n(r))|$ in (7.8), we exploit Kantorovich's duality and the Lipschitz continuity of K^{cp} to get that a.s.

$$\begin{aligned}
&\bigl|K^{\mathrm{cp}} * \nu_M^{\mathrm{p}}(r)\bigl(X_n^M(r)\bigr) - K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\bigr| \\
&\quad = \left|\int_{\mathbb{R}^2} K^{\mathrm{cp}}\bigl(X_n^M(r) - y\bigr)\,\mathrm{d}\nu_M^{\mathrm{p}}(r)(y) - \int_{\mathbb{R}^2} K^{\mathrm{cp}}\bigl(\bar{X}_n(r) - y\bigr)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(r)(y)\right| \\
&\quad \le \left|\int_{\mathbb{R}^2} K^{\mathrm{cp}}\bigl(X_n^M(r) - y\bigr)\,\mathrm{d}\bigl(\nu_M^{\mathrm{p}}(r) - \bar{\mu}^{\mathrm{p}}(r)\bigr)(y)\right| \\
&\qquad + \left|\int_{\mathbb{R}^2} \bigl(K^{\mathrm{cp}}\bigl(X_n^M(r) - y\bigr) - K^{\mathrm{cp}}\bigl(\bar{X}_n(r) - y\bigr)\bigr)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\right| \\
&\quad \le C\mathcal{W}_1\bigl(\nu_M^{\mathrm{p}}(r), \bar{\mu}^{\mathrm{p}}(r)\bigr) + C\bigl|X_n^M(r) - \bar{X}_n(r)\bigr| \\
&\quad \le C\max_{n'}\bigl|X_{n'}^M(r) - \bar{X}_{n'}(r)\bigr| + C\mathcal{W}_1\bigl(\nu_M^{\mathrm{p}}(r), \bar{\nu}_M^{\mathrm{p}}(r)\bigr) + C\mathcal{W}_1\bigl(\bar{\nu}_M^{\mathrm{p}}(r), \bar{\mu}^{\mathrm{p}}(r)\bigr).
\end{aligned} \tag{7.9}$$

We bound $\mathcal{W}_1(\nu_M^{\mathrm{p}}(r), \bar{\nu}_M^{\mathrm{p}}(r))$ using for a.e. $\omega \in \Omega$ as an admissible transport plan the diagonal transport $\gamma(\omega) = \frac{1}{N}\sum_{n=1}^N \delta_{(Y_n^M(r,\omega), \bar{Y}_n^M(r,\omega))}$ to obtain that a.s.

$$\begin{aligned}
\mathcal{W}_1\bigl(\nu_M^{\mathrm{p}}(r), \bar{\nu}_M^{\mathrm{p}}(r)\bigr) &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \bigl|y - y'\bigr|\,\mathrm{d}\gamma\bigl(y, y'\bigr) = \frac{1}{M}\sum_{m=1}^M \bigl|Y_m^M(r) - \bar{Y}_m^M(r)\bigr| \\
&\le \max_m \bigl|Y_m^M(r) - \bar{Y}_m^M(r)\bigr|.
\end{aligned} \tag{7.10}$$

To estimate the term involving $K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)(\bar{X}_n(r))$ in (7.8), we use the fact that $|K^{\mathrm{cp}}(z)| \le |K^{\mathrm{cp}}(0)| + C|z|$ to get that

$$\begin{aligned}
&\bigl|K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\bigr| \\
&\quad \le \int_{\mathbb{R}^2} \bigl|K^{\mathrm{cp}}\bigl(\bar{X}_n(r) - y\bigr)\bigr|\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(r)(y) \\
&\quad \le \int_{\mathbb{R}^2} C\bigl(1 + \bigl|\bar{X}_n(r)\bigr| + |y|\bigr)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(r)(y) \\
&\quad \le C\left(1 + \max_{n'}\|\bar{X}_{n'}\|_\infty + \sup_{0\le r\le T}\left(\int_{\mathbb{R}^2} |y|\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(r)(y)\right)\right) \\
&\quad \le C\left(1 + \max_{n'}\|\bar{X}_{n'}\|_\infty + \int_{C^0} \|\varphi\|_\infty\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(\varphi)\right),
\end{aligned}$$

where in the last inequality we used that

$$\int_{\mathbb{R}^2} |y|\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(r)(y) = \int_{\mathbb{R}^2} |y|\,\mathrm{d}\bigl((\mathrm{ev}_r)_\#\bar{\mu}^{\mathrm{p}}\bigr)(y) = \int_{C^0} \bigl|\mathrm{ev}_r(\varphi)\bigr|\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(\varphi) \le \int_{C^0} \|\varphi\|_\infty\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(\varphi),$$

which is finite since $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0)$ by Proposition 5.2. By Remark 5.3 we recall that $\max_n \|\bar{X}_n\|_\infty$ is bounded by a constant depending on $\max_n \|X_n^0\|_\infty$, T, $\|\nu^N\|_\infty$, $\mathbf{r}$, and K^{cp}. Hence

$$\bigl|K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(r)\bigl(\bar{X}_n(r)\bigr)\bigr| \le C. \tag{7.11}$$

Then we can proceed with the estimate in (7.8): By (7.9)–(7.11) and by exploiting also the Lipschitz continuity of $\mathbf{r}$ and ν^N, we obtain that

$$\begin{aligned}
&\left|X_n^M(s) - \bar{X}_n(s)\right| \\
&\quad\le C\int_0^s \max_{n'}\left|X_{n'}^M(r) - \bar{X}_{n'}(r)\right| \mathrm{d}r + C\int_0^s \max_m\left|Y_m^M(r) - \bar{Y}_m^M(r)\right| \mathrm{d}r \\
&\qquad + C\int_0^s \mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(r), \bar{\mu}^{\mathrm{p}}(r)\big)\,\mathrm{d}r \\
&\quad\le C\int_0^t \max_{n'}\sup_{0\le r\le s}\left|X_{n'}^M(r) - \bar{X}_{n'}(r)\right| \mathrm{d}s + C\int_0^t \max_m\sup_{0\le r\le s}\left|Y_m^M(r) - \bar{Y}_m^M(r)\right| \mathrm{d}s \\
&\qquad + C\int_0^T \mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(s), \bar{\mu}^{\mathrm{p}}(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Taking the supremum in s, the maximum in n, and then the expectation, we obtain that for every $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\Big(\max_n \sup_{0\le s\le t}\left|X_n^M(s) - \bar{X}_n(s)\right|\Big) \\
&\quad\le C\int_0^t \mathbb{E}\Big(\max_n\sup_{0\le r\le s}\left|X_n^M(r) - \bar{X}_n(r)\right|\Big)\,\mathrm{d}s \\
&\qquad + C\int_0^t \mathbb{E}\Big(\max_m\sup_{0\le r\le s}\left|Y_m^M(r) - \bar{Y}_m^M(r)\right|\Big)\,\mathrm{d}s + C\int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big)\,\mathrm{d}s.
\end{aligned} \tag{7.12}$$

Step 3. (Grönwall's inequality) Putting together (7.6) and (7.12), we have that for every $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\Big(\max_n \sup_{0\le s\le t}\left|X_n^M(s) - \bar{X}_n(s)\right| + \max_m \sup_{0\le s\le t}\left|Y_m^M(s) - \bar{Y}_m^M(s)\right|\Big) \\
&\quad\le C\int_0^t \mathbb{E}\Big(\max_n\sup_{0\le r\le s}\left|X_n^M(r) - \bar{X}_n(r)\right| + \max_m\sup_{0\le r\le s}\left|Y_m^M(r) - \bar{Y}_m^M(r)\right|\Big)\,\mathrm{d}s \\
&\qquad + CT\left\|Z^M - Z\right\|_\infty + C\int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big)\,\mathrm{d}s.
\end{aligned}$$

By Grönwall's inequality, we deduce that for every $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\Big(\max_n \sup_{0\le s\le t}\left|X_n^M(s) - \bar{X}_n(s)\right| + \max_m \sup_{0\le s\le t}\left|Y_m^M(s) - \bar{Y}_m^M(s)\right|\Big) \\
&\quad\le Ce^{Ct}\Big(CT\left\|Z^M - Z\right\|_\infty + \int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big)\,\mathrm{d}s\Big).
\end{aligned}$$

In particular,

$$\begin{aligned}
&\mathbb{E}\Big(\max_n\left\|X_n^M - \bar{X}_n^M\right\|_\infty + \max_m\left\|Y_m^M - \bar{Y}_m^M\right\|_\infty\Big) \\
&\quad\le C\Big(\left\|Z^M - Z\right\|_\infty + \int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{p}}(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big)\,\mathrm{d}s\Big) =: \alpha(M),
\end{aligned}$$

where the constant depends additionally on T.

Step 4. (Convergence to zero of $\alpha(M)$) To conclude the proof, we show that $\alpha(M) \to 0$ as $M \to +\infty$.

Substep 4.1. Let us show that $\|Z^M - Z\|_\infty \to 0$ as $M \to +\infty$. We start by observing that by (7.2) and (7.4)

$$\begin{aligned}
&\left|Z_\ell^M(t) - Z_\ell(t)\right| \\
&\quad\le \frac{1}{L}\sum_{\ell'=1}^{L}\int_0^t \left|K^{\mathrm{gg}}\big(Z_\ell^M(s) - Z_{\ell'}^M(s)\big) - K^{\mathrm{gg}}\big(Z_\ell(s) - Z_{\ell'}(s)\big)\right| \mathrm{d}s + \left|\int_0^t \big(u_\ell^M(s) - u_\ell(s)\big)\,\mathrm{d}s\right| \\
&\quad\le \int_0^t C\left|Z^M(s) - Z(s)\right| \mathrm{d}s + \left|\int_0^t \big(u^M(s) - u(s)\big)\,\mathrm{d}s\right|,
\end{aligned}$$

where the constant C depends on K^{gg}. By Grönwall's inequality, it follows that

$$\left|Z^M(t) - Z(t)\right| \le R_M(t) + \int_0^t R_M(s) C e^{C(t-s)}\,\mathrm{d}s \le R_M(t) + Ce^{CT}\int_0^T R_M(s)\,\mathrm{d}s,$$

where $R_M(t) = |\int_0^t (u^M(s) - u(s))\,\mathrm{d}s|$, hence

$$\left\|Z^M - Z\right\|_\infty \le C\|R_M\|_\infty.$$

Since $u^M \overset{*}{\rightharpoonup} u$ weakly* in $L^\infty([0,T];\mathcal{U})$, we have that $R_M(t) \to 0$ for every $t \in [0,T]$. Moreover, by the boundedness of $\mathcal{U}$, $R_M(t)$ are equibounded and equi-Lipschitz. It follows that $\|R_M\|_\infty \to 0$, thus $\|Z^M - Z\|_\infty \to 0$.

Substep 4.2. Let us show that $\int_0^T \mathbb{E}(\mathcal{W}_1(\bar{\nu}_M^{\mathrm{P}}(s), \bar{\mu}^{\mathrm{P}}(s)))\,\mathrm{d}s \to 0$ as $M \to +\infty$.

To show this, we apply the discussion in Sect. 2.6 about the approximation of a law (here played by $\bar{\mu}^{\mathrm{P}}(s)$) with empirical measures on independent samples of the law (here played by $\bar{\nu}_M^{\mathrm{P}}(s)$). Let us check that all the assumptions hold true. For every $s \in [0,T]$, we have that $\bar{\mu}^{\mathrm{P}}(s) \in \mathcal{P}_1(\mathbb{R}^d)$. This follows from the fact that, by Proposition 6.1, $\bar{\mu}^{\mathrm{P}} = \mathrm{Law}(\bar{Y}_1^M) = \cdots = \mathrm{Law}(\bar{Y}_M^M) = \mathrm{Law}(\bar{Y})$, thus

$$\begin{aligned}
\int_{\mathbb{R}^2} |y|\,\mathrm{d}\bar{\mu}^{\mathrm{P}}(s)(y) &= \int_{\mathbb{R}^2} |y|\,\mathrm{d}\big((\mathrm{ev}_s)_\#(\bar{Y})_\#\mathbb{P}\big)(y) = \int_\Omega \left|\bar{Y}(s,\omega)\right| \mathrm{d}\mathbb{P}(\omega) \\
&= \mathbb{E}\big(\left|\bar{Y}(s)\right|\big) \le \mathbb{E}\big(\|\bar{Y}\|_\infty\big) < +\infty,
\end{aligned} \tag{7.13}$$

where the finiteness of $\mathbb{E}(\|\bar{Y}\|_\infty)$ follows from Proposition 5.2. Moreover, the random variables $\bar{Y}_1^M(s), \dots, \bar{Y}_M^M(s)$ are i.d. with law $(\bar{Y}_m^M(s,\cdot))_\#\mathbb{P} = (\mathrm{ev}_s)_\#(\bar{Y}_m^M)_\#\mathbb{P} = (\mathrm{ev}_s)_\#\bar{\mu}^{\mathrm{P}} = \bar{\mu}^{\mathrm{P}}(s)$. Finally, by Proposition 6.2 we have that $(\bar{Y}_1^M(t))_{t\in[0,T]}, \dots, (\bar{Y}_M^M(t))_{t\in[0,T]}$ are independent stochastic processes, thus, in particular, $\bar{Y}_1^M(s), \dots, \bar{Y}_M^M(s)$ are independent random variables. By [33, Lemma 4.7.1] we conclude that

$$\mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}_M^{\mathrm{P}}(s), \bar{\mu}^{\mathrm{P}}(s)\big)\big) \to 0 \quad \text{for every } s \in [0,T],$$

as $M \to +\infty$. Let us now show that $s \mapsto \mathbb{E}(\mathcal{W}_1(\bar{\nu}^{\mathrm{p}}_M(s), \bar{\mu}^{\mathrm{p}}(s)))$ is dominated. Indeed, since $\bar{Y}_1(s), \dots, \bar{Y}_M(s)$ are identically distributed and by (7.13), for every $s \in [0, T]$, we have that

$$\begin{aligned}\mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}^{\mathrm{p}}_M(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big) &\le \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}^{\mathrm{p}}_M(s), \delta_0\big)\big) + \mathcal{W}_1\big(\bar{\mu}^{\mathrm{p}}(s), \delta_0\big) \\ &\le \mathbb{E}\left(\int_{\mathbb{R}^2} |y| \,\mathrm{d}\bar{\nu}^{\mathrm{p}}_M(s)(y)\right) + \int_{\mathbb{R}^2} |y| \,\mathrm{d}\bar{\mu}^{\mathrm{p}}(s)(y) \\ &\le \frac{1}{M}\sum_{m=1}^{M} \mathbb{E}\big(\big|\bar{Y}^M_m(s)\big|\big) + \int_{\mathbb{R}^2} |y| \,\mathrm{d}\bar{\mu}^{\mathrm{p}}(s)(y) \\ &\le \mathbb{E}\big(\big|\bar{Y}(s)\big|\big) + \int_{\mathbb{R}^2} |y| \,\mathrm{d}\bar{\mu}^{\mathrm{p}}(s)(y) \le 2\mathbb{E}\big(\|\bar{Y}\|_\infty\big) < +\infty,\end{aligned}$$

where the finiteness of the last term follows from Proposition 5.2. We conclude that

$$\int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\bar{\nu}^{\mathrm{p}}_M(s), \bar{\mu}^{\mathrm{p}}(s)\big)\big) \,\mathrm{d}s \to 0 \tag{7.14}$$

as $M \to +\infty$. This concludes the proof of (7.5).

Step 5. (Conclusion with the proof of (7.3)) By (7.10), we have that

$$\int_0^T \mathbb{E}\big(\mathcal{W}_1\big(\nu^{\mathrm{p}}_M(s), \bar{\nu}^{\mathrm{p}}_M(s)\big)\big) \,\mathrm{d}s \le T\mathbb{E}\Big(\max_m \big\|Y^M_m - \bar{Y}^M_m\big\|_\infty\Big).$$

Combining this with (7.14) and (7.5), we obtain (7.3) and we conclude the proof. □

Proposition 7.2 *Under the assumptions of Theorem 7.1, the curve $\bar{X} = (\bar{X}_1, \dots, \bar{X}_N)$, the law $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0([0, T]; \mathbb{R}^2))$, and the curve $Z = (Z_1, \dots, Z_L)$ from (7.4) are solutions to the ODE/PDE/ODE system*

$$\begin{cases} \frac{\mathrm{d}\bar{X}_n}{\mathrm{d}t}(t) = v^N(\bar{X}(t))(\mathbf{r}(\bar{X}_n(t)) + K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}(t)(\bar{X}_n(t))), \\ \partial_t \bar{\mu}^{\mathrm{p}} - \kappa \Delta_y \bar{\mu}^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\cdot - Z_\ell(t)) \\ \quad - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\cdot - \bar{X}_n(t)))\bar{\mu}^{\mathrm{p}}) = 0, \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = (\frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t))\,\mathrm{d}t, \\ \bar{X}_n(0) = X^0_n, \qquad Z_\ell(0) = Z^0_\ell, \quad n = 1, \dots, N, \ell = 1, \dots, L, \\ \bar{\mu}^{\mathrm{p}}(0) = \mathrm{Law}(\bar{Y}^0), \end{cases} \tag{7.15}$$

where the parabolic PDE is understood in the sense of distributions.[16]

Proof We exploit the fact that $\bar{\mu}^{\mathrm{p}}$ is the law of stochastic processes $(\bar{Y}(t))_{t\in[0,T]}$, where $(\bar{Y}(t))_{t\in[0,T]}$ solves the SDE

$$\begin{cases} \mathrm{d}\bar{Y}(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}(t) - Z_\ell(t)) - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\bar{Y}(t) - \bar{X}_n(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \bar{Y}(0) = \bar{Y}^0 \quad \text{a.s.} \end{cases}$$

[16]To be precise, we regard $\bar{\mu}^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$ as the distribution defined by the duality

$$\int_{-\infty}^0 \int_{\mathbb{R}^2} \xi(t,y)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(0)(y)\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2} \xi(t,y)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t \quad \text{for every } \xi \in C^\infty_c((-\infty,T)\times\mathbb{R}^2).$$

Let us fix a test function $\xi \in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$. By Itô's formula [28, Theorem 6.4], we have that $(\xi(t,\bar{Y}(t)))_{t\in[0,T]}$ is an Itô process solving the SDE

$$\begin{aligned}
&\mathrm{d}\big(\xi\big(t,\bar{Y}(t)\big)\big)\\
&\quad= \partial_t\xi\big(t,\bar{Y}(t)\big)\,\mathrm{d}t + \kappa\,\Delta_y\xi\big(t,\bar{Y}(t)\big)\,\mathrm{d}t\\
&\qquad+\left(\frac{1}{L}\sum_{\ell=1}^{L}K^{\mathrm{pg}}\big(\bar{Y}(t)-Z_\ell(t)\big)-\frac{1}{N}\sum_{n=1}^{N}K^{\mathrm{pc}}\big(\bar{Y}(t)-\bar{X}_n(t)\big)\right)\cdot\nabla_y\xi\big(t,\bar{Y}(t)\big)\,\mathrm{d}t\\
&\qquad+\nabla_y\xi\big(t,\bar{Y}(t)\big)\cdot\mathrm{d}W(t)
\end{aligned}$$

with initial datum $\xi(0,\bar{Y}^0)$. This means that a.s. for every $t\in[0,T]$

$$\begin{aligned}
&\xi\big(t,\bar{Y}(t)\big)\\
&\quad=\xi\big(0,\bar{Y}^0\big)+\int_0^t\Bigg[\partial_t\xi\big(s,\bar{Y}(s)\big)+\kappa\,\Delta_y\xi\big(s,\bar{Y}(s)\big)\\
&\qquad+\left(\frac{1}{L}\sum_{\ell=1}^{L}K^{\mathrm{pg}}\big(\bar{Y}(s)-Z_\ell(s)\big)-\frac{1}{N}\sum_{n=1}^{N}K^{\mathrm{pc}}\big(\bar{Y}(s)-\bar{X}_n(s)\big)\right)\cdot\nabla_y\xi\big(s,\bar{Y}(s)\big)\Bigg]\,\mathrm{d}s\\
&\qquad+\int_0^t\nabla_y\xi\big(s,\bar{Y}(s)\big)\cdot\mathrm{d}W(s).
\end{aligned}$$

By [28, Lemma 5.4] we have that

$$\mathbb{E}\left(\int_0^t\nabla_y\xi\big(s,\bar{Y}(s)\big)\cdot\mathrm{d}W(s)\right)=0.$$

Thus, taking the expectation, we obtain in particular that

$$\begin{aligned}
&\mathbb{E}\big(\xi\big(T,\bar{Y}(T)\big)\big)\\
&\quad=\mathbb{E}\big(\xi\big(0,\bar{Y}^0\big)\big)+\int_0^T\mathbb{E}\Bigg[\partial_t\xi\big(t,\bar{Y}(t)\big)+\kappa\,\Delta_y\xi\big(t,\bar{Y}(t)\big)\\
&\qquad+\left(\frac{1}{L}\sum_{\ell=1}^{L}K^{\mathrm{pg}}\big(\bar{Y}(t)-Z_\ell(t)\big)-\frac{1}{N}\sum_{n=1}^{N}K^{\mathrm{pc}}\big(\bar{Y}(t)-\bar{X}_n(t)\big)\right)\cdot\nabla_y\xi\big(t,\bar{Y}(t)\big)\Bigg]\,\mathrm{d}t.
\end{aligned}$$

Note that the function $t\mapsto\int_{\mathbb{R}^2}\xi(t,y)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)=\int_{C^0([0,T];\mathbb{R}^2)}\xi(t,\varphi(t))\,\mathrm{d}\mu(\varphi)$ is continuous in t, *e.g.*, by the dominated convergence theorem. A solution to the PDE in the sense of distributions satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^2}\xi(0,y)\,\mathrm{dLaw}(\bar{Y}_1^0)(y)+\int_0^T\int_{\mathbb{R}^2}\Bigg[\partial_t\xi(t,y)+\kappa\,\Delta_y\xi(t,y)\\
&\quad+\left(\frac{1}{L}\sum_{\ell=1}^{L}K^{\mathrm{pg}}(y-Z_\ell(t))-\frac{1}{N}\sum_{n=1}^{N}K^{\mathrm{pc}}(y-\bar{X}_n(t))\right)\cdot\nabla_y\xi(t,y)\Bigg]\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t=0
\end{aligned}$$

for every $\xi\in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$).

Using the fact that $\bar{\mu}^{\mathrm{p}}(t) = \mathrm{Law}(\bar{Y}(t))$ and $\xi(T,\cdot) \equiv 0$, we get that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^2} \xi(T,y)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(T)(y) \\
&= \int_{\mathbb{R}^2} \xi(0,y)\,\mathrm{dLaw}\big(\bar{Y}^0\big)(y) + \int_0^T \int_{\mathbb{R}^2} \Bigg[\partial_t \xi(t,y) + \kappa\,\Delta_y \xi(t,y) \\
&\quad + \Bigg(\frac{1}{L} \sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(y - Z_\ell(t)\big) + \frac{1}{N} \sum_{n=1}^{N} K^{\mathrm{pc}}\big(y - \bar{X}_n(t)\big) \Bigg) \cdot \nabla_y \xi(t,y) \Bigg] \,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t.
\end{aligned}$$

This concludes the proof. □

7.2 Limit of optimal control problems as $M \to +\infty$

Let us consider the following cost functional for the limit problem obtained in (7.1). Let $\mathcal{J}_N : L^\infty([0,T];\mathcal{U}) \to \mathbb{R}$ be defined for every $u \in L^\infty([0,T];\mathcal{U})$ by

$$\mathcal{J}_N(u) := \frac{1}{2} \int_0^T \big|u(t)\big|^2 \,\mathrm{d}t + \frac{1}{N} \sum_{n=1}^{N} \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t, \tag{7.16}$$

where $\bar{X} = (\bar{X}_1,\dots,\bar{X}_N)$ and $(\bar{Y}(t))_{t\in[0,T]}$ are the unique strong solutions to (5.1) provided by Proposition 5.2. Notice that the definition of $\mathcal{J}_N$ does not depend on the particular initial random datum $\bar{Y}^0$ but only on its law since this is also the case for $\bar{\mu}^{\mathrm{p}}$ by Proposition 6.1.

Theorem 7.3 *Let us fix $N \geq 1$. Under the assumptions of Theorem 7.1, the sequence of functionals $(\mathcal{J}_{N,M})_{M\geq 1}$ Γ-converges to $\mathcal{J}_N$ as $M \to +\infty$ with respect to the weak* topology in $L^\infty([0,T];\mathcal{U})$.*[17]

Proof *Step 1*. (Asymptotic lower bound). Let us fix a sequence of controls $(u^M)_{M\geq 1}$, $u^M \in L^\infty([0,T];\mathcal{U})$ such that $u^M \overset{*}{\rightharpoonup} u$ weakly* in $L^\infty([0,T];\mathcal{U})$ as $M \to +\infty$. Let us show that

$$\mathcal{J}_N(u) \leq \liminf_{M\to+\infty} \mathcal{J}_{N,M}\big(u^M\big). \tag{7.17}$$

On the one hand, by Definition (3.3), we have that

$$\mathcal{J}_{N,M}\big(u^M\big) = \frac{1}{2} \int_0^T \big|u^M(t)\big|^2 \,\mathrm{d}t + \mathbb{E}\Bigg(\int_0^T \frac{1}{N} \frac{1}{M} \sum_{n=1}^{N} \sum_{m=1}^{M} H^{\mathrm{d}}\big(X_n^M(t) - Y_m^M(t)\big)\,\mathrm{d}t \Bigg),$$

where the stochastic processes $(X^M(t))_{t\in[0,T]} = (X_1^M(t),\dots,X_N^M(t))_{t\in[0,T]}$, $(Y^M(t))_{t\in[0,T]} = (Y_1^M(t),\dots,Y_M^M(t))_{t\in[0,T]}$ (and the curve $Z^M = (Z_1^M,\dots,Z_L^M)$) are the unique strong solution to (7.2). On the other hand, we have that

$$\mathcal{J}_N(u) := \frac{1}{2} \int_0^T \big|u(t)\big|^2 \,\mathrm{d}t + \frac{1}{N} \sum_{n=1}^{N} \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t,$$

[17] Note that the weak* convergence in $L^\infty([0,T];\mathcal{U})$ is metrizable since $\mathcal{U}$ is bounded, hence we can use the sequential characterization of Γ-limits, *cf.* [20, Proposition 8.1].

where the curve $\bar{X} = (\bar{X}_1, \dots, \bar{X}_N)$, the stochastic process $(\bar{Y}(t))_{t\in[0,T]}$ with law $\bar{\mu}^{\mathrm{p}}$ (and the curve $Z = (Z_1, \dots, Z_L)$) are the unique strong solution to (5.1).

By the weak sequential lower semicontinuity of the L^2-norm, we have that

$$\int_0^T |u(t)|^2 \, \mathrm{d}t \le \liminf_{M\to+\infty} \int_0^T |u^M(t)|^2 \, \mathrm{d}t.$$

Let us prove the convergence

$$\begin{aligned} &\mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n=1}^N \sum_{m=1}^M H^{\mathrm{d}}\big(X_n^M(t) - Y_m^M(t)\big)\, \mathrm{d}t\right) \\ &\quad \to \frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\, \mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\, \mathrm{d}t, \end{aligned} \tag{7.18}$$

as $M \to +\infty$. This will conclude the proof of (7.17).

We exploit the equality

$$\frac{1}{M}\sum_{m=1}^M H^{\mathrm{d}}\big(X_n^M(t) - Y_m^M(t)\big) = H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(X_n^M(t)\big)$$

to deduce that

$$\begin{aligned} &\Bigg|\mathbb{E}\left(\int_0^T \frac{1}{N}\frac{1}{M}\sum_{n=1}^N \sum_{m=1}^M H^{\mathrm{d}}\big(X_n^M(t) - Y_m^M(t)\big)\, \mathrm{d}t\right) \\ &\qquad - \frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\, \mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\, \mathrm{d}t\Bigg| \\ &\quad \le \frac{1}{N}\sum_{n=1}^N \int_0^T \big|\mathbb{E}\big(H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(X_n^M(t)\big)\big) - H^{\mathrm{d}} * \bar{\mu}^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\big|\, \mathrm{d}t \\ &\quad \le \frac{1}{N}\sum_{n=1}^N \int_0^T \mathbb{E}\big(\big|H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(X_n^M(t)\big) - H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\big|\big)\, \mathrm{d}t \\ &\qquad + \frac{1}{N}\sum_{n=1}^N \int_0^T \mathbb{E}\big(\big|H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big) - H^{\mathrm{d}} * \bar{\mu}^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\big|\big)\, \mathrm{d}t. \end{aligned} \tag{7.19}$$

We estimate the first term on the right-hand side of (7.19) by using the fact that, by the Lipschitz continuity of H^{d}, a.s. for every $t \in [0, T]$

$$\begin{aligned} &\big|H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(X_n^M(t)\big) - H^{\mathrm{d}} * \nu_M^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\big| \\ &\quad \le \int_{\mathbb{R}^2} \big|H^{\mathrm{d}}\big(X_n^M(t) - y\big) - H^{\mathrm{d}}\big(\bar{X}_n(t) - y\big)\big|\, \mathrm{d}\nu_M^{\mathrm{p}}(t)(y) \\ &\quad \le C\big|X_n^M(t) - \bar{X}_n(t)\big| \le C \max_{n'} \big\|X_{n'}^M - \bar{X}_{n'}\big\|_\infty. \end{aligned}$$

We estimate the second term on the right-hand side of (7.19) by Kantorovich's duality, which by the Lipschitz continuity of $H^{\mathrm{d}}(\bar{X}_n(t)-\cdot)$ yields a.s. for every $t\in[0,T]$

$$\begin{aligned}&\left|H^{\mathrm{d}}*\nu_M^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)-H^{\mathrm{d}}*\bar{\mu}^{\mathrm{p}}(t)\big(\bar{X}_n(t)\big)\right|\\&\quad=\left|\int_{\mathbb{R}^2}H^{\mathrm{d}}\big(\bar{X}_n(t)-y\big)\,\mathrm{d}\big(\nu_M^{\mathrm{p}}(t)-\bar{\mu}^{\mathrm{p}}(t)\big)(y)\right|\\&\quad\le C\mathcal{W}_1\big(\nu_M^{\mathrm{p}}(t),\bar{\mu}^{\mathrm{p}}(t)\big).\end{aligned}$$

Putting together the previous inequalities, we conclude that

$$\begin{aligned}&\left|\mathbb{E}\left(\int_0^T\frac{1}{N}\frac{1}{M}\sum_{n=1}^N\sum_{m=1}^M H^{\mathrm{d}}\big(X_n^M(t)-Y_m^M(t)\big)\,\mathrm{d}t\right)\right.\\&\qquad\left.-\frac{1}{N}\sum_{n=1}^N\int_0^T\int_{\mathbb{R}^2}H^{\mathrm{d}}\big(\bar{X}_n(t)-y\big)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t\right|\\&\quad\le C\mathbb{E}\Big(\max_n\big\|X_n^M-\bar{X}_n\big\|_\infty\Big)+C\int_0^T\mathbb{E}\big(\mathcal{W}_1\big(\nu_M^{\mathrm{p}}(t),\bar{\mu}^{\mathrm{p}}(t)\big)\big)\,\mathrm{d}t,\end{aligned}$$

whence (7.18) by Theorem 7.1.

Step 2. (Asymptotic upper bound). Let us fix $u\in L^\infty([0,T];\mathcal{U})$. For every $M\ge 1$, let us set $u^M=u$. As in *Step 1*, we have that

$$\mathcal{J}_{N,M}\big(u^M\big)=\frac{1}{2}\int_0^T\big|u(t)\big|^2\,\mathrm{d}t+\mathbb{E}\left(\int_0^T\frac{1}{N}\frac{1}{M}\sum_{n=1}^N\sum_{m=1}^M H^{\mathrm{d}}\big(X_n^M(t)-Y_m^M(t)\big)\,\mathrm{d}t\right),$$

where the stochastic processes $(X^M(t))_{t\in[0,T]}$, $(Y^M(t))_{t\in[0,T]}$ (and the curve Z^M) are the unique strong solution to (7.2) corresponding to the control $u^M=u$ and

$$\mathcal{J}_N(u):=\frac{1}{2}\int_0^T\big|u(t)\big|^2\,\mathrm{d}t+\frac{1}{N}\sum_{n=1}^N\int_0^T\int_{\mathbb{R}^2}H^{\mathrm{d}}\big(\bar{X}_n(t)-y\big)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}(t)(y)\,\mathrm{d}t,$$

where the curve $\bar{X}$, the stochastic process $(\bar{Y}(t))_{t\in[0,T]}$ with law $\bar{\mu}^{\mathrm{p}}$ (and the curve Z) are the unique strong solution to (5.1). Trivially, we have $u^M\overset{*}{\rightharpoonup}u$, hence we deduce (7.18) once again and, in particular, the asymptotic upper bound

$$\lim_{M\to+\infty}\mathcal{J}_{N,M}(u)=\mathcal{J}_N(u).$$

This concludes the proof. □

As a byproduct, we obtain the following result.

Proposition 7.4 *Under the assumptions of Proposition 5.2, there exists an optimal control $u^*\in L^\infty([0,T];\mathcal{U})$, i.e.,*

$$\mathcal{J}_N\big(u^*\big)=\min_{u\in L^\infty([0,T];\mathcal{U})}\mathcal{J}_N(u).$$

Proof The proof is standard in the theory of Γ-convergence. Let us consider a sequence of independent Brownian motions $(W_m(t))_{t\in[0,T]}$, $m \geq 1$ and $Y_1^0, \dots, Y_M^0$ i.i.d. random variable with the same law of $\bar{Y}^0$. Let $(u^M)_{M\geq 1}$ be a sequence such that $\mathcal{J}_{N,M}(u^M) = \inf \mathcal{J}_{N,M}$. Since $(u^M)_{M\geq 1}$ is bounded in $L^\infty([0,T];\mathcal{U})$, there exist u^* and a subsequence (not relabeled) such that $u^M \overset{*}{\rightharpoonup} u^*$ weakly-* in $L^\infty([0,T];\mathcal{U})$. By Theorem 7.3 we have that

$$\begin{aligned}\mathcal{J}_N\big(u^*\big) &\leq \liminf_{M\to+\infty} \mathcal{J}_{N,M}\big(u^M\big) \\ &= \liminf_{M\to+\infty} \inf \mathcal{J}_{N,M} \leq \limsup_{M\to+\infty} \inf \mathcal{J}_{N,M} \\ &\leq \limsup_{M\to+\infty} \mathcal{J}_{N,M}\big(u^*\big) = \mathcal{J}_N\big(u^*\big).\end{aligned}$$

(Here we used the fact that the recovery sequence for u^* is the constant sequence given by u^*, see the proof of Theorem 7.3.) □

8 Mean-field limit for a large number of commercial ships

In this section we study the limit of the problem as $N \to +\infty$. For this reason we will stress the dependence of initial data and solutions on N.

8.1 Mean-field limit as $N \to +\infty$

In this section, we will use the explicit formula for the velocity correction

$$\begin{aligned}v_n^N(X) &= v\left(\frac{1}{N-1}\sum_{n'=1}^{N} \eta(X_n, X_n - X_{n'})\right) \\ &= v\left(\frac{1}{N}\sum_{n'=1}^{N} \eta_N(X_n, X_n - X_{n'})\right),\end{aligned}$$

where we set

$$\eta_N = \frac{N}{N-1}\eta. \tag{8.1}$$

In what follows, we shall use the symbol $*_2$ to indicate that the convolution is done with respect to the second variable, *i.e.*, $\eta *_2 \nu(x) = \int_{\mathbb{R}^2} \eta(x, x - x')\, \mathrm{d}\nu(x')$.

Theorem 8.1 *Assume the following*:

- *Let* $(W(t))_{t\in[0,T]}$ *be an* $\mathbb{R}^2$*-valued Brownian motion*;
- *Let* $X^{N,0} = (X_1^0, \dots, X_N^0) \in \mathbb{R}^{2\times N}$ *and assume that* $\max_n \|X_n^{N,0}\|_\infty \leq R_0$ *with* R_0 *independent of* N;
- *Let* $\bar{Y}^0$ *be an* $\mathbb{R}^2$*-valued random variable with* $\mathbb{E}(|\bar{Y}^0|) < +\infty$;
- *Let* $Z^0 = (Z_1^0, \dots, Z_L^0) \in \mathbb{R}^{2\times L}$;
- *Let* $\mu_0^c \in \mathcal{P}_1(\mathbb{R}^2)$ *with* $\operatorname{supp}(\mu_0^c) \subset \bar{B}_{R_0}$ *be such that* $\mathcal{W}_1(\frac{1}{N}\sum_{n=1}^N \delta_{X_n^0}, \mu_0^c) \to 0$ *as* $N \to +\infty$;

Let $u^N, u \in L^\infty([0,T];\mathcal{U})$ *be such that* $u^N \overset{*}{\rightharpoonup} u$ *weakly* in* $L^\infty([0,T];\mathcal{U})$.[18] *Let* $\bar{X}^N = (\bar{X}_1^N,\dots,\bar{X}_N^N)$, $(\bar{Y}^N(t))_{t\in[0,T]}$, *and* $Z^N = (Z_1^N,\dots,Z_L^N)$ *be the unique strong solution to*[19]

$$\begin{cases} \frac{\mathrm{d}\bar{X}_n^N}{\mathrm{d}t}(t) = v_n^N(\bar{X}^N(t))(\mathbf{r}(\bar{X}_n^N(t)) + K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(\bar{X}_n^N(t))), \\ \mathrm{d}\bar{Y}^N(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}^N(t) - Z_\ell^N(t)) - \frac{1}{N}\sum_{n=1}^N K^{\mathrm{pc}}(\bar{Y}^N(t) - \bar{X}_n^N(t)))\,\mathrm{d}t \\ \qquad + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \frac{\mathrm{d}Z_\ell^N}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell^N(t) - Z_{\ell'}^N(t)) + u_\ell^N(t), \\ \bar{X}_n^N(0) = X_n^0, \qquad Z_\ell^N(0) = Z_\ell^0, \quad n = 1,\dots,N, \ell = 1,\dots,L, \\ \bar{Y}^N(0) = \bar{Y}^0 \quad a.s., \qquad \bar{\mu}_N^{\mathrm{p}} = \mathrm{Law}(\bar{Y}^N). \end{cases} \tag{8.2}$$

Let us consider the measures

$$\nu_N^{\mathrm{c}}(t) := \frac{1}{N}\sum_{n=1}^N \delta_{\bar{X}_n^N(t)}. \tag{8.3}$$

Then there exist $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$, *and* $Z = (Z_1,\dots,Z_L)$ *such that*

$$\sup_{t\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(t), \mu^{\mathrm{c}}(t)\big) + \mathbb{E}\big(\big\|\bar{Y}^N - \bar{Y}\big\|_\infty\big) + \big\|Z^N - Z\big\|_\infty \to 0 \quad as\ N \to +\infty.$$

Moreover, $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$, *and* $Z = (Z_1,\dots,Z_L)$ *provide the unique solution to*

$$\begin{cases} \partial_t \mu^{\mathrm{c}} + \mathrm{div}_x(v(\eta *_2 \mu^{\mathrm{c}})(\mathbf{r} + K^{\mathrm{cp}} * \mu^{\mathrm{p}})\mu^{\mathrm{c}}) = 0, \\ \mathrm{d}\bar{Y}(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}(t) - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(\bar{Y}(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \mu^{\mathrm{c}}(0) = \mu_0^{\mathrm{c}}, \\ \bar{Y}(0) = \bar{Y}^0 \quad a.s., \qquad \mu^{\mathrm{p}} = \mathrm{Law}(\bar{Y}), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L. \end{cases} \tag{8.4}$$

Proof Step 1. (PDE solved by the empirical measures) In terms of $\nu_N^{\mathrm{c}}(t)$, $v_n^N(\bar{X}^N(t))$ reads

$$\begin{aligned} v_n^N\big(\bar{X}^N(t)\big) &= v\left(\frac{1}{N}\sum_{n'=1}^N \eta_N\big(\bar{X}_n^N(t), \bar{X}_n^N(t) - \bar{X}_{n'}^N(t)\big)\right) \\ &= v\left(\int_{\mathbb{R}^2} \eta_N\big(\bar{X}_n^N(t), \bar{X}_n^N(t) - x'\big)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)\big(x'\big)\right) = v\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)\big(\bar{X}_n^N(t)\big)\big). \end{aligned} \tag{8.5}$$

[18] In fact, by the boundedness of $\mathcal{U}$, this is equivalent to requiring that $u^N \rightharpoonup u$ weakly in $L^1([0,T];\mathcal{U})$.

[19] This corresponds to the averaged ODE/SDE/ODE system (5.1) with initial data $X^{N,0}$, $\bar{Y}^0$, Z^0, with Brownian motion W and control u^N. The solution is provided by Proposition 5.2.

Let us derive the PDE solved by $\nu_N^c(t)$ in the sense of distributions.[20] Let us fix $\xi \in C_c^\infty((-\infty,T)\times\mathbb{R}^2)$. By (8.2) and (8.5) we have that

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{0}\int_{\mathbb{R}^2}\xi(t,x)\,\mathrm{d}\nu_N^c(0)(x)\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\xi(t,x)\,\mathrm{d}\nu_N^c(t)(x)\,\mathrm{d}t\right)\\
&= \frac{1}{N}\sum_{n=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{0}\xi\left(t,X_n^0\right)\mathrm{d}t + \int_0^T\xi\left(t,\bar{X}_n^N(t)\right)\mathrm{d}t\right)\\
&= \frac{1}{N}\sum_{n=1}^{N}\Big(\int_{-\infty}^{0}\partial_t\xi\left(t,X_n^0\right)\mathrm{d}t\\
&\quad + \int_0^T\left(\partial_t\xi\left(t,\bar{X}_n^N(t)\right) + \frac{\mathrm{d}\bar{X}_n^N}{\mathrm{d}t}(t)\cdot\nabla_x\xi\left(t,\bar{X}_n^N(t)\right)\mathrm{d}t\right)\\
&= \frac{1}{N}\sum_{n=1}^{N}\bigg(\xi\left(0,X_n^0\right) + \int_0^T\Big(\partial_t\xi\left(t,\bar{X}_n^N(t)\right) + \nu\big(\eta_N *_2 \nu_N^c(t)\big(\bar{X}_n^N(t)\big)\big)\\
&\quad \times\left(\mathbf{r}\big(\bar{X}_n^N(t)\big) + K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)\big(\bar{X}_n^N(t)\big)\right)\cdot\nabla_x\xi\left(t,\bar{X}_n^N(t)\right)\Big)\mathrm{d}t\bigg)\\
&= \int_{\mathbb{R}^2}\xi(0,x)\,\mathrm{d}\left(\frac{1}{N}\sum_{n=1}^{N}\delta_{X_n^0}\right)(x) + \int_0^T\int_{\mathbb{R}^2}\Big(\partial_t\xi(t,x) + \nu\big(\eta_N *_2 \nu_N^c(t)(x)\big)\\
&\quad \times\left(\mathbf{r}(x) + K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x)\right)\cdot\nabla_x\xi(t,x)\Big)\,\mathrm{d}\nu_N^c(t)(x)\,\mathrm{d}t.
\end{aligned}\tag{8.6}$$

This means that ν_N^c is a distributional solution to

$$\begin{cases}\partial_t\nu_N^c + \mathrm{div}_x(\nu(\eta_N *_2 \nu_N^c)(\mathbf{r} + K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}})\nu_N^c) = 0,\\ \nu_N^c(0) = \frac{1}{N}\sum_{n=1}^{N}\delta_{X_n^0}.\end{cases}\tag{8.7}$$

Step 2. (Convergence of empirical measures ν_N^c) To show the compactness of the sequence of curves $\nu_N^c \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, we rely on the Arzelà–Ascoli theorem for metric-valued functions. We split the proof in substeps.

Substep 2.1. (Equiboundedness of supports) By Remark 5.3, we have that $\max_n \|\bar{X}_n^N\|_\infty \le R$, where the constant R depending on the initial datum X^0, the final time T, $\|\nu^N\|_\infty$, $\mathbf{r}$, and K^{cp}. This implies that $\mathrm{supp}(\nu_N^c(t))$ are contained in the closed ball $\bar{B}_R$ for every $t \in [0,T]$.

Substep 2.2. (Equicontinuity) Let us prove that $\nu_N^c \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$ are equicontinuous.

We observe that the sequence $\|Z^N\|_\infty$ is bounded. Indeed, by (8.2),

$$\begin{aligned}
\left|Z_\ell^N(t)\right| &\le \left|Z_\ell^0\right| + \int_0^t\left(\left|\frac{1}{L}\sum_{\ell'=1}^{L}K^{\mathrm{gg}}\big(Z_\ell^N(s) - Z_{\ell'}^N(s)\big)\right| + \left|u_\ell^N(s)\right|\right)\mathrm{d}s\\
&\le \left|Z^0\right| + \int_0^t C\big(1 + \left|Z^N(s)\right|\big)\,\mathrm{d}s \le \left|Z^0\right| + CT + \int_0^t C\left|Z^N(s)\right|\mathrm{d}s,
\end{aligned}$$

[20] We use here the duality introduced in Footnote 16.

the constant C depending on K^{gg} and the set of admissible controls $\mathcal{U}$ (bounded). Taking the norm of Z^N and by Grönwall's inequality, we obtain that

$$\left|Z^N(t)\right| \le \left(\left|Z^0\right| + CT\right)e^{Ct} \le R',$$

where the constant R' depends on $K^{\mathrm{gg}}, \mathcal{U}$, and T.

By Remark 5.3, for every $r \in [0,T]$, we have that

$$\int_{\mathbb{R}^2} |y| \, \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(r)(y) = \mathbb{E}\left(\left|\bar{Y}^N(r)\right|\right) \le C\left(1 + \mathbb{E}\left(\left|\bar{Y}^0\right|\right)\right), \tag{8.8}$$

where the constant C depends on K^{pg}, K^{pc}, $\|Z^N\|_\infty$ (bounded by R'), $\max_n \|\bar{X}_n^N\|_\infty$ (bounded by R), T, and W. Then the Lipschitz continuity of K^{cp} and (8.8) yield

$$\begin{aligned} \left|K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(r)(x)\right| &\le \int_{\mathbb{R}^2} \left|K^{\mathrm{cp}}(x-y)\right| \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(r)(y) \\ &\le \int_{\mathbb{R}^2} \left(\left|K^{\mathrm{cp}}(0)\right| + C|x| + C|y|\right) \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(r)(y) \\ &\le C\left(1 + \mathbb{E}\left(\left|\bar{Y}^0\right|\right) + |x|\right) \le C\left(1 + |x|\right), \end{aligned} \tag{8.9}$$

where the constant C additionally depends on $\mathbb{E}(|\bar{Y}^0|)$.

By (8.2) and (8.9), for $s \le t$ and $n = 1, \dots, N$, we have that

$$\begin{aligned} \left|\bar{X}_n^N(s) - \bar{X}_n^N(t)\right| &\le \int_s^t \left|v_n^N\left(\bar{X}^N(r)\right)\left(\mathbf{r}\left(\bar{X}_n^N(r)\right) + K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(r)\left(\bar{X}_n^N(r)\right)\right)\right| \mathrm{d}r \\ &\le \int_s^t C\left(1 + \left|\bar{X}_n^N(r)\right|\right) \mathrm{d}r \le C|t-s|, \end{aligned}$$

where the constant C depends on the constant obtained in (8.9) and additionally on $\|v^N\|_\infty$ and $\mathbf{r}$. Using as a transport plan between $\nu_N^{\mathrm{c}}(s)$ and $\nu_N^{\mathrm{c}}(t)$ the measure $\gamma = \frac{1}{N}\sum_{n=1}^N \delta_{(\bar{X}_n^N(s), \bar{X}_n^N(t))}$, we obtain that

$$\mathcal{W}_1\left(\nu_N^{\mathrm{c}}(s), \nu_N^{\mathrm{c}}(t)\right) \le \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|x - x'\right| \mathrm{d}\gamma\left(x, x'\right) = \frac{1}{N}\sum_{n=1}^N \left|\bar{X}_n^N(s) - \bar{X}_n^N(t)\right| \le C|t-s|,$$

i.e., the curves $\nu_N^{\mathrm{c}} \in C^0([0,T]; \mathcal{P}_1(\bar{B}_R))$ are equi-Lipschitz with respect to the 1-Wasserstein distance.

Substep 2.3. (Compactness) Since the ball $\bar{B}_R$ is compact, the Wasserstein space $\mathcal{P}_1(\bar{B}_R)$ is compact too [39, Remark 6.19].[21] Hence the Arzelà–Ascoli theorem for continuous functions with values in a metric space guarantees the existence of a curve $\mu^{\mathrm{c}} \in C^0([0,T]; \mathcal{P}_1(\bar{B}_R))$ and a subsequence N_k such that

$$\sup_{t \in [0,T]} \mathcal{W}_1\left(\nu_{N_k}^{\mathrm{c}}(t), \mu^{\mathrm{c}}(t)\right) \to 0 \quad \text{as } N_k \to +\infty. \tag{8.10}$$

[21] In fact, the curves $\nu_N^{\mathrm{c}} \in C^0([0,T]; \mathcal{P}_1(\mathbb{R}^2))$ take values in a compact set of $\mathcal{P}_1(\mathbb{R}^2)$ independent of N even under weaker assumptions. This is the case, *e.g.*, when q-moments of $\nu_N^{\mathrm{c}}(t)$ with $q > 1$ are uniformly bounded, *i.e.*, $\sup_N \sup_t \int_{\mathbb{R}^2} |x|^q \, \mathrm{d}\nu_N^{\mathrm{c}}(t)(x) < +\infty$ for some $q > 1$ (this can be proven based on [39, Theorem 6.9]. A uniform bound on the q-moments follows from the analogous assumption on the distribution of initial data by a Grönwall inequality.

Without loss of generality, we do not relabel this subsequence and denote it simply by N. This does not affect the proof as in Theorem 8.2 we shall prove the uniqueness of solutions for the limit problem.

Step 3. (Convergence of Z^N) We let $Z = (Z_1, \dots, Z_L)$ be the unique solution to

$$\begin{cases} \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ Z_\ell(0) = Z_\ell^0. \end{cases}$$

As in *Substep 4.1*, in the proof of Theorem 7.1 we get that

$$\|Z^N - Z\|_\infty \to 0 \quad \text{as } N \to +\infty. \tag{8.11}$$

Step 4. (Convergence of $\bar{Y}^N$) Let us consider the SDE

$$\begin{cases} \mathrm{d}\bar{Y}(t) = (\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\bar{Y}(t) - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(\bar{Y}(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \bar{Y}(0) = \bar{Y}^0 \quad \text{a.s.} \end{cases} \tag{8.12}$$

We will show that $\bar{Y}^N$ converges to $\bar{Y}$.

Substep 4.1. (Well-posedness of (8.12)) There exists a unique strong solution to (8.12). Indeed, let us consider the drift

$$b(t, Y) := \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(Y - Z_\ell(t)\big) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(Y)$$

and the constant dispersion matrix $\sigma = \sqrt{2\kappa}\,\mathrm{Id}_2$, so that

$$\begin{cases} \mathrm{d}\bar{Y}(t) = b(t, \bar{Y}(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ \bar{Y}(0) = \bar{Y}^0 \quad \text{a.s.} \end{cases}$$

Let us observe that b is continuous in t and Lipschitz continuous in Y (with Lipschitz constant independent of t). Indeed, Z is a continuous curve, while by Kantorovich's duality

$$\begin{aligned} &\big|K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(Y) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(s)(Y)\big| \\ &\quad = \left|\int_{\mathbb{R}^2} K^{\mathrm{pc}}(Y - x)\,\mathrm{d}\big(\mu^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(s)\big)(x)\right| \le C\mathcal{W}_1\big(\mu^{\mathrm{c}}(t), \mu^{\mathrm{c}}(s)\big), \end{aligned} \tag{8.13}$$

and $t \mapsto \mu^{\mathrm{c}}(t)$ is a continuous curve in the Wasserstein space $\mathcal{P}_1(\mathbb{R}^2)$. Moreover, the function $Y \mapsto K^{\mathrm{pg}}(Y - Z_\ell(t))$ is Lipschitz continuous, and so is $Y \mapsto K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(Y)$ since

$$\begin{aligned} \big|K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(Y) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)\big(Y'\big)\big| &\le \int_{\mathbb{R}^2} \big|K^{\mathrm{pc}}(Y - x) - K^{\mathrm{pc}}\big(Y' - x\big)\big|\,\mathrm{d}\mu^{\mathrm{c}}(t)(x) \\ &\le \int_{\mathbb{R}^2} C\big|Y - Y'\big|\,\mathrm{d}\mu^{\mathrm{c}}(t)(x) = C\big|Y - Y'\big|. \end{aligned}$$

Moreover, we have that

$$\big|b(t, Y)\big| \le \big|b(t, 0)\big| + \big|b(t, 0) - b(t, Y)\big| \le \big|b(t, 0)\big| + C|Y|$$

and

$$\begin{aligned}\big|b(t,0)\big| &\le \frac{1}{L}\sum_{\ell=1}^{L}\big|K^{\mathrm{pg}}\big(-Z_\ell(t)\big)\big| + \big|K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(0)\big| \\ &\le C\big(1+\big|Z(t)\big|\big) + \int_{\mathbb{R}^2} C\big(1+|x|\big)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x) \le C\big(1+\|Z\|_\infty\big) + C(1+R) \le C,\end{aligned}$$

where the last inequality follows from the fact that Z is bounded and $\mu^{\mathrm{c}}(t)$ has support in the ball $\bar{B}_R(0)$ for every $t \in [0,T]$. We conclude that

$$\big|b(t,Y)\big| \le C\big(1+|Y|\big), \tag{8.14}$$

where the constant C depends on K^{pg}, K^{pc}, $\|Z\|_\infty$, R. Thus the assumptions of Proposition 2.1 are satisfied. Proposition 2.1 also gives us that

$$\mathbb{E}\big(\|\bar{Y}\|_\infty\big) \le C, \tag{8.15}$$

where the constant C depends on K^{pg}, K^{pc}, $\|Z\|_\infty$, R, $\bar{Y}^0$, T, and W.

Substep 4.2. (Convergence of $\bar{Y}^N$ to $\bar{Y}$) Let us prove that

$$\mathbb{E}\big(\big\|\bar{Y}^N - \bar{Y}\big\|_\infty\big) \to 0 \quad \text{as } N \to +\infty. \tag{8.16}$$

We start by noticing that

$$\frac{1}{N}\sum_{n=1}^{N} K^{\mathrm{pc}}\big(\bar{Y}^N(t) - \bar{X}_n^N(t)\big) = K^{\mathrm{pc}} * \nu_N^{\mathrm{c}}(t)\big(\bar{Y}^N(t)\big).$$

Hence, by (8.2), (8.12), (8.13), and by Kantorovich's duality, we have a.s. for $0 \le s \le t \le T$

$$\begin{aligned}&\big|\bar{Y}^N(s) - \bar{Y}(s)\big| \\ &\quad\le \frac{1}{L}\sum_{\ell=1}^{L}\int_0^s \big|K^{\mathrm{pg}}\big(\bar{Y}^N(r) - Z_\ell^N(r)\big) - K^{\mathrm{pg}}\big(\bar{Y}(r) - Z_\ell(r)\big)\big|\,\mathrm{d}r \\ &\qquad + \int_0^s \big|K^{\mathrm{pc}} * \nu_N^{\mathrm{c}}(r)\big(\bar{Y}^N(r)\big) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(r)\big(\bar{Y}(r)\big)\big|\,\mathrm{d}r \\ &\quad\le \int_0^s C\big|\bar{Y}^N(r) - \bar{Y}(r)\big|\,\mathrm{d}r + CT\big\|Z^N - Z\big\|_\infty \\ &\qquad + \int_0^s \big|K^{\mathrm{pc}} * \nu_N^{\mathrm{c}}(r)\big(\bar{Y}^N(r)\big) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(r)\big(\bar{Y}^N(r)\big)\big|\,\mathrm{d}r \\ &\qquad + \int_0^s \big|K^{\mathrm{pc}} * \mu^{\mathrm{c}}(r)\big(\bar{Y}^N(r)\big) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(r)\big(\bar{Y}(r)\big)\big|\,\mathrm{d}r \\ &\quad\le \int_0^s C\big|\bar{Y}^N(r) - \bar{Y}(r)\big|\,\mathrm{d}r + CT\big\|Z^N - Z\big\|_\infty + CT\sup_{r\in[0,T]}\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(r),\mu^{\mathrm{c}}(r)\big).\end{aligned}$$

Taking the supremum and the expectation, we deduce that

$$\mathbb{E}\Big(\sup_{0\le s\le t}\big|\bar{Y}^N(s)-\bar{Y}(s)\big|\Big)\le\int_0^t C\mathbb{E}\Big(\sup_{0\le r\le s}\big|\bar{Y}^N(r)-\bar{Y}(r)\big|\Big)\,\mathrm{d}s$$
$$+CT\big\|Z^N-Z\big\|_\infty+CT\sup_{r\in[0,T]}\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(r),\mu^{\mathrm{c}}(r)\big)$$

and, by Grönwall's inequality,

$$\mathbb{E}\Big(\sup_{0\le s\le t}\big|\bar{Y}^N(s)-\bar{Y}(s)\big|\Big)\le CT\Big(\big\|Z^N-Z\big\|_\infty+\sup_{r\in[0,T]}\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(r),\mu^{\mathrm{c}}(r)\big)\Big)e^{Ct}.$$

In particular,

$$\mathbb{E}\big(\big\|\bar{Y}^N-\bar{Y}\big\|_\infty\big)\le C\Big(\big\|Z^N-Z\big\|_\infty+\sup_{r\in[0,T]}\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(r),\mu^{\mathrm{c}}(r)\big)\Big),$$

the constant C depending also on T. By (8.11) and (8.10), we obtain (8.16).

Step 5. (Limit problem) With (8.10), (8.11), and (8.16) at hand, we are in a position to pass to the limit as $N\to+\infty$ in (8.7) and prove that μ^{c} is a distributional solution to

$$\begin{cases}\partial_t\mu^{\mathrm{c}}+\operatorname{div}_x(v(\eta*_2\mu^{\mathrm{c}})(\mathbf{r}+K^{\mathrm{cp}}*\mu^{\mathrm{p}})\mu^{\mathrm{c}})=0,\\ \mu^{\mathrm{c}}(0)=\mu_0^{\mathrm{c}},\end{cases}\tag{8.17}$$

i.e.,

$$\begin{aligned}0={}&\int_{\mathbb{R}^2}\xi(0,x)\,\mathrm{d}\mu_0^{\mathrm{c}}(x)\\&+\int_0^T\int_{\mathbb{R}^2}\big(\partial_t\xi(t,x)+v\big(\eta*_2\mu^{\mathrm{c}}(t)(x)\big)\\&\times\big(\mathbf{r}(x)+K^{\mathrm{cp}}*\mu^{\mathrm{p}}(t)(x)\big)\cdot\nabla_x\xi(t,x)\big)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t.\end{aligned}\tag{8.18}$$

We divide the proof in substeps.

Substep 5.1. (Convergence of initial datum term) By the Lipschitz continuity of $x\mapsto\xi(0,x)$ and by Kantorovich's duality, we have that

$$\left|\int_{\mathbb{R}^2}\xi(0,x)\,\mathrm{d}\bigg(\frac{1}{N}\sum_{n=1}^N\delta_{X_n^0}-\mu_0^{\mathrm{c}}\bigg)(x)\right|\le C\mathcal{W}_1\bigg(\frac{1}{N}\sum_{n=1}^N\delta_{X_n^0},\mu_0^{\mathrm{c}}\bigg).$$

By the assumption on the initial data, we have that $\mathcal{W}_1(\frac{1}{N}\sum_{n=1}^N\delta_{X_n^0},\mu_0^{\mathrm{c}})\to0$, hence

$$\int_{\mathbb{R}^2}\xi(0,x)\,\mathrm{d}\bigg(\frac{1}{N}\sum_{n=1}^N\delta_{X_n^0}\bigg)(x)\to\int_{\mathbb{R}^2}\xi(0,x)\,\mathrm{d}\mu_0^{\mathrm{c}}.\tag{8.19}$$

Substep 5.2. (Convergence of time-derivative term) Since $x\mapsto\partial_t\xi(t,x)$ is Lipschitz continuous with a Lipschitz constant independent of t, by Kantorovich's duality we have that

$$\left|\int_{\mathbb{R}^2}\partial_t\xi(t,x)\,\mathrm{d}\big(\nu_N^{\mathrm{c}}(t)-\mu^{\mathrm{c}}(t)\big)(x)\right|\le C\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(t),\mu^{\mathrm{c}}(t)\big)$$

for every t. By (8.10) it follows that $\int_{\mathbb{R}^2} \partial_t \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x) \to \int_{\mathbb{R}^2} \partial_t \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)$ as $N \to +\infty$ uniformly in t, thus

$$\int_0^T \int_{\mathbb{R}^2} \partial_t \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^2} \partial_t \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t. \tag{8.20}$$

Substep 5.3. (Convergence of divergence term – I) Let us show that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \nu_N^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x)\cdot\nabla_x \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\
&\quad\to \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x)\cdot\nabla_x \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t \quad \text{as } N \to +\infty.
\end{aligned} \tag{8.21}$$

We start by splitting

$$\begin{aligned}
&\bigg|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \nu_N^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x)\cdot\nabla_x \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\
&\qquad - \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x)\cdot\nabla_x \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t\bigg| \\
&\quad\le \int_0^T \int_{\mathbb{R}^2} \big|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big|\big|\mathbf{r}(x)\nabla_x \xi(t,x)\big|\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\
&\qquad + \bigg|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x)\cdot\nabla_x \xi(t,x)\,\mathrm{d}\big(\nu_N^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(t)\big)(x)\,\mathrm{d}t\bigg|.
\end{aligned} \tag{8.22}$$

By the Lipschitz continuity of ν, by (8.1), by the Lipschitz continuity of η, and by Kantorovich's duality, we have that for every $x \in \mathbb{R}^2$ and $t \in [0,T]$

$$\begin{aligned}
&\big|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big| \\
&\quad\le C\big|\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x) - \eta *_2 \mu^{\mathrm{c}}(t)(x)\big| \\
&\quad\le C\big|\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x) - \eta *_2 \nu_N^{\mathrm{c}}(t)(x)\big| + C\big|\eta *_2 \nu_N^{\mathrm{c}}(t)(x) - \eta *_2 \mu^{\mathrm{c}}(t)(x)\big| \\
&\quad\le C\int_{\mathbb{R}^2} \big|\eta_N\big(x, x-x'\big) - \eta\big(x, x-x'\big)\big|\,\mathrm{d}\nu_N^{\mathrm{c}}(t)\big(x'\big) \\
&\qquad + C\bigg|\int_{\mathbb{R}^2} \eta\big(x, x-x'\big)\,\mathrm{d}\big(\nu_N^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(t)\big)\big(x'\big)\bigg| \\
&\quad\le C\bigg(\frac{1}{N-1} + \sup_{s\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big)\bigg),
\end{aligned} \tag{8.23}$$

where the constant C depends on ν and η. Integrating in time and space and using the fact that $|\mathbf{r}(x)| \le C(1+|x|)$, thus it is bounded on the compact support of ξ, we obtain that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \big|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big|\big|\mathbf{r}(x)\nabla_x \xi(t,x)\big|\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\
&\quad\le C\bigg(\frac{1}{N-1} + \sup_{s\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big)\bigg),
\end{aligned} \tag{8.24}$$

where the constant C depends on ν, η, $\mathbf{r}$, ξ, and T.

Moreover, the function $x \mapsto \nu(\eta *_2 \mu^{\mathrm{c}}(t)(x))\mathbf{r}(x) \cdot \nabla_x \xi(t,x)$ is Lipschitz continuous with a Lipschitz constant independent of t and depending on ν, η, $\mathbf{r}$, and ξ. For $x \mapsto \nu(\eta *_2 \mu^{\mathrm{c}}(t)(x))$ satisfies the latter property since

$$\begin{aligned} &\left|\nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)\big(x'\big)\big)\right| \\ &\quad \le C\left|\eta *_2 \mu^{\mathrm{c}}(t)(x) - \eta *_2 \mu^{\mathrm{c}}(t)\big(x'\big)\right| \\ &\quad \le C\int_{\mathbb{R}^2}\left|\eta\big(x,x-x''\big) - \eta\big(x',x'-x''\big)\right| \mathrm{d}\mu^{\mathrm{c}}(t)\big(x''\big) \le C\left|x-x'\right|, \end{aligned} \tag{8.25}$$

where the constant C depends on ν and η. As above, $\mathbf{r}$ is bounded on the support of ξ. By the Lipschitz continuity of $\mathbf{r}$ and $\nabla_x \xi$, we conclude that the product $x \mapsto \nu(\eta *_2 \mu^{\mathrm{c}}(t)(x))\mathbf{r}(x) \cdot \nabla_x \xi(t,x)$ is also Lipschitz continuous. Thus by Kantorovich's duality we obtain that for every $t \in [0,T]$

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\big(\nu_N^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(t)\big)(x)\right| \\ &\quad \le C \sup_{s\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big), \end{aligned} \tag{8.26}$$

where C depends on ν, η, $\mathbf{r}$, ξ. Combining (8.22), (8.24), and (8.26), by (8.10) it follows that

$$\begin{aligned} &\left|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \nu_N^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \right.\\ &\qquad \left. - \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t\right| \\ &\quad \le \int_0^T \int_{\mathbb{R}^2} C\left|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\right| \mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\ &\qquad + \left|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\mathbf{r}(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \right.\\ &\qquad \left. - \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)\big(x'\big)\big)\mathbf{r}\big(x'\big) \cdot \nabla_x \xi\big(t,x'\big)\,\mathrm{d}\mu^{\mathrm{c}}(t)\big(x'\big)\,\mathrm{d}t\right| \\ &\quad \le C\left(\frac{1}{N-1} + \sup_{t\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(t), \mu^{\mathrm{c}}(t)\big)\right) \to 0 \quad \text{as } N \to +\infty, \end{aligned}$$

where the constant C depends on ν, η, $\mathbf{r}$, ξ, and T.

Substep 5.4. (Convergence of divergence term – II) Let us prove that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big)K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\ &\to \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x)\,\mathrm{d}t \\ &\quad \text{as } N \to +\infty. \end{aligned} \tag{8.27}$$

We start by splitting

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\, \mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\, \mathrm{d}t \right.\\
&\qquad \left. - \int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big) K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\, \mathrm{d}\mu^{\mathrm{c}}(t)(x)\, \mathrm{d}t \right| \\
&\quad \le \int_0^T \int_{\mathbb{R}^2} \big|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big| \\
&\qquad \times \big|K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\big|\, \mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\, \mathrm{d}t \\
&\qquad + \int_0^T \int_{\mathbb{R}^2} \big|\nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big| \\
&\qquad \times \big|\big(K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x) - K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x)\big) \cdot \nabla_x \xi(t,x)\big|\, \mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\, \mathrm{d}t \\
&\qquad + \left|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big) K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x) \right. \\
&\qquad \left. \times \nabla_x \xi(t,x)\, \mathrm{d}\big(\nu_N^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(t)\big)(x)\, \mathrm{d}t \right|.
\end{aligned} \tag{8.28}$$

For the first term on the right-hand side of (8.28), we argue analogously to (8.24) to obtain that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \big|\nu\big(\eta_N *_2 \nu_N^{\mathrm{c}}(t)(x)\big) - \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\big| \big|K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x) \cdot \nabla_x \xi(t,x)\big|\, \mathrm{d}\nu_N^{\mathrm{c}}(t)(x)\, \mathrm{d}t \\
&\quad \le C\left(\frac{1}{N-1} + \sup_{s\in[0,T]} \mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big)\right) \to 0 \quad \text{as } N \to +\infty,
\end{aligned}$$

where the constant C depends on ν, η, K^{cp}, ξ, and T. The only difference consists in the fact that we have $K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x)$ in place of $\mathbf{r}(x)$. For this, we need to observe that

$$\begin{aligned}
\big|K^{\mathrm{cp}} * \bar{\mu}_N^{\mathrm{p}}(t)(x)\big| &= \left|\int_{\mathbb{R}^2} K^{\mathrm{cp}}(x-y)\, \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\right| \\
&\le \int_{\mathbb{R}^2} \big(\big|K^{\mathrm{cp}}(0)\big| + C|x| + C|y|\big)\, \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y) \\
&\le C\big(1 + |x|\big).
\end{aligned} \tag{8.29}$$

In the last inequality, we used the fact that, since $\bar{\mu}_N^{\mathrm{p}}(t)$ is the law of $\bar{Y}^N(t)$,

$$\int_{\mathbb{R}^2} |y|\, \mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y) \le \mathbb{E}\big(\big\|\bar{Y}^N\big\|_\infty\big) \le C,$$

where the boundedness follows from the convergence (8.16).

For the second term on the right-hand side of (8.28), we start by observing that K^{cp} is Lipschitz, thus we have for every $x \in \mathbb{R}^2$ and $t \in [0,T]$

$$\begin{aligned}
&\left|K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}_N(t)(x) - K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x)\right| \\
&\quad = \left|\int_{\mathbb{R}^2} K^{\mathrm{cp}}(x-y)\,\mathrm{d}\bar{\mu}^{\mathrm{p}}_N(t)(y) - \int_{\mathbb{R}^2} K^{\mathrm{cp}}\big(x-y'\big)\,\mathrm{d}\mu^{\mathrm{p}}(t)\big(y'\big)\right| \\
&\quad = \left|\mathbb{E}\big(K^{\mathrm{cp}}\big(x-\bar{Y}^N(t)\big) - K^{\mathrm{cp}}\big(x-\bar{Y}(t)\big)\big)\right| \le C\mathbb{E}\big(\big|\bar{Y}^N(t)-\bar{Y}(t)\big|\big) \le \mathbb{E}\big(\big\|\bar{Y}^N-\bar{Y}\big\|_\infty\big).
\end{aligned}$$

By (8.16), it follows that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \left|\nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)\right|\left|K^{\mathrm{cp}} * \bar{\mu}^{\mathrm{p}}_N(t)(x) - K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x)\right|\left|\nabla_x\xi(t,x)\right|\,\mathrm{d}\nu^{\mathrm{c}}_N(t)(x)\,\mathrm{d}t \\
&\quad \le CT\|\nu\|_\infty\|\nabla_x\xi\|_\infty\mathbb{E}\big(\big\|\bar{Y}^N-\bar{Y}\big\|_\infty\big) \to 0 \quad \text{as } N \to +\infty.
\end{aligned}$$

For the third term on the right-hand side of (8.28), we observe that the function $x \mapsto \nu(\eta *_2 \mu^{\mathrm{c}}(t)(x))K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x) \cdot \nabla_x\xi(t,x)$ is Lipschitz continuous with a Lipschitz constant independent of t and depending on ν, η, K^{cp}, μ^{p}, and ξ. This follows from (8.25), from the fact that ξ is compactly supported, and the inequality

$$\left|K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x)\right| \le C\big(1+|x|\big)$$

obtained as in (8.29). By Kantorovich's duality,

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \nu\big(\eta *_2 \mu^{\mathrm{c}}(t)(x)\big)K^{\mathrm{cp}} * \mu^{\mathrm{p}}(t)(x) \cdot \nabla_x\xi(t,x)\,\mathrm{d}\big(\nu^{\mathrm{c}}_N(t) - \mu^{\mathrm{c}}(t)\big)(x)\,\mathrm{d}t\right| \\
&\quad \le C\mathcal{W}_1\big(\nu^{\mathrm{c}}_N(t),\mu^{\mathrm{c}}(t)\big) \le \sup_{s\in[0,T]} C\mathcal{W}_1\big(\nu^{\mathrm{c}}_N(s),\mu^{\mathrm{c}}(s)\big) \to 0 \quad \text{as } N \to +\infty,
\end{aligned}$$

by (8.10).

Substep 5.5. (Conclusion) Combining (8.6), (8.19), (8.20), (8.21), and (8.27), we conclude the proof of (8.18).

We prove the uniqueness of the solution in Theorem 8.2. □

Theorem 8.2 *Under the assumptions of Theorem* 8.1, *the solution* $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$, *and* $Z = (Z_1,\dots,Z_L)$ *to* (8.4) *is unique.*

Proof The uniqueness of Z is direct as the ODE for Z is decoupled from the first two equations.

Assume now that $\mu^{\mathrm{c}}_i \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}_i(t))_{t\in[0,T]}$ for $i = 1,2$ are solutions to (8.4) with the same initial data, *i.e.*,

$$\begin{cases}
\partial_t\mu^{\mathrm{c}}_i + \mathrm{div}_x(\nu(\eta *_2 \mu^{\mathrm{c}}_i)(\mathbf{r} + K^{\mathrm{cp}} * \mu^{\mathrm{p}}_i)\mu^{\mathrm{c}}_i) = 0, \\
\mathrm{d}\bar{Y}_i(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_i(t) - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}_i(t)(\bar{Y}_i(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\
\frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = (\frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t))\,\mathrm{d}t, \\
\mu^{\mathrm{c}}_i(0) = \mu^{\mathrm{c}}_0, \\
\bar{Y}_i(0) = \bar{Y}^0 \quad \text{a.s.}, \qquad \mu^{\mathrm{p}}_i = \mathrm{Law}(\bar{Y}_i), \\
Z_\ell(0) = Z^0_\ell, \quad \ell = 1,\dots,L,
\end{cases} \tag{8.30}$$

where $\mathrm{supp}(\mu_i^{\mathrm{c}}(0)) = \mathrm{supp}(\mu_0^{\mathrm{c}}) \subset \bar{B}_R$. As customary in uniqueness proofs for evolutionary problems, we will temporary neglect the assumption that the initial data $\bar{Y}_1(0)$ and $\bar{Y}_2(0)$ are a.s. equal and $\mu_1^{\mathrm{c}}(0)$ and $\mu_2^{\mathrm{c}}(0)$ are equal in order to carry out a Grönwall-type argument to deduce stability with respect to the initial data. The objective is to prove the following pair of estimates:

$$\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) \le C\bigg(\mathbb{E}\big(\big|\bar{Y}_1(0) - \bar{Y}_2(0)\big|\big) + \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s\bigg), \tag{8.31}$$

$$\mathcal{W}_1\big(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t)\big) \le C\big(\mathcal{W}_1\big(\mu_1^{\mathrm{c}}(0), \mu_2^{\mathrm{c}}(0)\big) + \mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big)\big). \tag{8.32}$$

These two inequalities provide uniqueness when combined. Indeed, if $\bar{Y}_1(0) = \bar{Y}^0 = \bar{Y}_2(0)$ a.s. and $\mu_1^{\mathrm{c}}(0) = \mu_0^{\mathrm{c}} = \mu_2^{\mathrm{c}}(0)$, then (8.31) simply reads

$$\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) \le C \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s.$$

Substituting into (8.32), we get that

$$\mathcal{W}_1\big(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t)\big) \le C \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s,$$

which by Grönwall's inequality yields $\mathcal{W}_1(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t)) = 0$ for all $t \in [0, T]$. Then (8.31) gives $\mathbb{E}(\|\bar{Y}_1 - \bar{Y}_2\|_\infty) = 0$.

We divide the proof of (8.31)–(8.32) into several steps.

Step 1. (Estimate of $\mathbb{E}(\|\bar{Y}_1 - \bar{Y}_2\|_\infty)$) By (8.30) we have that a.s. for $0 \le s \le t \le T$

$$\begin{aligned}
&\big|\bar{Y}_1(s) - \bar{Y}_2(s)\big| \\
&\quad\le \big|\bar{Y}_1(0) - \bar{Y}_2(0)\big| + \int_0^s \Bigg|\Bigg(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(\bar{Y}_1(r) - Z_\ell(r)\big) - K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_1(r)\big)\Bigg) \\
&\qquad - \Bigg(\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(\bar{Y}_2(r) - Z_\ell(r)\big) - K^{\mathrm{pc}} * \mu_2^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big)\Bigg)\Bigg|\,\mathrm{d}r \\
&\quad\le \big|\bar{Y}_1(0) - \bar{Y}_2(0)\big| + \frac{1}{L}\sum_{\ell=1}^{K}\int_0^s \big|K^{\mathrm{pg}}\big(\bar{Y}_1(r) - Z_\ell(r)\big) - K^{\mathrm{pg}}\big(\bar{Y}_2(r) - Z_\ell(r)\big)\big|\,\mathrm{d}r \\
&\qquad + \int_0^s \big|K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_1(r)\big) - K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big)\big|\,\mathrm{d}r \\
&\qquad + \int_0^s \big|K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big) - K^{\mathrm{pc}} * \mu_2^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big)\big|\,\mathrm{d}r.
\end{aligned} \tag{8.33}$$

The first integrand in (8.33) is bounded using the Lipschitz continuity of K^{pg} by

$$\big|K^{\mathrm{pg}}\big(\bar{Y}_1(r) - Z_\ell(r)\big) - K^{\mathrm{pg}}\big(\bar{Y}_2(r) - Z_\ell(r)\big)\big| \le C\big|\bar{Y}_1(r) - \bar{Y}_2(r)\big|. \tag{8.34}$$

The second integrand in (8.33) is estimated using the Lipschitz continuity of K^{pc} as follows:

$$\begin{aligned}&\left|K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_1(r)\big) - K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big)\right| \\ &\quad \le \int_{\mathbb{R}^2} \left|K^{\mathrm{pc}}\big(\bar{Y}_1(r) - x\big) - K^{\mathrm{pc}}\big(\bar{Y}_2(r) - x\big)\right| \,\mathrm{d}\mu_1^{\mathrm{c}}(r)(x) \le C\left|\bar{Y}_1(r) - \bar{Y}_2(r)\right|.\end{aligned} \tag{8.35}$$

The third integrand in (8.33) is estimated using the Lipschitz continuity of K^{pc} by Kantorovich's duality

$$\begin{aligned}&\left|K^{\mathrm{pc}} * \mu_1^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big) - K^{\mathrm{pc}} * \mu_2^{\mathrm{c}}(r)\big(\bar{Y}_2(r)\big)\right| \\ &\quad = \left|\int_{\mathbb{R}^2} K^{\mathrm{pc}}\big(\bar{Y}_2(r) - x\big)\,\mathrm{d}\big(\mu_1^{\mathrm{c}}(r) - \mu_2^{\mathrm{c}}(r)\big)(x)\right| \\ &\quad \le C\mathcal{W}_1\big(\mu_1^{\mathrm{c}}(r), \mu_2^{\mathrm{c}}(r)\big).\end{aligned} \tag{8.36}$$

Combining (8.33)–(8.36), we infer that a.s. for $0 \le s \le t \le T$

$$\left|\bar{Y}_1(s) - \bar{Y}_2(s)\right| \le \left|\bar{Y}_1(0) - \bar{Y}_2(0)\right| + \int_0^s \left|\bar{Y}_1(r) - \bar{Y}_2(r)\right| \mathrm{d}r + C\int_0^s \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(r), \mu_2^{\mathrm{c}}(r)\big)\,\mathrm{d}r.$$

Taking the supremum in s and the expectation, we deduce that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}\left|\bar{Y}_1(s) - \bar{Y}_2(s)\right|\Big) &\le \mathbb{E}\big(\left|\bar{Y}_1(0) - \bar{Y}_2(0)\right|\big) + C\int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s \\ &\quad + C\int_0^t \sup_{0\le r\le s} \mathbb{E}\big(\left|\bar{Y}_1(r) - \bar{Y}_2(r)\right|\big)\,\mathrm{d}r.\end{aligned}$$

By Grönwall's inequality,

$$\mathbb{E}\Big(\sup_{0\le s\le t}\left|\bar{Y}_1(s) - \bar{Y}_2(s)\right|\Big) \le \left(\mathbb{E}\big(\left|\bar{Y}_1(0) - \bar{Y}_2(0)\right|\big) + C\int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s\right)e^{Ct}$$

for $t \in [0, T]$ and, in particular,

$$\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) \le C\left(\mathbb{E}\big(\left|\bar{Y}_1(0) - \bar{Y}_2(0)\right|\big) + \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s\right)$$

for $t \in [0, T]$, where the constant C also depends on T.

Step 2. (Introducing the flow for the transport equation) Following an idea in [35, 36], we prove uniqueness by regarding the solutions of the transport equation from a Lagrangian point of view. Let us consider for every $x \in \operatorname{supp}(\mu_i^{\mathrm{c}}(0))$ the flow

$$\begin{cases}\partial_t \Phi_i(t,x) = \nu(\eta *_2 \mu_i^{\mathrm{c}}(t)(\Phi_i(t,x)))(\mathbf{r}(\Phi_i(t,x)) + K^{\mathrm{cp}} * \mu_i^{\mathrm{p}}(t)(\Phi_i(t,x))), \\ \Phi_i(0,x) = x.\end{cases} \tag{8.37}$$

Then $\mu_i^{\mathrm{c}}(t) = \Phi_i(t,\cdot)_\# \mu_i^{\mathrm{c}}(0)$, see [38, Theorem 5.34].

Let us show that the flows Φ_i are bounded. We notice that

$$\begin{aligned}\left|K^{\mathrm{cp}}*\mu_i^{\mathrm{p}}(t)(X)\right| &\le \int_{\mathbb{R}^2}\left(\left|K^{\mathrm{cp}}(0)\right| + \left|K^{\mathrm{cp}}(X-y)-K^{\mathrm{cp}}(0)\right|\right)\mathrm{d}\mu_i^{\mathrm{p}}(t)(y)\\ &\le \int_{\mathbb{R}^2} C\big(1+|X|+|y|\big)\,\mathrm{d}\mu_i^{\mathrm{p}}(t)(y) \le C\big(1+|X|+\mathbb{E}\big(\|\bar{Y}_i\|_\infty\big)\big)\\ &\le C\big(1+|X|\big),\end{aligned}$$

where we used bound (8.15). By (8.37) and by the estimate $|r(X)| \le C(1+|X|)$, we deduce that for every $x \in \bar{B}_R$

$$\begin{aligned}\left|\Phi_i(t,x)\right| &\le |x| + \int_0^t \|\nu\|_\infty\left(\left|\mathbf{r}\big(\Phi_i(s,x)\big)\right| + \left|K^{\mathrm{cp}}*\mu_i^{\mathrm{p}}(s)\big(\Phi_i(s,x)\big)\right|\right)\\ &\le |x| + \int_0^t C\big(1+\left|\Phi_i(s,x)\right|\big)\,\mathrm{d}s = |x| + Ct + \int_0^t C\left|\Phi_i(s,x)\right|\mathrm{d}s.\end{aligned}$$

By Grönwall's inequality and since $x \in \bar{B}_R$, we obtain that

$$\left|\Phi_i(t,x)\right| \le \big(|x|+Ct\big)e^{Ct} \le (R+CT)e^{CT} \le C \quad \text{for } t\in[0,T], \tag{8.38}$$

where the constant C depends on $\|\nu\|_\infty$, $\mathbf{r}$, K^{cp}, R, and T (in addition to the constant in (8.15)).

In what follows we will show that

$$\begin{aligned}&\left|\Phi_1(t,x)-\Phi_2\big(t,x'\big)\right|\\ &\quad\le C\left|x-x'\right| + C\left(\int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{p}}(s),\mu_2^{\mathrm{p}}(s)\big)\,\mathrm{d}s + \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s),\mu_2^{\mathrm{c}}(s)\big)\,\mathrm{d}s\right)\end{aligned} \tag{8.39}$$

for $x, x' \in \bar{B}_R$ and $t\in[0,T]$.

We start by observing that

$$\left|\Phi_1(t,x)-\Phi_2\big(t,x'\big)\right| \le \left|\Phi_1(t,x)-\Phi_2(t,x)\right| + \left|\Phi_2(t,x)-\Phi_2\big(t,x'\big)\right| \tag{8.40}$$

for every $x, x' \in \mathbb{R}^2$ and $t \in [0,T]$.

<u>*Step 3*</u>. (Estimate of $|\Phi_1(t,x)-\Phi_2(t,x)|$) We estimate the first term on the right-hand side of (8.40) as follows:

$$\begin{aligned}&\left|\Phi_1(t,x)-\Phi_2(t,x)\right|\\ &\quad= \left|\int_0^t \nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big)\big(\mathbf{r}\big(\Phi_1(s,x)\big) + K^{\mathrm{cp}}*\mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big)\big)\,\mathrm{d}s\right.\\ &\qquad\left.-\int_0^t \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\big(\mathbf{r}\big(\Phi_2(s,x)\big) + K^{\mathrm{cp}}*\mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big)\,\mathrm{d}s\right|\\ &\quad\le \int_0^t \left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right|\left|\mathbf{r}\big(\Phi_1(s,x)\big)\right|\mathrm{d}s\\ &\qquad+\int_0^t \left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right|\end{aligned} \tag{8.41}$$

$$\begin{aligned}
&\times \left|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big)\right| \mathrm{d}s \\
&+ \int_0^t \|\nu\|_\infty \left|\mathbf{r}\big(\Phi_1(s,x)\big) - \mathbf{r}\big(\Phi_2(s,x)\big)\right| \mathrm{d}s \\
&+ \int_0^t \|\nu\|_\infty \left|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big) - K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\right| \mathrm{d}s \\
&+ \int_0^t \|\nu\|_\infty \left|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big) - K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\right| \mathrm{d}s.
\end{aligned}$$

In the following substeps we estimate the five terms on the right-hand side of (8.41).

Substep 3.1. Let us estimate the first term on the right-hand side of (8.41). For $x \in \operatorname{supp}(\mu_i^{\mathrm{c}}(0))$ and $s \in [0,T]$, we split

$$\begin{aligned}
&\left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right| \\
&\quad \le \left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right| \\
&\qquad + \left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right|.
\end{aligned} \tag{8.42}$$

We exploit the Lipschitz continuity of ν and η to obtain that

$$\begin{aligned}
&\left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right| \\
&\quad \le C\left|\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big) - \eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\right| \le \left|\Phi_1(s,x) - \Phi_2(s,x)\right|.
\end{aligned} \tag{8.43}$$

Moreover, we use the Lipschitz continuity of $x' \mapsto \eta(\Phi_2(s,x), \Phi_2(s,x) - x')$ and Kantorovich's duality to deduce that

$$\begin{aligned}
&\left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right| \\
&\quad \le C\left|\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big) - \eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\right| \\
&\quad \le C\left|\int_{\mathbb{R}^2} \eta\big(\Phi_2(s,x), \Phi_2(s,x) - x'\big)\, \mathrm{d}\big(\mu_1^{\mathrm{c}}(s) - \mu_2^{\mathrm{c}}(s)\big)(x')\right| \\
&\quad \le C\mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big).
\end{aligned} \tag{8.44}$$

By (8.38) we have that for $x \in \bar{B}_R$ and $t \in [0,T]$

$$\left|\mathbf{r}\big(\Phi_1(s,x)\big)\right| \le C\big(1 + \left|\Phi_1(s,x)\right|\big) \le C. \tag{8.45}$$

By (8.42)–(8.45) we get that for every $x \in \bar{B}_R$ and $t \in [0,T]$

$$\begin{aligned}
&\int_0^t \left|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\right| \left|\mathbf{r}\big(\Phi_1(s,x)\big)\right| \mathrm{d}s \\
&\quad \le C\int_0^t \Big(\left|\Phi_1(s,x) - \Phi_2(s,x)\right| + \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\Big)\, \mathrm{d}s.
\end{aligned} \tag{8.46}$$

Substep 3.2. Let us estimate the second term on the right-hand side of (8.41). By the Lipschitz continuity of K^{cp}, we observe that for $x \in \bar{B}_R$ and $s \in [0,T]$

$$\begin{aligned}\big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big)\big| &\leq \int_{\mathbb{R}^2} \big|K^{\mathrm{cp}}\big(\Phi_1(s,x)-y\big)\big| \,\mathrm{d}\mu_1^{\mathrm{p}}(s)(y) \\ &\leq \int_{\mathbb{R}^2} \big(\big|K^{\mathrm{cp}}(0)\big| + C\big|\Phi_1(s,x)\big| + C|y|\big)\,\mathrm{d}\mu_1^{\mathrm{p}}(s)(y) \\ &\leq C\big(1+\big|\Phi_1(s,x)\big|\big) \leq C,\end{aligned}$$

where we used (8.15) and (8.38). Then, as for (8.46), we have that

$$\begin{aligned}&\int_0^t \big|\nu\big(\eta *_2 \mu_1^{\mathrm{c}}(s)\big(\Phi_1(s,x)\big)\big) - \nu\big(\eta *_2 \mu_2^{\mathrm{c}}(s)\big(\Phi_2(s,x)\big)\big)\big|\big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big)\big|\,\mathrm{d}s \\ &\quad\leq C\int_0^t \big(\big|\Phi_1(s,x)-\Phi_2(s,x)\big| + \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s),\mu_2^{\mathrm{c}}(s)\big)\big)\,\mathrm{d}s.\end{aligned} \tag{8.47}$$

Substep 3.3. Let us estimate the third term on the right-hand side of (8.41). By the Lipschitz continuity of $\mathbf{r}$, we get that

$$\int_0^t \|\nu\|_\infty \big|\mathbf{r}\big(\Phi_1(s,x)\big) - \mathbf{r}\big(\Phi_2(s,x)\big)\big|\,\mathrm{d}s \leq C\int_0^t \big|\Phi_1(s,x)-\Phi_2(s,x)\big|\,\mathrm{d}s. \tag{8.48}$$

Substep 3.4. Let us estimate the fourth term on the right-hand side of (8.41). We exploit the Lipschitz continuity of K^{cp} to deduce that

$$\begin{aligned}&\big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big) - K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big| \\ &\quad\leq \int_{\mathbb{R}^2} \big|K^{\mathrm{cp}}\big(\Phi_1(s,x)-y\big) - K^{\mathrm{cp}}\big(\Phi_2(s,x)-y\big)\big|\,\mathrm{d}\mu_1^{\mathrm{p}}(s) \leq C\big|\Phi_1(s,x)-\Phi_2(s,x)\big|,\end{aligned}$$

from which it follows that

$$\begin{aligned}&\int_0^t \|\nu\|_\infty \big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_1(s,x)\big) - K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big|\,\mathrm{d}s \\ &\quad\leq C\int_0^t \big|\Phi_1(s,x)-\Phi_2(s,x)\big|\,\mathrm{d}s.\end{aligned} \tag{8.49}$$

Substep 3.5. Let us estimate the fifth term on the right-hand side of (8.41). By the Lipschitz continuity of $y \mapsto K^{\mathrm{cp}}(\Phi_2(s,x)-y)$, we have that

$$\begin{aligned}&\big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big) - K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big| \\ &\quad= \left|\int_{\mathbb{R}^2} K^{\mathrm{cp}}\big(\Phi_2(s,x)-y\big)\,\mathrm{d}\big(\mu_1^{\mathrm{p}}(s)-\mu_2^{\mathrm{p}}(s)\big)(y)\right| \\ &\quad\leq \mathbb{E}\big(\big|K^{\mathrm{cp}}\big(\Phi_2(s,x)-\bar{Y}_1(s)\big) - K^{\mathrm{cp}}\big(\Phi_2(s,x)-\bar{Y}_2(s)\big)\big|\big) \\ &\quad\leq C\mathbb{E}\big(\big|\bar{Y}_1(s)-\bar{Y}_2(s)\big|\big) \\ &\quad\leq C\mathbb{E}\big(\|\bar{Y}_1-\bar{Y}_2\|_\infty\big),\end{aligned}$$

from which we infer that

$$\int_0^t \|v\|_\infty \big|K^{\mathrm{cp}} * \mu_1^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big) - K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big| \, \mathrm{d}s \le C\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big), \tag{8.50}$$

the constant C depending also on T.

Substep 3.6. Combining (8.41), (8.46), (8.47), (8.48), (8.49), and (8.50), we obtain that

$$\begin{aligned}\big|\Phi_1(t,x) - \Phi_2(t,x)\big| &\le C\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) + C\int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\, \mathrm{d}s \\ &\quad + C\int_0^t \big|\Phi_1(s,x) - \Phi_2(s,x)\big| \, \mathrm{d}s.\end{aligned}$$

By Grönwall's inequality this yields

$$\big|\Phi_1(t,x) - \Phi_2(t,x)\big| \le Ce^{Ct}\bigg(\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) + \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\, \mathrm{d}s\bigg)$$

for $t \in [0,T]$ and, in particular,

$$\big|\Phi_1(t,x) - \Phi_2(t,x)\big| \le C\bigg(\mathbb{E}\big(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\big) + \int_0^t \mathcal{W}_1\big(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\big)\, \mathrm{d}s\bigg) \tag{8.51}$$

for $t \in [0,T]$ with C depending also on T.

Step 4. (Estimate of $|\Phi_2(t,x) - \Phi_2(t,x')|$) Let us estimate the second term on the right-hand side of (8.40). By (8.37), we have that

$$\begin{aligned}&\big|\Phi_2(t,x) - \Phi_2\big(t,x'\big)\big| \le \big|x - x'\big| \\ &\quad + \bigg|\int_0^t v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2(s,x)\big)\big)\big(\mathbf{r}\big(\Phi_2(s,x)\big) + K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big)\, \mathrm{d}s \\ &\quad - \int_0^t v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2\big(s,x'\big)\big)\big)\big(\mathbf{r}\big(\Phi_2\big(s,x'\big)\big) + K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2\big(s,x'\big)\big)\big)\, \mathrm{d}s\bigg| \\ &\le \big|x - x'\big| + \int_0^t \big|v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2(s,x)\big)\big) - v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2\big(s,x'\big)\big)\big)\big|\big|\mathbf{r}\big(\Phi_2(s,x)\big)\big|\, \mathrm{d}s \\ &\quad + \int_0^t \big|v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2(s,x)\big)\big) - v\big(\eta *_2 \mu_2^{\mathrm{c}}(t)\big(\Phi_2\big(s,x'\big)\big)\big)\big|\big|K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big)\big|\, \mathrm{d}s \\ &\quad + \int_0^t \|v\|_\infty \big|\mathbf{r}\big(\Phi_2(s,x)\big) - \mathbf{r}\big(\Phi_2\big(s,x'\big)\big)\big|\, \mathrm{d}s \\ &\quad + \int_0^t \|v\|_\infty \big|K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2(s,x)\big) - K^{\mathrm{cp}} * \mu_2^{\mathrm{p}}(s)\big(\Phi_2\big(s,x'\big)\big)\big|\, \mathrm{d}s.\end{aligned}$$

Reasoning similarly to *Step 2* (*i.e.*, exploiting the Lipschitz continuity of v, η, $\mathbf{r}$, and K^{cp}), one shows that for $x, x' \in \bar{B}_R$ and $t \in [0,T]$

$$\big|\Phi_2(t,x) - \Phi_2\big(t,x'\big)\big| \le \big|x - x'\big| + C\int_0^t \big|\Phi_2(s,x) - \Phi_2\big(s,x'\big)\big|\, \mathrm{d}s,$$

which by Grönwall's inequality yields $|\Phi_2(t,x) - \Phi_2(t,x')| \leq |x - x'|e^{Ct}$ for $x, x' \in \bar{B}_R$ and $t \in [0,T]$ and, in particular,

$$\left|\Phi_2(t,x) - \Phi_2\left(t,x'\right)\right| \leq C\left|x - x'\right| \tag{8.52}$$

for $x, x' \in \bar{B}_R$, where the constant C depends also on T.

Putting together (8.40), (8.51), and (8.52), we conclude that (8.39) holds true.

Step 5. (Estimate of $\mathcal{W}_1(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t))$) Let $\gamma \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ be an optimal transport plan satisfying $\pi^i_{\#}\gamma = \mu_i^{\mathrm{c}}(0)$ and

$$\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(0), \mu_2^{\mathrm{c}}(0)\right) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|x - x'\right| \mathrm{d}\gamma\left(x,x'\right). \tag{8.53}$$

We observe that since $\mu_1^{\mathrm{c}}(0)$ and $\mu_2^{\mathrm{c}}(0)$ have both supports contained in the closed ball $\bar{B}_R$, the measure γ is also concentrated on a set contained in $\bar{B}_R \times \bar{B}_R$, see [39, Theorem 5.10-(ii)-(e)]. We consider the map $(x,x') \mapsto (\Phi_1(t,\pi^1(x,x')), \Phi_2(t,\pi^2(x,x')))$ and the transport plan $(\Phi_1(t,\pi^1), \Phi_2(t,\pi^2))_{\#}\gamma$, observing that it has marginals $\mu_1^{\mathrm{c}}(t)$ and $\mu_2^{\mathrm{c}}(t)$ since we have that $\pi^i_{\#}(\Phi_1(t,\pi^1), \Phi_2(t,\pi^2))_{\#}\gamma = \Phi_i(t,\cdot)_{\#}\pi^i_{\#}\gamma = \Phi_i(t,\cdot)_{\#}\mu_i^{\mathrm{c}}(0) = \mu_i^{\mathrm{c}}(t)$. From (8.39) and (8.53) it follows that

$$\begin{aligned}
&\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t)\right) \\
&\quad\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|X - X'\right| \mathrm{d}\left(\left(\Phi_1\left(t,\pi^1\right), \Phi_2\left(t,\pi^2\right)\right)_{\#}\gamma\right)\left(X,X'\right) \\
&\quad= \int_{\bar{B}_R \times \bar{B}_R} \left|\Phi_1(t,x) - \Phi_2\left(t,x'\right)\right| \mathrm{d}\gamma\left(x,x'\right) \\
&\quad\leq C\int_{\bar{B}_R \times \bar{B}_R} \left|x - x'\right| \mathrm{d}\gamma\left(x,x'\right) + C\mathbb{E}\left(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\right) + C\int_0^t \mathcal{W}_1\left(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\right) \mathrm{d}s \\
&\quad= C\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(0), \mu_2^{\mathrm{c}}(0)\right) + C\mathbb{E}\left(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\right) + C\int_0^t \mathcal{W}_1\left(\mu_1^{\mathrm{c}}(s), \mu_2^{\mathrm{c}}(s)\right) \mathrm{d}s.
\end{aligned}$$

By Grönwall's inequality

$$\begin{aligned}
&\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(t), \mu_2^{\mathrm{c}}(t)\right) \\
&\quad\leq Ce^{Ct}\left(\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(0), \mu_2^{\mathrm{c}}(0)\right) + \mathbb{E}\left(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\right)\right) \\
&\quad\leq C\left(\mathcal{W}_1\left(\mu_1^{\mathrm{c}}(0), \mu_2^{\mathrm{c}}(0)\right) + \mathbb{E}\left(\|\bar{Y}_1 - \bar{Y}_2\|_\infty\right)\right)
\end{aligned}$$

for $t \in [0,T]$, where the constant C also depends on T. This concludes the proof of (8.32) and of the theorem. □

Proposition 8.3 *Under the assumptions of Theorem 8.1, the curve* $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, *the law* $\mu^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$, *and the curve* $Z = (Z_1, \dots, Z_L)$ *from* (8.4) *are solutions to* *the ODE/PDE/ODE system*

$$\begin{cases} \partial_t \mu^{\mathrm{c}} + \mathrm{div}_x(\nu(\eta *_2 \mu^{\mathrm{c}})(\mathbf{r} + K^{\mathrm{cp}} * \mu^{\mathrm{p}})\mu^{\mathrm{c}}) = 0, \\ \partial_t \mu^{\mathrm{p}} - \kappa\, \Delta_y \mu^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\cdot - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}})\mu^{\mathrm{p}}) = 0, \\ \mathrm{d}Z_\ell(t) = (\frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t))\,\mathrm{d}t, \\ \mu^{\mathrm{c}}(0) = \mu^{\mathrm{c}}_0, \\ \mu^{\mathrm{p}}(0) = \mu^{\mathrm{p}}_0, \\ Z_\ell(0) = Z^0_\ell, \quad \ell = 1, \dots, L, \end{cases} \tag{8.54}$$

where the PDEs are understood in the sense of distributions and μ^{p}_0 *is the law of* $\bar{Y}^0$.

Proof The proof consists in deriving the PDE solved by μ^{p} using Itô's lemma and is obtained as in the proof of Proposition 7.2 *mutatis mutandis*. □

Theorem 8.4 *Under the assumptions of Theorem* 8.1 *and further assuming that*:

- $\mu^{\mathrm{p}}_0 \in \mathcal{P}_2(\mathbb{R}^2)$;
- μ^{p}_0 *has finite entropy, i.e.,* $\mu^{\mathrm{p}}_0 = \rho^{\mathrm{p}}_0(x)\,\mathrm{d}x$ *for some function* $\rho^{\mathrm{p}}_0 \in L^1(\mathbb{R}^2)$ *satisfying* $\int_{\mathbb{R}^2} \rho^{\mathrm{p}}_0(x) \log(\rho^{\mathrm{p}}_0(x))\,\mathrm{d}x < +\infty$;
- $\mu^{\mathrm{p}}_0 = \mathrm{Law}(\widetilde{Y}^0)$;

the solution to (8.54) *is unique.*

Proof Step 1. Let us fix $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$ and $Z = (Z_1, \dots, Z_L) \in C^0([0,T];\mathbb{R}^2)$ solution to the ODE in (8.54) involving Z. We start by observing that the PDE

$$\begin{cases} \partial_t \mu^{\mathrm{p}} - \kappa\, \Delta_y \mu^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\cdot - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}})\mu^{\mathrm{p}}) = 0, \\ \mu^{\mathrm{p}}(0) = \mu^{\mathrm{p}}_0, \end{cases} \tag{8.55}$$

has at most one solution $\mu^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\bar{B}_R))$, where $\bar{B}_R$ is a closed ball of radius $R > 0$. As done in [9, Theorem 3.7], we apply the result [10, Theorem 3.3]. To check all the conditions, let us write the PDE using the same notation of [10]. Let $A(t,y) = \mathrm{Id}_2$ and $b(t,y) = \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(y - Z_\ell(t)) - K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(y)$. Let us define the operator $\mathcal{L}\xi = \partial_t \xi + \mathrm{tr}(A\nabla^2\xi) + b \cdot \nabla\xi$. If $\mu^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$ solves (8.55), then it is a Radon measure[22] on $(0,T)\times\mathbb{R}^2$ solving $\mathcal{L}^*\mu^{\mathrm{p}} = 0$, *i.e.*, $\int_{(0,T)\times\mathbb{R}^2} \mathcal{L}\xi\,\mathrm{d}\mu^{\mathrm{p}} = 0$ for every $\xi \in C^\infty_c((0,T)\times\mathbb{R}^2)$. Trivially, A is bounded and Lipschitz in the y variable. By the Lipschitz continuity of K^{pg} and the boundedness of Z,

$$\left| \frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}\big(y - Z_\ell(t)\big) \right| \le C\frac{1}{L}\sum_{\ell=1}^{L} \big|y - Z_\ell(t)\big| \le C|y| + C\|Z\|_\infty \le C\big(1 + |y|\big).$$

Moreover, by the Lipschitz continuity of K^{pc},

$$\big|K^{\mathrm{pc}} * \mu^{\mathrm{c}}(t)(y)\big| \le \int_{\mathbb{R}^2} \big|K^{\mathrm{pc}}(y - x)\big|\,\mathrm{d}\mu^{\mathrm{c}}(t)(x) \le \int_{\mathbb{R}^2} C\big(|y| + |x|\big)\,\mathrm{d}\mu^{\mathrm{c}}(t)(x) \le C\big(1 + |y|\big),$$

[22] Indeed, μ^{p} can be seen as a Radon measure on $(0,T)\times\mathbb{R}^2$ since the duality $\xi \in C^0_c((0,T)\times\mathbb{R}^2) \mapsto \int_0^T \int_{\mathbb{R}^2} \xi(t,y)\mu^{\mathrm{p}}(t)(y)\,\mathrm{d}t$ is a linear and continuous operator, *cf.* [8, Corollary 1.55].

where the constant C also depends on R. In conclusion,

$$\left|b(t,y)\cdot y\right| \le C\left(1+|y|^2\right).$$

By the assumption $\mu_0^{\mathrm{p}} \in \mathcal{P}_2(\mathbb{R}^2)$, we have that $\int_{\mathbb{R}^2} |y|^2 \,\mathrm{d}\mu_0^{\mathrm{p}}(y) < +\infty$. Finally, the initial condition is attained also in the sense

$$\lim_{t\to 0}\int_{\mathbb{R}^2} \xi(y)\,\mathrm{d}\mu^{\mathrm{p}}(t)(y) = \int_{\mathbb{R}^2} \xi(y)\,\mathrm{d}\mu_0^{\mathrm{p}}(y)$$

since $t \in [0,T] \mapsto \int_{\mathbb{R}^2} \xi(y)\,\mathrm{d}\mu^{\mathrm{p}}(t)(y)$ is a continuous function, *cf.* also Footnote 16.

By [10, Theorem 3.3] we conclude that there is at most one family $(\mu^{\mathrm{p}}(t))_{t\in[0,T]}$ that solves (8.55).

Step 2. Let now $\mu_i^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $\mu_i^{\mathrm{p}} \in \mathcal{P}_1(C^0([0,T];\mathbb{R}^2))$, $i = 1,2$ (and $Z = (Z_1,\dots,Z_L) \in C^0([0,T];\mathbb{R}^2)$) be two solutions of (8.54), *i.e.*, for $i = 1,2$

$$\begin{cases} \partial_t\mu_i^{\mathrm{c}} + \mathrm{div}_x(\nu(\eta *_2 \mu_i^{\mathrm{c}})(\mathbf{r} + K^{\mathrm{cp}} * \mu_i^{\mathrm{p}})\mu_i^{\mathrm{c}}) = 0, \\ \partial_t\mu_i^{\mathrm{p}} - \kappa\,\Delta_y\mu_i^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\cdot - Z_\ell(t)) - K^{\mathrm{pc}} * \mu_i^{\mathrm{c}})\mu_i^{\mathrm{p}}) = 0, \\ \mathrm{d}Z_\ell(t) = (\frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t))\,\mathrm{d}t, \\ \mu_i^{\mathrm{c}}(0) = \mu_0^{\mathrm{c}}, \\ \mu_i^{\mathrm{p}}(0) = \mu_0^{\mathrm{p}}, \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L. \end{cases} \tag{8.56}$$

Let $\bar{Y}^0$ be an $\mathbb{R}^2$-valued random variable with law $\mu_1^{\mathrm{p}}(0) = \mu_2^{\mathrm{p}}(0) = \mu_0^{\mathrm{p}}$, and let us consider for $i = 1,2$ the solutions to

$$\begin{cases} \mathrm{d}\bar{Y}_i(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_i(t) - Z_\ell(t)) - K^{\mathrm{pc}} * \mu_i^{\mathrm{c}}(t)(\bar{Y}_i(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \bar{Y}_i(0) = \bar{Y}^0 \quad \text{a.s.}, \end{cases} \tag{8.57}$$

which exist and are unique by *Substep 4.1* in the proof of Theorem 8.1. Let us set $\bar{\mu}_i^{\mathrm{p}} = \mathrm{Law}(\bar{Y}_i)$. (Notice the temporary difference between $\bar{\mu}_i^{\mathrm{p}}$ and μ_i^{p}.) By deriving the PDE solved by the law $\bar{\mu}_i^{\mathrm{p}}$ using Itô's lemma (see the proof of Proposition 7.2), we obtain that for $i = 1,2$

$$\begin{cases} \partial_t\bar{\mu}_i^{\mathrm{p}} - \kappa\,\Delta_y\bar{\mu}_i^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\cdot - Z_\ell(t)) - K^{\mathrm{pc}} * \mu_i^{\mathrm{c}})\bar{\mu}_i^{\mathrm{p}}) = 0, \\ \bar{\mu}_i^{\mathrm{p}}(0) = \mu_0^{\mathrm{p}}. \end{cases}$$

However, by (8.56) and by the uniqueness proven in *Step 1*, we deduce that $\bar{\mu}_i^{\mathrm{p}} = \mu_i^{\mathrm{p}}$ for $i = 1,2$. Combining (8.56) and (8.57), we obtain that, for $i = 1,2$,

$$\begin{cases} \partial_t\mu_i^{\mathrm{c}} + \mathrm{div}_x(\nu(\eta *_2 \mu_i^{\mathrm{c}})(\mathbf{r} + K^{\mathrm{cp}} * \mu_i^{\mathrm{p}})\mu_i^{\mathrm{c}}) = 0, \\ \mathrm{d}\bar{Y}_i(t) = (\frac{1}{L}\sum_{\ell=1}^L K^{\mathrm{pg}}(\bar{Y}_i(t) - Z_\ell(t)) - K^{\mathrm{pc}} * \mu_i^{\mathrm{c}}(t)(\bar{Y}_i(t)))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}W(t), \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^L K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \mu_i^{\mathrm{c}}(0) = \mu_0^{\mathrm{c}}, \\ \bar{Y}_i(0) = \bar{Y}^0 \quad \text{a.s.}, \qquad \mu_i^{\mathrm{p}} = \mathrm{Law}(\bar{Y}_i), \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L. \end{cases}$$

By Theorem 8.2, the problem above has a unique solution, hence $\mu_1^{\mathrm{c}} = \mu_2^{\mathrm{c}}$ and $\bar{Y}_1 = \bar{Y}_2$ a.s., yielding $\mu_1^{\mathrm{p}} = \mathrm{Law}(\bar{Y}_1) = \mathrm{Law}(\bar{Y}_2) = \mu_2^{\mathrm{p}}$. This concludes the proof. □

Remark 8.5 Thanks to Theorem 8.2, if there exist absolutely continuous solutions to (8.54), then μ^{c} and μ^{p} are *a fortiori* absolutely continuous by uniqueness. Under suitable conditions, the solutions are, in fact, absolutely continuous.

If $\mu^{\mathrm{c}}(0) = \rho_0^{\mathrm{c}}\,\mathrm{d}x$, then by [36, Theorem 2] the measure $\mu^{\mathrm{c}}(t)$ is absolutely continuous for all $t \in [0,T]$, hence there exists a density $\rho^{\mathrm{c}}(t) \in L^1(\mathbb{R}^2)$ such that $\mu^{\mathrm{c}}(t) = \rho^{\mathrm{c}}(t)\,\mathrm{d}x$. This is a consequence of the Lagrangian representation of the solution to the nonlocal transport equation.

By [10, Theorem 2.5] (see also [11, Corollary 3.9]) there exists a function ρ^{p} locally Hölder continuous on $(0,T)\times\mathbb{R}^2$ such that $\mu^{\mathrm{p}} = \rho^{\mathrm{p}}(t,y)\,\mathrm{d}t\,\mathrm{d}y$.

In conclusion,

$$\begin{cases} \partial_t \rho^{\mathrm{c}} + \mathrm{div}_x(\nu(\eta *_2 \rho^{\mathrm{c}})(\mathbf{r} + K^{\mathrm{cp}} * \rho^{\mathrm{p}})\rho^{\mathrm{c}}) = 0, \\ \partial_t \rho^{\mathrm{p}} - \kappa\,\Delta_y \rho^{\mathrm{p}} + \mathrm{div}_y((\frac{1}{L}\sum_{\ell=1}^{L} K^{\mathrm{pg}}(\cdot - Z_\ell(t)) - K^{\mathrm{pc}} * \rho^{\mathrm{c}})\rho^{\mathrm{p}}) = 0, \\ \frac{\mathrm{d}Z_\ell}{\mathrm{d}t}(t) = \frac{1}{L}\sum_{\ell'=1}^{L} K^{\mathrm{gg}}(Z_\ell(t) - Z_{\ell'}(t)) + u_\ell(t), \\ \rho^{\mathrm{c}}(0) = \rho_0^{\mathrm{c}}, \\ \rho^{\mathrm{p}}(t)\,\mathrm{d}y \rightharpoonup \mu_0^{\mathrm{p}} \quad \text{as } t \to 0, \\ Z_\ell(0) = Z_\ell^0, \quad \ell = 1,\dots,L. \end{cases}$$

8.2 Limit of optimal control problems as $N \to +\infty$

Let us consider the following cost functional for the limit problem obtained in (8.4). Let $\mathcal{J}\colon L^\infty([0,T];\mathcal{U}) \to \mathbb{R}$ be defined for every $u \in L^\infty([0,T];\mathcal{U})$ by

$$\mathcal{J}(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t, \tag{8.58}$$

where $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$ are obtained by Theorem 8.1 as the unique solution to (8.4) and μ^{p} is the law of $(\bar{Y}(t))_{t\in[0,T]}$.

Theorem 8.6 *Under the assumptions of Theorem 8.1, the sequence of functionals $(\mathcal{J}_N)_{N\geq 1}$ Γ-converges to $\mathcal{J}$ as $N \to +\infty$ with respect to the weak* topology in $L^\infty([0,T];\mathcal{U})$.*[23]

Proof *Step 1*. (Asymptotic lower bound). Let us fix a sequence of controls $(u^N)_{N\geq 1}$, $u^N \in L^\infty([0,T];\mathcal{U})$ such that $u^N \overset{*}{\rightharpoonup} u$ weakly* in $L^\infty([0,T];\mathcal{U})$ as $N \to +\infty$. Let us show that

$$\mathcal{J}(u) \leq \liminf_{N\to+\infty} \mathcal{J}_N(u^N). \tag{8.59}$$

On the one hand, by Definition (7.16), we have that

$$\mathcal{J}_N(u^N) := \frac{1}{2}\int_0^T \left|u^N(t)\right|^2 \mathrm{d}t + \frac{1}{N}\sum_{n=1}^{N}\int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}(\bar{X}_n^N(t) - y)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t,$$

[23] *cf.* Footnote 17.

where $\bar{X}^N = (\bar{X}_1^N, \dots, \bar{X}_N^N)$, $(\bar{Y}^N(t))_{t\in[0,T]}$, and $Z^N = (Z_1^N, \dots, Z_L^N)$ are the unique strong solution to (8.2) and $\bar{\mu}_N^{\mathrm{p}}$ is the law of $(\bar{Y}^N(t))_{t\in[0,T]}$. On the other hand,

$$\mathcal{J}(u) := \frac{1}{2}\int_0^T \left|u(t)\right|^2 \mathrm{d}t + \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t,$$

where $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$ are obtained by the unique solution to (8.4) and μ^{p} is the law of $(\bar{Y}(t))_{t\in[0,T]}$.

By the weak sequential lower semicontinuity of the L^2-norm, we have that

$$\int_0^T \left|u(t)\right|^2 \mathrm{d}t \le \liminf_{N\to+\infty}\int_0^T \left|u^N(t)\right|^2 \mathrm{d}t.$$

Let us prove the convergence

$$\begin{aligned}&\frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n^N(t)-y\big)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t \\ &\quad\to \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t,\end{aligned} \tag{8.60}$$

as $N\to+\infty$. This will conclude the proof of (8.59).

Setting $\check{H}^{\mathrm{d}}(w) = H^{\mathrm{d}}(-w)$ and considering the empirical measures in (8.3), we get that

$$\frac{1}{N}\sum_{n=1}^N H^{\mathrm{d}}\big(\bar{X}_n^N(t)-y\big) = \check{H}^{\mathrm{d}} * \nu_N^{\mathrm{c}}(t)(y).$$

Moreover, by Fubini's theorem

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y) = \int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\,\mathrm{d}\mu^{\mathrm{p}}(t)(y).$$

These equations allow us to estimate

$$\begin{aligned}&\bigg|\frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n^N(t)-y\big)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t \\ &\qquad - \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t\bigg| \\ &\quad = \left|\int_0^T \left(\int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \nu_N^{\mathrm{c}}(t)(y)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y) - \int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\,\mathrm{d}\mu^{\mathrm{p}}(t)(y)\right)\mathrm{d}t\right| \\ &\quad \le \int_0^T \int_{\mathbb{R}^2} \left|\check{H}^{\mathrm{d}} * \nu_N^{\mathrm{c}}(t)(y) - \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\right|\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t \\ &\qquad + \int_0^T \left|\int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y) - \int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)\big(y'\big)\,\mathrm{d}\mu^{\mathrm{p}}(t)\big(y'\big)\right|\mathrm{d}t.\end{aligned} \tag{8.61}$$

We estimate the first term on the right-hand side of (8.61) using the Lipschitz continuity of $\check{H}^{\mathrm{d}}(y-\cdot)$ and Kantorovich's duality by

$$\begin{aligned}\left|\check{H}^{\mathrm{d}} * \nu_N^{\mathrm{c}}(t)(y) - \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\right| &= \left|\int_{\mathbb{R}^2} \check{H}^{\mathrm{d}}(y-x)\,\mathrm{d}\big(\nu_N^{\mathrm{c}}(t) - \mu^{\mathrm{c}}(t)\big)(x)\right| \\ &\leq C\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(t), \mu^{\mathrm{c}}(t)\big) \leq \sup_{s\in[0,T]} C\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big).\end{aligned}$$

For the second term on the right-hand side of (8.61), we observe that $\check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)$ is Lipschitz continuous as

$$\left|\check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y) - \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y')\right| \leq \int_{\mathbb{R}^2} \left|\check{H}^{\mathrm{d}}(y-x) - \check{H}^{\mathrm{d}}(y'-x)\right| \mathrm{d}\mu^{\mathrm{c}}(t)(x) \leq C|y-y'|.$$

Hence, since $\bar{\mu}_N^{\mathrm{p}}(t)$ is the law of $\bar{Y}^N(t)$ and $\mu^{\mathrm{p}}(t)$ is the law of $\bar{Y}(t)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y) - \int_{\mathbb{R}^2} \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)(y')\,\mathrm{d}\mu^{\mathrm{p}}(t)(y')\right| \\ &\quad\leq \mathbb{E}\Big(\big|\check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)\big(\bar{Y}^N(t)\big) - \check{H}^{\mathrm{d}} * \mu^{\mathrm{c}}(t)\big(\bar{Y}(t)\big)\big|\Big) \\ &\quad\leq \mathbb{E}\big(\big|\bar{Y}^N(t) - \bar{Y}(t)\big|\big) \leq \mathbb{E}\big(\big\|\bar{Y}^N - \bar{Y}\big\|_\infty\big).\end{aligned}$$

Combining the previous inequalities, (8.61) reads

$$\begin{aligned}&\left|\frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n^N(t) - y\big)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t\right. \\ &\qquad\left. - \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t\right| \\ &\quad\leq CT\Big(\sup_{s\in[0,T]} C\mathcal{W}_1\big(\nu_N^{\mathrm{c}}(s), \mu^{\mathrm{c}}(s)\big) + \mathbb{E}\big(\big\|\bar{Y}^N - \bar{Y}\big\|_\infty\big)\Big) \to 0 \quad \text{as } N \to +\infty,\end{aligned}$$

where the convergence follows from Theorem 8.1. This proves (8.60).

Step 2. (Asymptotic upper bound). Let us fix $u \in L^\infty([0,T];\mathcal{U})$. For every $N \geq 1$, let us set $u^N = u$. As in *Step 1*, we have that

$$\mathcal{J}_N\big(u^N\big) := \frac{1}{2}\int_0^T |u(t)|^2\,\mathrm{d}t + \frac{1}{N}\sum_{n=1}^N \int_0^T \int_{\mathbb{R}^2} H^{\mathrm{d}}\big(\bar{X}_n^N(t) - y\big)\,\mathrm{d}\bar{\mu}_N^{\mathrm{p}}(t)(y)\,\mathrm{d}t,$$

where $\bar{X}^N = (\bar{X}_1^N, \dots, \bar{X}_N^N)$, $(\bar{Y}^N(t))_{t\in[0,T]}$, and $Z^N = (Z_1^N, \dots, Z_L^N)$ are the unique strong solution to (8.2) corresponding to the control $u^N = u$ and $\bar{\mu}_N^{\mathrm{p}}$ is the law of $(\bar{Y}^N(t))_{t\in[0,T]}$. Moreover,

$$\mathcal{J}(u) := \frac{1}{2}\int_0^T |u(t)|^2\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} H^{\mathrm{d}}(x-y)\,\mathrm{d}\mu^{\mathrm{c}}(t)\times\mu^{\mathrm{p}}(t)(x,y)\,\mathrm{d}t,$$

where $\mu^{\mathrm{c}} \in C^0([0,T];\mathcal{P}_1(\mathbb{R}^2))$, $(\bar{Y}(t))_{t\in[0,T]}$ are obtained by the unique solution to (8.4) and μ^{p} is the law of $(\bar{Y}(t))_{t\in[0,T]}$. Trivially, we have $u^N \overset{*}{\rightharpoonup} u$, hence we deduce (8.60) once again and, in particular, the asymptotic upper bound

$$\lim_{N\to+\infty} \mathcal{J}_N(u) = \mathcal{J}(u),$$

concluding the proof. □

Proposition 8.7 *Under the assumptions of Theorem 8.1, there exists an optimal control $u^* \in L^\infty([0,T];\mathcal{U})$, i.e.,*

$$\mathcal{J}(u^*) = \min_{u\in L^\infty([0,T];\mathcal{U})} \mathcal{J}(u).$$

Proof The proof is completely analogous to the proof of Proposition 7.4, as it is a general result about Γ-convergence. □

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The author declares that he has no competing interests.

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GO is the unique author of the manuscript. GO prepared the manuscript initially and performed all the steps of the proofs in this research. GO read and approved the final manuscript.

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