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Existence of positive periodic solutions for a periodic predator–prey model with fear effect and general functional responses

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Abstract

This paper investigates the existence of positive periodic solutions for a periodic predator-prey model with fear effect and general functional responses. The general functional responses can cover the Holling types II and III functional response, the Beddington–DeAngelis functional response, the Crowley–Martin functional response, the ratio-dependent type with Michaelis–Menten type functional response, etc. Some new sufficient conditions for the existence of positive periodic solutions of the model are obtained by employing the continuation theorem of coincidence degree theory and some ingenious estimation techniques for the upper and lower bounds of the a priori solutions of the corresponding operator equation. Our results considerably improve and extend some known results.

Keywords: Periodic solution; Continuation theorem; Predator–prey model; Fear effect; General functional responses

1 Introduction

The dynamic relationship between predators and prey is very common and essential in ecological environments. Consequently, many scholars have studied different types of predator–prey models based on some practical problems. Many scholars have studied the important dynamic properties of the autonomous and nonautonomous predator–prey models such as stability, permanence, extinction, global attractivity, and the existence of periodic and almost periodic solutions. These studies are valuable in exploring and predicting the relationships and patterns of changes between predators and prey. Periodic phenomena, such as seasonal effects of weather, food supply, mating habits, hunting or harvesting seasons [1], are widespread in ecosystems. In the predator–prey model, a wide variety of functional responses are available reflecting how direct killing may occur. The periodic predator–prey models with different functional responses and practical factors have been studied by many scholars. For example, the ratio-dependent functional responses [2, 3], the Holling type functional responses [4–6], the Beddington–DeAngelis functional responses [4, 7–11], the Crowley–Martin functional responses [12–14] (see also the references therein).

Recently, Tripathi et al. [14] studied the following nonautonomous predator–prey model with Crowley–Martin functional response:

$$\begin{cases} \dot{x}(t) = x(t)[a(t) - b(t)x(t) - \frac{c(t)y(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)+q(t)x(t)y(t)}], \\ \dot{y}(t) = y(t)[-d(t) - r(t)y(t) + \frac{f(t)x(t)}{\alpha(t)+\beta(t)x(t)+\gamma(t)y(t)+q(t)x(t)y(t)}], \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the population densities of the prey and predators at time t, respectively. In model (1.1), it is assumed that all parameters are continuous and have positive upper and lower bounds. The function $a(t)$ denotes the intrinsic rate of prey; $a(t)/b(t)$ denotes carrying capacity in the absence of predation; $c(t)$ denotes the capturing rate; $f(t)$ denotes the conversion rate (the coefficient of conversion from prey to predator); $d(t)$ denotes the death rate of predators; $r(t)$ denotes the predator density dependence rate (predator population decreases due to competition among the predators). Predators consume prey with a Crowley–Martin type functional response $(c(t)y(t))/(\alpha(t) + \beta(t)x(t) + \gamma(t)y(t) + q(t)x(t)y(t))$ and contribute to its growth with rate $(f(t)x(t))/(\alpha(t) + \beta(t)x(t) + \gamma(t)y(t) + q(t)x(t)y(t))$. The function $\alpha(t)$ measures the half saturation of prey species; $\beta(t)$ measures the handling time; $\gamma(t)$ denotes the coefficient of interference among predators; $q(t)$ denotes the coefficient of interference among predators at the high density of prey. More detailed biological explanations can be found in [10, 14, 15] and the references therein. Tripathi et al. [14] studied the permanence, extinction, global attractivity, and the existence of periodic and almost periodic solutions of model (1.1) in detail. If $q(t) \equiv 0$, then the functional response of model (1.1) becomes the Beddington–DeAngelis type, and then model (1.1) was studied in [10, 11]; further if $r(t) \equiv 0$, then model (1.1) was studied in [4, 7–9]. Moreover, if $\alpha(t) \equiv 0$ and $q(t) \equiv 0$, the functional response of model (1.1) becomes the ratio-dependent type, then model (1.1) was studied in [3]; further if $r(t) \equiv 0$, then model (1.1) was studied in [2].

Many biologists realized that the cost of fear should be incorporated along with direct predation in prey–predator interactions [16]. Experiments by Zenette et al. [17] showed that fear of predators alone led to a 40% reduction in the number of offspring that song sparrow parents could produce. In [18], Wang et al. first formulated and investigated a predator–prey model incorporating the cost of fear (indirect effects) and observed that the cost of fear plays a crucial role in changing the dynamics of predator–prey interactions. Further, some predator–prey models with fear effects and different functional responses and practical factors have been studied by many scholars (see, e.g., [19–23]).

Motivated by the above research works, in this paper, we further consider the following periodic predator–prey model with fear effect and general functional responses:

$$\begin{cases} \dot{x}(t) = x(t)[a(t)F(k(t), y(t)) - b(t)x(t)] - c(t)G(t, x(t), y(t))y(t), \\ \dot{y}(t) = y(t)[-d(t) - r(t)y(t) + f(t)G(t, x(t), y(t))]. \end{cases} \tag{1.2}$$

In model (1.2), the predators follow general functional responses to hunt the prey population, and the function $G(t, x, y)$ satisfies some assumptions which will be given below. In this paper, we always assume that the functions $a(t)$, $b(t)$, $d(t)$, and $f(t)$ are continuous, positive, and ω-periodic ($\omega > 0$); the functions $k(t)$, $c(t)$, and $r(t)$ are continuous, nonnegative, and ω-periodic. In addition, some additional restrictions on the parameter functions will be given in our theorem conditions. Here, the term $F(k(t), y(t)) \le 1$ denotes the cost of anti-predator defense due to fear; $k(t)$ reflects the level of fear which drives anti-predator behaviors of the prey [18]. We assume that the function $F(k,y)$ satisfies the following condition (see [18]):

(H) $F(k,y)$ is continuous on $\mathbb{R}_+^2$ and continuously differentiable with respect to $(k,y)^T \in \mathbb{R}_+^2$; $F(k,y) > 0$, $F(0,y) = 1$, $F(k,0) = 1$ for $k \ge 0$ and $y \ge 0$; the partial derivatives $\frac{\partial F(k,y)}{\partial k} \le 0$ and $\frac{\partial F(k,y)}{\partial y} \le 0$ for $k \ge 0$ and $y \ge 0$.

Clearly, the function $F(k,y)$ can cover the following forms: $1/(1+ky)$, $1/(1+ky^2)$, e^{-ky}, etc.

For convenience, in this paper, we always assume that $m(t)$, $\alpha_1(t)$, $\alpha_3(t)$, and $\alpha_5(t)$ are continuous, positive, and ω-periodic, $\alpha_2(t)$ and $\alpha_4(t)$ are continuous, nonnegative, and ω-periodic. In addition, we assume that the function $G(t,x,y) \equiv G_1(t,x,y)$ or $G(t,x,y) \equiv G_2(t,x)$, or $G(t,x,y) \equiv G_3(m(t), \frac{x}{y})$, where $G_1(t,x,y)$, $G_2(t,x)$ and $G_3(m(t), \frac{x}{y})$ satisfy some of the given assumptions.

Assume that the function $G_1(t,x,y)$ satisfies the following conditions:

(P1) $G_1(t,x,y)$ is nonnegative and continuous on $\mathbb{R} \times \mathbb{R}_+^2$ and continuously differentiable with respect to $(x,y)^T \in \mathbb{R}_+^2$, and ω-periodic in t.

(P2) $G_1(t,x,y) > 0$ and $G_1(t,0,y) = 0$ for $t \in \mathbb{R}$, $x > 0$, $y \ge 0$; for each $(t,y)^T \in \mathbb{R} \times \mathbb{R}_+$, $G_1(t,x,y)$ is increasing with respect to x on $\mathbb{R}_+$; for each $(t,x)^T \in \mathbb{R} \times \mathbb{R}_+$, $G_1(t,x,y)$ is nonincreasing with respect to y on $\mathbb{R}_+$.

(P3) There exists a continuous ω-periodic function $\Theta_1(t) > 0$ such that $G_1(t,x,0) \le \Theta_1(t)x$ for $t \in \mathbb{R}$, $x \in \mathbb{R}_+$.

(P4) For each $x \in (0,+\infty)$, there exists a continuous function $\widetilde{G_1}(t,x) > 0$, which is ω-periodic in t, such that $yG_1(t,x,y) \le \widetilde{G_1}(t,x)$ for $t \in \mathbb{R}$, $y \ge 0$.

(P5) The partial derivatives $\frac{\partial G_1(t,x,y)}{\partial x} \ge 0$ and $\frac{\partial G_1(t,x,y)}{\partial y} < 0$ for $t \in \mathbb{R}$, $x > 0$, $y > 0$; for each $(t,x)^T \in \mathbb{R} \times \mathbb{R}_+$, $\lim_{y\to\infty} G_1(t,x,y) = 0$.

It is not difficult to find that $G_1(t,x,y)$ can cover some common forms such as the Beddington–DeAngelis functional response

$$\frac{x}{\alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y},$$

the Crowley–Martin functional response

$$\frac{x}{\alpha_1(t) + \alpha_2(t)x + \alpha_3(t)y + \alpha_4(t)xy},$$

and other forms of functional response, such as

$$\frac{x}{\alpha_1(t) + \alpha_2(t)y + \alpha_3(t)y^2}, \qquad \frac{x^2}{\alpha_1(t) + \alpha_3(t)y + \alpha_5(t)x^2}.$$

Assume that the function $G_2(t,x)$ satisfies the following conditions:

(Q1) $G_2(t,x)$ is nonnegative and continuous on $\mathbb{R} \times \mathbb{R}_+$ and continuously differentiable with respect to $x \in \mathbb{R}_+$, and ω-periodic in t.

(Q2) $G_2(t,x) > 0$ and $G_2(t,0) = 0$ for $t \in \mathbb{R}$, $x > 0$; for each $t \in \mathbb{R}$, $G_2(t,x)$ is increasing with respect to x on $\mathbb{R}_+$.

(Q3) There exists a continuous ω-periodic function $\Theta_2(t) > 0$ such that $G_2(t,x) \le \Theta_2(t)x$ for $t \in \mathbb{R}$, $x \in \mathbb{R}_+$.

(Q4) The partial derivative $\frac{\partial G_2(t,x)}{\partial x} > 0$ for $t \in \mathbb{R}$, $x > 0$.

Clearly, the function $G_2(t,x)$ can cover the classical Holling type II functional response

$$\frac{x}{\alpha_1(t) + \alpha_2(t)x}.$$

Note that, for $x \geq 0$, $n \in N^+$, and $n \geq 2$,

$$\overline{G_2}(t,x) := \frac{x^n}{\alpha_1(t) + \alpha_3(t)x^n} = \frac{x^{n-1}}{\alpha_1(t) + \alpha_3(t)x^n}x \leq \frac{(\zeta(t))^{n-1}}{\alpha_1(t) + \alpha_3(t)(\zeta(t))^n}x,$$

where

$$\zeta(t) = \left(\frac{\alpha_1(t)}{\alpha_3(t)}(n-1)\right)^{\frac{1}{n}}.$$

In addition, it is not difficult to verify that the function $\overline{G_2}(t,x)$ satisfies conditions (Q1)–(Q4). Note that, when $n = 2$, the function $\overline{G_2}(t,x)$ becomes the classical Holling type III functional response

$$\frac{x^2}{\alpha_1(t) + \alpha_3(t)x^2}.$$

Assume that the function $G_3(m,z)$ ($z = \frac{x}{y}$) satisfies the following conditions:

(H1) $G_3(m,z)$ is nonnegative and continuous on $\mathbb{R}_+^2$ and continuously differentiable with respect to $z \in \mathbb{R}_+$.

(H2) $G_3(m,z) > 0$ and $G_3(m,0) = 0$ for $m > 0$, $z > 0$; for each $z > 0$, $G_3(m,z)$ is nonincreasing with respect to m on $(0,+\infty)$; for each $m > 0$, $G_3(m,z)$ is increasing with respect to z on $(0,+\infty)$.

(H3) For each $m > 0$, $\frac{\partial G_3(m,z)}{\partial z} > 0$ for $z > 0$, and $\lim_{z\to\infty} G_3(m,z) = \Theta_3(m) > 0$.

(H4) For each $m > 0$, $\frac{G_3(m,z)}{z}$ is nonincreasing with respect to z on $(0,+\infty)$, and $\lim_{z\to 0^+} \frac{G_3(m,z)}{z} = \Theta_4(m) > 0$.

It is not difficult to find that $G_3(m(t),\frac{x}{y})$ can cover the ratio-dependent type with Michaelis–Menten functional response

$$\frac{\frac{x}{y}}{m(t) + \frac{x}{y}} = \frac{x}{m(t)y + x}.$$

The main purpose of this paper is to study the existence of positive periodic solutions for model (1.2) by using the continuation theorem of coincidence degree theory [24]. The most crucial aspect of using the coincidence theorem is to estimate the upper and lower bounds of the a priori solutions of the corresponding operator equation (see $L\varphi = \mu N\varphi$, $\mu \in (0,1)$ in Sect. 2). The existence of positive periodic solutions for the special cases of model (1.2) has attracted the attention of many scholars and has yielded plentiful results (see, e.g., [2–4, 7–11, 14]). For model (1.2), our main results (see Theorems 3.1 and 3.2, Corollary 3.1) extend and improve Theorem 4.2 in Li and She [10], Theorem 3.1 in Li and She [3], Theorem 8 in Tripathi [14]. In addition, we obtain different results compared to some of the known ones (see Theorem 3.5 in Fan et al. [2], Theorems 3.1 and 3.2 in Fan and Kuang [7], Theorems 3.1 and 3.2 in Bohner et al. [4], Theorems 1 and

2 in Fazly and Hesaaraki [9], Theorem 3.1 in Jiang [11], Theorem 9 in Tripathi [14]). It is worth mentioning that the continuation theorem of coincidence degree theory [24] is very effective to study the existence of periodic solutions of predator–prey models (see, e.g., [2, 4–7, 9, 11–14, 25]) and other biological models (see, e.g., [26–28]).

The rest of this paper is organized as follows. In Sect. 2, we first review the continuation theorem of coincidence degree theory [24] and then study the existence of positive periodic solutions of model (1.2). In Sect. 3, we give some applications of our results and compare them with some known results. The last section contains the conclusions and some numerical simulations of this paper.

2 Existence of positive periodic solutions of the model

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, it follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 ([24]) *Assume that $\Omega \subset X$ is an open bounded set. Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Assume that*

(i) $Lu \neq \mu Nu$, $\forall u \in \partial\Omega \cap \operatorname{Dom} L$, $\mu \in (0, 1)$;

(ii) $QNu \neq 0$, $\forall u \in \partial\Omega \cap \operatorname{Ker} L$;

(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

For any continuous ω-periodic function $\varrho(t)$ defined on $\mathbb{R}$, we denote

$$\widehat{\varrho} = \frac{1}{\omega} \int_0^\omega \varrho(t)\,dt, \qquad \varrho^u = \max_{t\in[0,\omega]} \varrho(t), \qquad \varrho^l = \min_{t\in[0,\omega]} \varrho(t).$$

For convenience, for any $\nu, \widetilde{\nu} \in \mathbb{R}$, we define

$$\Phi_1(\nu) = \frac{1}{\omega} \int_0^\omega f(t)\widetilde{G_1}\bigl(t, \exp\{\nu\}\bigr)\,dt, \qquad \Phi_2(\nu, \widetilde{\nu}) = \frac{1}{\omega} \int_0^\omega \frac{c(t)G_2(t, \exp\{\widetilde{\nu}\})}{\exp\{\nu\}}\,dt,$$

$$\Phi_3(\nu) = \Theta_4\bigl(m^l\bigr)\widehat{f}\exp\{\nu\}, \qquad \Upsilon(\nu) = \frac{(\frac{c}{a})^u(\frac{d}{f})^l}{F(k^u, \exp\{\nu\})},$$

$$M_1 = \min\left\{\ln\left(\frac{\widehat{a}}{\widehat{b}}\right) + 2\widehat{a}\omega, \ln\left[\left(\frac{a}{b}\right)^u\right]\right\},$$

$$M_2 = \ln\left(\frac{\Phi_3(M_1)}{\widehat{d}}\right) + 2\Theta_3\bigl(m^l\bigr)\widehat{f}\omega.$$

The main results of this paper are as follows.

Theorem 2.1 *Assume that $\widehat{c} > 0$ and one of the following conditions holds*:

(A1) $G(t, x, y) \equiv G_1(t, x, y)$, $\Pi_1 := \frac{1}{\omega} \int_0^\omega f(t)G_1(t, (\frac{a}{b})^l, 0)\,dt > \widehat{d}$;

(A2) $G(t,x,y) \equiv G_2(t,x)$, $\Pi_2 := \frac{1}{\omega}\int_0^{\omega} f(t)G_2(t,(\frac{a}{b})^l) > \widehat{d}$;
(A3) $G(t,x,y) \equiv G_3(m(t),\frac{x}{y})$, $\Pi_3 := \Theta_3(m^u)\widehat{f} > \widehat{d}$, $G_3(m^l,\Upsilon(M_2)) < (\frac{d}{f})^l$.

Then model (1.2) *has at least one positive ω-periodic solution.*

Proof Assume that $G(t,x,y) \equiv G_1(t,x,y)$ or $G(t,x,y) \equiv G_2(t,x)$, or $G(t,x,y) \equiv G_3(m(t),\frac{x}{y})$. Let $x(t) = \exp\{\varphi_1(t)\}$ and $y(t) = \exp\{\varphi_2(t)\}$, then model (1.2) can be transformed into

$$\begin{cases} \dot{\varphi}_1(t) = a(t)F(k(t),\exp\{\varphi_2(t)\}) - b(t)\exp\{\varphi_1(t)\} \\ \qquad - \frac{c(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}, \\ \dot{\varphi}_2(t) = -d(t) - r(t)\exp\{\varphi_2(t)\} + f(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\}). \end{cases} \tag{2.1}$$

Clearly, it is only necessary to prove that model (2.1) has an ω-periodic solution.

Let

$$X = Z = \{\varphi = (\varphi_1(t),\varphi_2(t))^T \in C(\mathbb{R},\mathbb{R}^2) \mid \varphi(t) = \varphi(t+\omega)\}$$

with the norm $\|\varphi\| = \max_{t\in[0,\omega]}|\varphi_1(t)| + \max_{t\in[0,\omega]}|\varphi_2(t)|$. Clearly, both X and Z are Banach spaces. Define

$$P\varphi = \frac{1}{\omega}\int_0^{\omega}\varphi(t)\,dt(\varphi \in X), \qquad Q\varphi = \frac{1}{\omega}\int_0^{\omega}\varphi(t)\,dt(\varphi \in Z),$$

$$L\varphi = \dot{\varphi}(t), \qquad N\varphi = \begin{bmatrix} N_1(t) \\ N_2(t) \end{bmatrix},$$

where

$$\begin{aligned} N_1(t) &= a(t)F(k(t),\exp\{\varphi_2(t)\}) - b(t)\exp\{\varphi_1(t)\} \\ &\quad - \frac{c(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}, \\ N_2(t) &= -d(t) - r(t)\exp\{\varphi_2(t)\} + f(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\}). \end{aligned}$$

Then, it has $\operatorname{Ker} L = \{\varphi \in X \mid \varphi \in \mathbb{R}^2\}$ and $\operatorname{Im} L = \{\varphi \in Z \mid \int_0^{\omega}\varphi(t)\,dt = 0\}$. Clearly, $\operatorname{Im} L$ is closed in Z, and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$. Thus, L is a Fredholm mapping of index zero. Moreover, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists and is given by $K_P\varphi = \int_0^t \varphi(s)\,ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t \varphi(s)\,ds\,dt$. Then, similar to the proof of Theorem 2.1 in [28], we can obtain that N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $L\varphi = \mu N\varphi$, $\mu \in (0,1)$, we have

$$\begin{cases} \dot{\varphi}_1(t) = \mu[a(t)F(k(t),\exp\{\varphi_2(t)\}) - b(t)\exp\{\varphi_1(t)\} \\ \qquad - \frac{c(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}], \\ \dot{\varphi}_2(t) = \mu[-d(t) - r(t)\exp\{\varphi_2(t)\} + f(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})]. \end{cases} \tag{2.2}$$

Assume that $(\varphi_1(t),\varphi_2(t))^T \in X$ is an arbitrary solution of model (2.2) for a certain $\mu \in (0,1)$. Since $(\varphi_1(t),\varphi_2(t))^T \in X$, there exist ξ_1, ξ_2, η_1, $\eta_2 \in [0,\omega]$ such that

$$\varphi_1(\xi_1) = \min_{t\in[0,\omega]}\varphi_1(t), \qquad \varphi_1(\eta_1) = \max_{t\in[0,\omega]}\varphi_1(t),$$

$$\varphi_2(\xi_2) = \min_{t\in[0,\omega]} \varphi_2(t), \qquad \varphi_2(\eta_2) = \max_{t\in[0,\omega]} \varphi_2(t).$$

Clearly, $\dot{\varphi}_1(\xi_1) = \dot{\varphi}_1(\eta_1) = \dot{\varphi}_2(\xi_2) = \dot{\varphi}_2(\eta_2) = 0$. Integrating on both sides of (2.2) over the interval $[0,\omega]$, we have

$$\begin{aligned}&\int_0^\omega \bigg[a(t)F\big(k(t),\exp\{\varphi_2(t)\}\big) - b(t)\exp\{\varphi_1(t)\} \\ &\quad - \frac{c(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}\bigg]\,dt = 0\end{aligned} \tag{2.3}$$

and

$$\int_0^\omega \big[d(t) + r(t)\exp\{\varphi_2(t)\} - f(t)G\big(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\}\big)\big]\,dt = 0. \tag{2.4}$$

Note that

$$\int_0^\omega \dot{\varphi}_2(t)\exp\{\varphi_2(t)\}\,dt = \exp\{\varphi_2(\omega)\} - \exp\{\varphi_2(0)\} = 0,$$

then we have

$$\begin{aligned}&\int_0^\omega \big[d(t)\exp\{\varphi_2(t)\} + r(t)\exp\{2\varphi_2(t)\}\big]\,dt \\ &\quad = \int_0^\omega f(t)\exp\{\varphi_2(t)\}G\big(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\}\big)\,dt.\end{aligned} \tag{2.5}$$

From (2.2), (2.3), and condition (H), we have

$$\begin{aligned}&\int_0^\omega \big|\dot{\varphi}_1(t)\big|\,dt \\ &\quad \le \int_0^\omega a(t)F\big(k(t),\exp\{\varphi_2(t)\}\big)\,dt \\ &\qquad + \int_0^\omega \bigg[b(t)\exp\{\varphi_1(t)\} + \frac{c(t)G(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}\bigg]\,dt \\ &\quad = 2\int_0^\omega a(t)F\big(k(t),\exp\{\varphi_2(t)\}\big)\,dt \le 2\widehat{a}\omega.\end{aligned} \tag{2.6}$$

From (2.3), we have

$$\widehat{a}\omega - \widehat{b}\exp\{\varphi_1(\xi_1)\}\omega \ge \int_0^\omega \big[a(t)F\big(k(t),\exp\{\varphi_2(t)\}\big) - b(t)\exp\{\varphi_1(\xi_1)\}\big]\,dt \ge 0,$$

which implies that

$$\varphi_1(\xi_1) \le \ln\left(\frac{\widehat{a}}{\widehat{b}}\right) := m_1.$$

Then, from (2.6), we have

$$\varphi_1(t) \le \varphi_1(\xi_1) + \int_0^\omega \big|\dot{\varphi}_1(t)\big|\,dt \le m_1 + 2\widehat{a}\omega := M_1^*.$$

Also, from $\dot{\varphi}_1(\eta_1) = 0$, we can easily obtain that

$$\varphi_1(t) \le \varphi_1(\eta_1) \le \ln\left(\frac{a(\eta_1)F(k(\eta_1), \exp\{\varphi_2(\eta_1)\})}{b(\eta_1)}\right) \le \ln\left[\left(\frac{a}{b}\right)^u\right] := \widetilde{M}_1^*.$$

Thus, we have

$$\varphi_1(t) \le \min\{M_1^*, \widetilde{M}_1^*\} = M_1. \tag{2.7}$$

From (2.2) and (2.4), we have

$$\begin{aligned}
&\int_0^\omega |\dot{\varphi}_2(t)|\,dt \\
&\quad \le \int_0^\omega [d(t) + r(t)\exp\{\varphi_2(t)\}]\,dt + \int_0^\omega f(t)G(t, \exp\{\varphi_1(t)\}, \exp\{\varphi_2(t)\})\,dt \\
&\quad = 2\int_0^\omega f(t)G(t, \exp\{\varphi_1(t)\}, \exp\{\varphi_2(t)\})\,dt.
\end{aligned} \tag{2.8}$$

We consider the following three cases.

Case (i). Condition (A1) holds.

From (2.7), (2.8), and condition (P2), we have

$$\int_0^\omega |\dot{\varphi}_2(t)|\,dt \le 2\int_0^\omega f(t)G_1(t, \exp\{M_1\}, 0)\,dt = 2\widehat{\Delta}_1\omega, \tag{2.9}$$

where $\Delta_1(t) = f(t)G_1(t, \exp\{M_1\}, 0)$. From (2.5), (2.7), and conditions (P2) and (P4), we have

$$\begin{aligned}
\widehat{d}\exp\{\varphi_2(\xi_2)\} &\le \frac{1}{\omega}\int_0^\omega f(t)G_1(t, \exp\{M_1\}, \exp\{\varphi_2(t)\})\exp\{\varphi_2(t)\}\,dt \\
&\le \frac{1}{\omega}\int_0^\omega f(t)\widetilde{G}_1(t, \exp\{M_1\})\,dt \\
&= \Phi_1(M_1) > 0,
\end{aligned}$$

which implies that

$$\varphi_2(\xi_2) \le \ln\left(\frac{\Phi_1(M_1)}{\widehat{d}}\right).$$

Then, from (2.9), we have

$$\varphi_2(t) \le \varphi_2(\xi_2) + \int_0^\omega |\dot{\varphi}_2(t)|\,dt \le \ln\left(\frac{\Phi_1(M_1)}{\widehat{d}}\right) + 2\widehat{\Delta}_1\omega := M_2^{(1)}.$$

From (2.4) and conditions (P2) and (P3), we have

$$\begin{aligned}\widehat{d}\omega &\le \int_0^\omega f(t)G_1\bigl(t,\exp\{\varphi_1(t)\},\exp\{\varphi_2(t)\}\bigr)\,dt\\ &\le \int_0^\omega f(t)G_1\bigl(t,\exp\{\varphi_1(\eta_1)\},0\bigr)\,dt\\ &\le \int_0^\omega f(t)\Theta_1(t)\exp\{\varphi_1(\eta_1)\}\,dt,\end{aligned}$$

which implies that

$$\varphi_1(\eta_1)\ge \ln\biggl(\frac{\widehat{d}}{\widehat{(f\Theta_1)}}\biggr):=l_1^{(1)}.$$

Then, from (2.6), we have

$$\varphi_1(t)\ge\varphi_1(\eta_1)-\int_0^\omega\bigl|\dot{\varphi}_1(t)\bigr|\,dt\ge l_1^{(1)}-2\widehat{a}\omega:=L_1^{(1)}.$$

If $\Pi_1>\widehat{d}$, then there exists a sufficiently small constant $\delta_1>0$ such that

$$\begin{aligned}&\Gamma_1(\delta_1):=\biggl(\frac{a}{b}\biggr)^l F\bigl(k^u,\delta_1\bigr)-\biggl(\frac{c\Theta_1}{b}\biggr)^u\delta_1>0,\\ &\int_0^\omega f(t)G_1\bigl(t,\Gamma(\delta_1),\delta_1\bigr)\,dt>\widehat{d}\omega+\widehat{r}\delta_1\omega.\end{aligned}\tag{2.10}$$

Claim (i). If $\Pi_1>\widehat{d}$, then $\exp\{\varphi_2(\eta_2)\}\ge\delta_1$.
If the claim is not true, then

$$\max_{t\in[0,\omega]}\exp\bigl\{\varphi_2(t)\bigr\}=\exp\bigl\{\varphi_2(\eta_2)\bigr\}<\delta_1.$$

From $\dot{\varphi}_1(\xi_1)=0$, $\exp\{\varphi_2(\xi_1)\}\le\exp\{\varphi_2(\eta_2)\}<\delta_1$, conditions (P2) and (P3), we have

$$\begin{aligned}\exp\bigl\{\varphi_1(\xi_1)\bigr\}&=\frac{a(\xi_1)F(k(\xi_1),\exp\{\varphi_2(\xi_1)\})}{b(\xi_1)}\\ &\quad-\frac{c(\xi_1)G_1(\xi_1,\exp\{\varphi_1(\xi_1)\},\exp\{\varphi_2(\xi_1)\})\exp\{\varphi_2(\xi_1)\}}{b(\xi_1)\exp\{\varphi_1(\xi_1)\}}\\ &\ge\biggl(\frac{a}{b}\biggr)^l F\bigl(k^u,\delta_1\bigr)-\frac{c(\xi_1)G_1(\xi_1,\exp\{\varphi_1(\xi_1)\},0)}{b(\xi_1)\exp\{\varphi_1(\xi_1)\}}\delta_1\\ &\ge\biggl(\frac{a}{b}\biggr)^l F\bigl(k^u,\delta_1\bigr)-\frac{c(\xi_1)\Theta_1(\xi_1)}{b(\xi_1)}\delta_1\\ &\ge\biggl(\frac{a}{b}\biggr)^l F\bigl(k^u,\delta_1\bigr)-\biggl(\frac{c\Theta_1}{b}\biggr)^u\delta_1\\ &=\Gamma_1(\delta_1)>0.\end{aligned}$$

Further, from (2.4) and condition (P2), we have

$$\begin{aligned}
\widehat{d}\omega + \widehat{r}\delta_1\omega &\ge \int_0^\omega \bigl[d(t) + r(t)\exp\{\varphi_2(t)\}\bigr]\,dt \\
&= \int_0^\omega f(t)G_1\bigl(t, \exp\{\varphi_1(t)\}, \exp\{\varphi_2(t)\}\bigr)\,dt \\
&\ge \int_0^\omega f(t)G_1\bigl(t, \exp\{\varphi_1(\xi_1)\}, \exp\{\varphi_2(\eta_2)\}\bigr)\,dt \\
&\ge \int_0^\omega f(t)G_1\bigl(t, \Gamma_1(\delta_1), \delta_1\bigr)\,dt,
\end{aligned}$$

which is a contradiction to (2.10). This proves the claim.

From *Claim* (i) and (2.9), we can obtain

$$\varphi_2(t) \ge \varphi_2(\eta_2) - \int_0^\omega \bigl|\dot{\varphi}_2(t)\bigr|\,dt \ge \ln(\delta_1) - 2\widehat{\Delta_1}\omega := L_2^{(1)}.$$

Now, let us consider the following algebraic equations:

$$\begin{cases}
[\frac{1}{\omega}\int_0^\omega a(t)F(k(t), \exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\} \\
\quad - \frac{1}{\omega}\int_0^\omega \mu\frac{c(t)G_1(t,\exp\{\varphi_1\},\exp\{\varphi_2\})\exp\{\varphi_2\}}{\exp\{\varphi_1\}}\,dt] = 0, \\
\frac{1}{\omega}\int_0^\omega f(t)G_1(t, \exp\{\varphi_1\}, \exp\{\varphi_2\})\,dt - \widehat{d} - \mu\widehat{r}\exp\{\varphi_2\} = 0
\end{cases} \tag{2.11}$$

for $(\varphi_1, \varphi_2)^T \in \mathbb{R}^2$, where $\mu \in [0,1]$ is a parameter. By using the similar arguments as above, we can show that any solution $(\varphi_{11}^*, \varphi_{21}^*)^T \in \mathbb{R}^2$ of (2.11) with $\mu \in [0,1]$ satisfies

$$\begin{aligned}
&l_1^{(1)} = \ln\biggl(\frac{\widehat{d}}{(\widehat{f\Theta_1})}\biggr) \le \varphi_{11}^* \le \ln\biggl(\frac{\widehat{a}}{\widehat{b}}\biggr) = m_1, \\
&l_2^{(1)} := \ln\bigl(\delta_1^*\bigr) \le \varphi_{21}^* \le \ln\biggl(\frac{\Phi_1(m_1)}{\widehat{d}}\biggr) := m_2^{(1)},
\end{aligned} \tag{2.12}$$

where $\delta_1^* > 0$ satisfies

$$\begin{aligned}
&\Gamma_1^*\bigl(\delta_1^*\bigr) := \frac{\widehat{a}}{\widehat{b}}F\bigl(k^u, \delta_1^*\bigr) - \frac{1}{\widehat{b}}\biggl(\frac{1}{\omega}\int_0^\omega c(t)\Theta_1(t)\,dt\biggr)\delta_1^* > 0, \\
&\frac{1}{\omega}\int_0^\omega f(t)G_1\bigl(t, \Gamma_1^*\bigl(\delta_1^*\bigr), \delta_1^*\bigr)\,dt > \widehat{d} + \widehat{r}\delta_1^*.
\end{aligned}$$

Note that M_1, $M_2^{(1)}$, $L_1^{(1)}$, $L_2^{(1)}$, m_1, $m_2^{(1)}$, $l_1^{(1)}$, and $l_2^{(1)}$ are independent of μ. Define

$$\Omega^{(1)} = \bigl\{\varphi \in X \mid \|\varphi\| < U^{(1)}\bigr\},$$

where

$$U^{(1)} = 1 + \max\bigl\{\bigl|l_1^{(1)}\bigr|, \bigl|L_1^{(1)}\bigr|, |m_1|, |M_1|\bigr\} + \max\bigl\{\bigl|l_2^{(1)}\bigr|, \bigl|L_2^{(1)}\bigr|, \bigl|m_2^{(1)}\bigr|, \bigl|M_2^{(1)}\bigr|\bigr\}.$$

Clearly, $\Omega^{(1)}$ satisfies condition (i) in Lemma 2.1. When $\varphi = (\varphi_1,\varphi_2)^T \in \partial\Omega^{(1)} \cap \operatorname{Ker} L = \partial\Omega^{(1)} \cap \mathbb{R}^2$, then φ is a constant vector in $\mathbb{R}^2$ with $|\varphi_1| + |\varphi_2| = U^{(1)}$. Then

$$QN\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix}\frac{1}{\omega}\int_0^\omega a(t)F(k(t),\exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\} - W_1(\varphi_1,\varphi_2)\\ \frac{1}{\omega}\int_0^\omega f(t)G_1(t,\exp\{\varphi_1\},\exp\{\varphi_2\})\,dt - \widehat{d} - \widehat{r}\exp\{\varphi_2\}\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix},$$

where

$$W_1(\varphi_1,\varphi_2) = \frac{1}{\omega}\int_0^\omega \frac{c(t)G_1(t,\exp\{\varphi_1\},\exp\{\varphi_2\})\exp\{\varphi_2\}}{\exp\{\varphi_1\}}\,dt.$$

Here, we have proved that condition (ii) in Lemma 2.1 is satisfied. To compute the Brouwer degree, let us consider the homotopy

$$\Xi_\mu^{(1)}\big((\varphi_1,\varphi_2)^T\big) = \mu QN\big((\varphi_1,\varphi_2)^T\big) + (1-\mu)\Psi^{(1)}\big((\varphi_1,\varphi_2)^T\big), \quad \mu \in [0,1],$$

where

$$\Psi^{(1)}\big((\varphi_1,\varphi_2)^T\big) = \begin{bmatrix}\frac{1}{\omega}\int_0^\omega a(t)F(k(t),\exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\}\\ \frac{1}{\omega}\int_0^\omega f(t)G_1(t,\exp\{\varphi_1\},\exp\{\varphi_2\})\,dt - \widehat{d}\end{bmatrix}.$$

From (2.12), it follows that $0 \notin \Xi_\mu^{(1)}(\partial\Omega^{(1)} \cap \mathbb{R}^2)$ for $\mu \in [0,1]$. In addition, from conditions (H), (P2), and (P5), one can easily show that $\Psi^{(1)}((\varphi_1,\varphi_2)^T) = 0$ has a unique solution $(\widetilde{\varphi_{11}^*},\widetilde{\varphi_{21}^*})^T$ in $\mathbb{R}^2$ if $\Pi_1 > \widehat{d}$. Let $e_{11} = \exp\{\widetilde{\varphi_{11}^*}\} > 0$, $e_{21} = \exp\{\widetilde{\varphi_{21}^*}\} > 0$. From conditions (H) and (P5), we have

$$\begin{aligned}
\varsigma_1^{(1)} &:= \frac{1}{\omega}\int_0^\omega a(t)\frac{\partial F(k(t),e_{21})}{\partial y}e_{21}\,dt \le 0,\\
\varsigma_2^{(1)} &:= \frac{1}{\omega}\int_0^\omega f(t)\frac{\partial G_1(t,e_{11},e_{21})}{\partial x}e_{11}\,dt \ge 0,\\
\varsigma_3^{(1)} &:= \frac{1}{\omega}\int_0^\omega f(t)\frac{\partial G_1(t,e_{11},e_{21})}{\partial y}e_{21}\,dt < 0.
\end{aligned}$$

A direct calculation produces

$$\begin{aligned}
\deg\{\Psi^{(1)},\partial\Omega^{(1)} \cap \operatorname{Ker} L, 0\} &= \operatorname{sign}\begin{vmatrix}-\widehat{b}e_{11} & \varsigma_1^{(1)}\\ \varsigma_2^{(1)} & \varsigma_3^{(1)}\end{vmatrix}\\
&= \operatorname{sign}\{-\widehat{b}e_{11}\varsigma_3^{(1)} - \varsigma_1^{(1)}\varsigma_2^{(1)}\}\\
&= 1 \neq 0.
\end{aligned}$$

Since $\operatorname{Im} Q = \operatorname{Ker} L$, then we have $J = I$. Furthermore, by the invariance property of homotopy, we have

$$\begin{aligned}
\deg\{JQN,\partial\Omega^{(1)} \cap \operatorname{Ker} L, 0\} &= \deg\{QN,\partial\Omega^{(1)} \cap \operatorname{Ker} L, 0\}\\
&= \deg\{\Psi^{(1)},\partial\Omega^{(1)} \cap \operatorname{Ker} L, 0\} \neq 0.
\end{aligned}$$

By Lemma 2.1, if condition (A1) holds, then model (2.1) admits at least one ω-periodic solution.

Case (ii). Condition (A2) holds.

From (2.7), (2.8), and condition (Q2), we have

$$\begin{aligned}\int_0^{\omega}\left|\dot{\varphi}_2(t)\right|dt &\le 2\int_0^{\omega} f(t)G_2\bigl(t,\exp\bigl\{\varphi_1(t)\bigr\}\bigr)\,dt \\ &\le 2\int_0^{\omega} f(t)G_2\bigl(t,\exp\{M_1\}\bigr)\,dt \\ &= 2\widehat{\Delta_2}\omega,\end{aligned} \tag{2.13}$$

where $\Delta_2(t)=f(t)G_2(t,\exp\{M_1\})$. From (2.4) and condition (Q3), we have

$$\widehat{d}\omega \le \int_0^{\omega} f(t)G_2\bigl(t,\exp\bigl\{\varphi_1(\eta_1)\bigr\}\bigr)\,dt \le \exp\bigl\{\varphi_1(\eta_1)\bigr\}\int_0^{\omega} f(t)\Theta_2(t)\,dt,$$

which implies that

$$\varphi_1(\eta_1)\ge \ln\left(\frac{\widehat{d}}{\widehat{(f\Theta_2)}}\right):=l_1^{(2)}.$$

Then, from (2.6), we have

$$\varphi_1(t)\ge\varphi_1(\eta_1)-\int_0^{\omega}\left|\dot{\varphi}_1(t)\right|dt\ge l_1^{(2)}-2\widehat{a}\omega:=L_1^{(2)}.$$

From (2.3) and condition (Q2), we have

$$\begin{aligned}\Phi_2\bigl(M_1,L_1^{(2)}\bigr)\exp\bigl\{\varphi_2(\xi_2)\bigr\}\omega &= \int_0^{\omega}\frac{c(t)G_2(t,\exp\{L_1^{(2)}\})\exp\{\varphi_2(\xi_2)\}}{\exp\{M_1\}}\,dt \\ &\le \int_0^{\omega}\frac{c(t)G_2(t,\exp\{\varphi_1(t)\})\exp\{\varphi_2(t)\}}{\exp\{\varphi_1(t)\}}\,dt \\ &\le \int_0^{\omega} a(t)F\bigl(k(t),\exp\bigl\{\varphi_2(t)\bigr\}\bigr)\,dt \\ &\le \widehat{a}\omega,\end{aligned}$$

which implies that

$$\exp\bigl\{\varphi_2(\xi_2)\bigr\}\le\frac{\widehat{a}}{\Phi_2(M_1,L_1^{(2)})}.$$

Then, from (2.13), we have

$$\varphi_2(t)\le\varphi_2(\xi_2)+\int_0^{\omega}\left|\dot{\varphi}_2(t)\right|dt\le\ln\left(\frac{\widehat{a}}{\Phi_2(M_1,L_1^{(2)})}\right)+2\widehat{\Delta_2}\omega:=M_2^{(2)}.$$

If $\Pi_2>\widehat{d}$, then there exists a sufficiently small constant $\delta_2>0$ such that

$$\Gamma_2(\delta_2):=\left(\frac{a}{b}\right)^l F\bigl(k^u,\delta_2\bigr)-\left(\frac{c\Theta_2}{b}\right)^u\delta_2>0,$$

$$\int_0^{\omega} f(t)G_2\bigl(t,\Gamma_2(\delta_2)\bigr)\,dt > \widehat{d}\omega + \widehat{r}\delta_2\omega.$$

Claim (ii). If $\Pi_2 > \widehat{d}$, then $\exp\{\varphi_2(\eta_2)\} \geq \delta_2$.

We omit the proof of *Claim* (ii) here since it is very similar to that of *Claim* (i).

From *Claim* (ii) and (2.13), we can obtain

$$\varphi_2(t) \geq \varphi_2(\eta_2) - \int_0^{\omega} \bigl|\dot{\varphi}_2(t)\bigr|\,dt \geq \ln(\delta_2) - 2\widehat{\Delta_2}\omega := L_2^{(2)}.$$

Now, let us consider the following algebraic equations:

$$\begin{cases} [\frac{1}{\omega}\int_0^{\omega} a(t)F(k(t),\exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\} \\ \quad - \frac{1}{\omega}\int_0^{\omega} \frac{c(t)G_2(t,\exp\{\varphi_1\})\exp\{\varphi_2\}}{\exp\{\varphi_1\}}\,dt] = 0, \\ \frac{1}{\omega}\int_0^{\omega} f(t)G_2(t,\exp\{\varphi_1\})\,dt - \widehat{d} - \mu\widehat{r}\exp\{\varphi_2\} = 0 \end{cases} \tag{2.14}$$

for $(\varphi_1,\varphi_2)^T \in \mathbb{R}^2$, where $\mu \in [0,1]$ is a parameter. By using similar arguments as above, we can show that any solution $(\varphi_{12}^*,\varphi_{22}^*)^T \in \mathbb{R}^2$ of (2.14) with $\mu \in [0,1]$ satisfies

$$\begin{aligned} l_1^{(2)} &= \ln\biggl(\frac{\widehat{d}}{\widehat{(f\Theta_2)}}\biggr) \leq \varphi_{12}^* \leq \ln\biggl(\frac{\widehat{a}}{\widehat{b}}\biggr) = m_1, \\ l_2^{(2)} &:= \ln\bigl(\delta_2^*\bigr) \leq \varphi_{22}^* \leq \ln\biggl(\frac{\widehat{a}}{\Phi_2(m_1,l_1^{(2)})}\biggr) := m_2^{(2)}, \end{aligned} \tag{2.15}$$

where $\delta_2^* > 0$ satisfies

$$\begin{aligned} &\Gamma_2^*\bigl(\delta_2^*\bigr) := \frac{\widehat{a}}{\widehat{b}}F\bigl(k^u,\delta_2^*\bigr) - \frac{1}{\widehat{b}}\biggl(\frac{1}{\omega}\int_0^{\omega} c(t)\Theta_2(t)\,dt\biggr)\delta_2^* > 0, \\ &\frac{1}{\omega}\int_0^{\omega} f(t)G_2\bigl(t,\Gamma_2^*\bigl(\delta_2^*\bigr)\bigr)\,dt > \widehat{d} + \widehat{r}\delta_2^*. \end{aligned}$$

Note that M_1, $M_2^{(2)}$, $L_1^{(2)}$, $L_2^{(2)}$, m_1, $m_2^{(2)}$, $l_1^{(2)}$, and $l_2^{(2)}$ are independent of μ. Define

$$\Omega^{(2)} = \bigl\{\varphi \in X \mid \|\varphi\| < U^{(2)}\bigr\},$$

where

$$U^{(2)} = 1 + \max\bigl\{\bigl|l_1^{(2)}\bigr|,\bigl|L_1^{(2)}\bigr|,|m_1|,|M_1|\bigr\} + \max\bigl\{\bigl|l_2^{(2)}\bigr|,\bigl|L_2^{(2)}\bigr|,\bigl|m_2^{(2)}\bigr|,\bigl|M_2^{(2)}\bigr|\bigr\}.$$

Clearly, $\Omega^{(2)}$ satisfies conditions (i) and (ii) in Lemma 2.1. Let us consider the homotopy

$$\Xi_{\mu}^{(2)}\bigl((\varphi_1,\varphi_2)^T\bigr) = \mu QN\bigl((\varphi_1,\varphi_2)^T\bigr) + (1-\mu)\Psi^{(2)}\bigl((\varphi_1,\varphi_2)^T\bigr), \quad \mu \in [0,1],$$

where

$$\Psi^{(2)}\bigl((\varphi_1,\varphi_2)^T\bigr) = \begin{bmatrix} \frac{1}{\omega}\int_0^{\omega} a(t)F(k(t),\exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\} - W_2(\varphi_1,\varphi_2) \\ \frac{1}{\omega}\int_0^{\omega} f(t)G_2(t,\exp\{\varphi_1\})\,dt - \widehat{d}, \end{bmatrix},$$

where

$$W_2(\varphi_1,\varphi_2)=\frac{1}{\omega}\int_0^{\omega}\frac{c(t)G_2(t,\exp\{\varphi_1\})\exp\{\varphi_2\}}{\exp\{\varphi_1\}}\,dt.$$

From (2.15), it follows that $0\notin\Xi_\mu^{(2)}(\partial\Omega^{(2)}\cap\mathbb{R}^2)$ for $\mu\in[0,1]$. In addition, from conditions (H) and (Q2), one can easily show that $\Psi^{(2)}((\varphi_1,\varphi_2)^T)=0$ has a unique solution $(\widetilde{\varphi_{12}^*},\widetilde{\varphi_{22}^*})^T$ in $\mathbb{R}^2$ if $\Pi_2>\widehat{d}$. Let $e_{12}=\exp\{\widetilde{\varphi_{12}^*}\}>0$, $e_{22}=\exp\{\widetilde{\varphi_{22}^*}\}>0$. From conditions (H) and (Q4), we have

$$\begin{aligned}
\varsigma_1^{(2)}&:=\frac{1}{\omega}\int_0^{\omega}a(t)\frac{\partial F(k(t),e_{22})}{\partial y}e_{22}\,dt\le 0,\\
\varsigma_2^{(2)}&:=\frac{1}{\omega}\int_0^{\omega}f(t)\frac{\partial G_2(t,e_{12})}{\partial x}e_{12}\,dt>0.
\end{aligned}$$

Note that $J=I$, by the invariance property of homotopy, we have

$$\begin{aligned}
&\deg\{JQN,\partial\Omega^{(2)}\cap\operatorname{Ker}L,0\}\\
&\quad=\deg\{\Psi^{(2)},\partial\Omega^{(2)}\cap\operatorname{Ker}L,0\}\\
&\quad=\operatorname{sign}\left\{-\left(\varsigma_1^{(2)}-\frac{1}{\omega}\int_0^{\omega}\frac{c(t)G_2(t,e_{12})e_{22}}{e_{12}}\,dt\right)\varsigma_2^{(2)}\right\}\\
&\quad=1\ne 0.
\end{aligned}$$

By Lemma 2.1, if condition (A2) holds, then model (2.1) admits at least one ω-periodic solution.

Case (iii). Condition (A3) holds.

From (2.7), (2.8), and conditions (H2) and (H3), we have

$$\int_0^{\omega}|\dot{\varphi}_2(t)|\,dt\le 2\int_0^{\omega}f(t)G_3\left(m^l,\frac{\exp\{M_1\}}{\exp\{\varphi_2(t)\}}\right)dt\le 2\Theta_3(m^l)\widehat{f}\omega:=2\Delta_3\omega.\tag{2.16}$$

From (2.5) and condition (H4), we have

$$\begin{aligned}
\widehat{d}\exp\{\varphi_2(\xi_2)\}&\le\frac{1}{\omega}\int_0^{\omega}f(t)G_3\left(m(t),\frac{\exp\{\varphi_1(t)\}}{\exp\{\varphi_2(t)\}}\right)\exp\{\varphi_2(t)\}\,dt\\
&\le\frac{1}{\omega}\int_0^{\omega}f(t)G_3\left(m^l,\frac{\exp\{M_1\}}{\exp\{\varphi_2(t)\}}\right)\frac{\exp\{\varphi_2(t)\}}{\exp\{M_1\}}\exp\{M_1\}\,dt\\
&\le\frac{1}{\omega}\int_0^{\omega}f(t)\Theta_4(m^l)\exp\{M_1\}\,dt\\
&=\Phi_3(M_1)>0,
\end{aligned}$$

which implies that

$$\exp\{\varphi_2(\xi_2)\}\le\frac{\Phi_3(M_1)}{\widehat{d}}.$$

Then, from (2.16), we have

$$\varphi_2(t)\le\varphi_2(\xi_2)+\int_0^{\omega}|\dot{\varphi}_2(t)|\,dt\le\ln\left(\frac{\Phi_3(M_1)}{\widehat{d}}\right)+2\Delta_3\omega=M_2:=M_2^{(3)}.$$

From $\dot{\varphi}_2(\eta_2) = 0$, we can obtain

$$f(\eta_2)G_3\left(m(\eta_2), \frac{\exp\{\varphi_1(\eta_2)\}}{\exp\{\varphi_2(\eta_2)\}}\right) = d(\eta_2) + r(\eta_2)\exp\{\varphi_2(\eta_2)\} \geq d(\eta_2),$$

which implies that

$$G_3\left(m^l, \frac{\exp\{\varphi_1(\eta_2)\}}{\exp\{\varphi_2(\eta_2)\}}\right) \geq \left(\frac{d}{f}\right)^l.$$

Note that

$$G_3\big(m^l, \Upsilon\big(M_2^{(3)}\big)\big) < \left(\frac{d}{f}\right)^l,$$

then from condition (H2), we have

$$\frac{\exp\{\varphi_1(\eta_2)\}}{\exp\{\varphi_2(\eta_2)\}} \geq \Upsilon(M_2) = \Upsilon\big(M_2^{(3)}\big).$$

Note that

$$\frac{\exp\{\varphi_1(\eta_1)\}}{\exp\{\varphi_2(\eta_1)\}} \geq \frac{\exp\{\varphi_1(\eta_2)\}}{\exp\{\varphi_2(\eta_2)\}} \geq \Upsilon\big(M_2^{(3)}\big) = \frac{(\frac{c}{a})^u(\frac{d}{f})^l}{F(k^u, \exp\{M_2^{(3)}\})},$$

then from $\dot{\varphi}_1(\eta_1) = 0$ and conditions (H), (H2), and (H4), we can obtain

$$\begin{aligned}
&\exp\{\varphi_1(\eta_1)\} \\
&\quad = \frac{a(\eta_1)}{b(\eta_1)}F\big(k(\eta_1), \exp\{\varphi_2(\eta_1)\}\big) \\
&\qquad - \frac{c(\eta_1)}{b(\eta_1)}G_3\left(m(\eta_1), \frac{\exp\{\varphi_1(\eta_1)\}}{\exp\{\varphi_2(\eta_1)\}}\right)\frac{\exp\{\varphi_2(\eta_1)\}}{\exp\{\varphi_1(\eta_1)\}} \\
&\quad \geq \frac{a(\eta_1)}{b(\eta_1)}F\big(k(\eta_1), \exp\{\varphi_2(\eta_1)\}\big) \\
&\qquad - \frac{c(\eta_1)}{b(\eta_1)}G_3\left(m(\eta_1), \frac{\exp\{\varphi_1(\eta_2)\}}{\exp\{\varphi_2(\eta_2)\}}\right)\frac{\exp\{\varphi_2(\eta_2)\}}{\exp\{\varphi_1(\eta_2)\}} \\
&\quad \geq \frac{a(\eta_1)}{b(\eta_1)}F\big(k^u, \exp\{M_2^{(3)}\}\big) - \frac{c(\eta_1)}{b(\eta_1)}\frac{G_3(m^l, \Upsilon(M_2^{(3)}))}{\Upsilon(M_2^{(3)})} \\
&\quad \geq \frac{a(\eta_1)}{b(\eta_1)}F\big(k^u, \exp\{M_2^{(3)}\}\big)\left[1 - \left(\frac{c}{a}\right)^u\frac{G_3(m^l, \Upsilon(M_2^{(3)}))}{F(k^u, \exp\{M_2^{(3)}\})\Upsilon(M_2^{(3)})}\right] \\
&\quad = \frac{a(\eta_1)}{b(\eta_1)(\frac{d}{f})^l}F\big(k^u, \exp\{M_2^{(3)}\}\big)\left[\left(\frac{d}{f}\right)^l - G_3\big(m^l, \Upsilon\big(M_2^{(3)}\big)\big)\right] \\
&\quad > 0,
\end{aligned}$$

which implies that

$$\varphi_1(\eta_1) \geq \ln\left\{\frac{(\frac{a}{b})^l}{(\frac{d}{f})^l}F\big(k^u, \exp\{M_2^{(3)}\}\big)\left[\left(\frac{d}{f}\right)^l - G_3\big(m^l, \Upsilon\big(M_2^{(3)}\big)\big)\right]\right\} := \widetilde{l_1^{(3)}}. \tag{2.17}$$

Then, from (2.6) and (2.17), we have

$$\varphi_1(t) \geq \varphi_1(\eta_1) - \int_0^\omega \left|\dot{\varphi}_1(t)\right| dt \geq \widetilde{l_1^{(3)}} - 2\widehat{a}\omega := L_1^{(3)}.$$

If $\Theta_3(m^u)\widehat{f} > \widehat{d}$, then there exists sufficiently small $\delta_3 > 0$ such that

$$G_3\left(m^u, \frac{\exp\{L_1^{(3)}\}}{\delta_3}\right)\widehat{f} > \widehat{d} + \widehat{r}\delta_3. \tag{2.18}$$

Claim (iii). If $\Theta_3(m^u)\widehat{f} > \widehat{d}$, then $\exp\{\varphi_2(\eta_2)\} \geq \delta_3$.

If the claim is not true, then $\exp\{\varphi_2(\eta_2)\} < \delta_3$. From (2.4) and condition (H2), we have

$$\begin{aligned}
\widehat{d}\omega + \widehat{r}\delta_3\omega &\geq \int_0^\omega \left[d(t) + r(t)\exp\{\varphi_2(t)\}\right] dt \\
&= \int_0^\omega f(t)G_3\left(m(t), \frac{\exp\{\varphi_1(t)\}}{\exp\{\varphi_2(t)\}}\right) dt \\
&\geq \int_0^\omega f(t)G_3\left(m^u, \frac{\exp\{L_1^{(3)}\}}{\delta_3}\right) dt \\
&= G_3\left(m^u, \frac{\exp\{L_1^{(3)}\}}{\delta_3}\right)\widehat{f}\omega,
\end{aligned}$$

which is a contradiction to (2.18). This proves the claim.

From *Claim* (iii) and (2.16), we can obtain

$$\varphi_2(t) \geq \varphi_2(\eta_2) - \int_0^\omega \left|\dot{\varphi}_2(t)\right| dt \geq \ln(\delta_3) - 2\Delta_3\omega := L_2^{(3)}.$$

Let $(\varphi_1, \varphi_2)^T \in \mathbb{R}^2$ satisfy the following algebraic equations:

$$\begin{cases}
[\frac{1}{\omega}\int_0^\omega a(t)F(k(t), \exp\{\varphi_2\})\, dt - \widehat{b}\exp\{\varphi_1\} \\
\quad - \frac{1}{\omega}\int_0^\omega \mu c(t)G_3(m(t), \frac{\exp\{\varphi_1\}}{\exp\{\varphi_2\}})\frac{\exp\{\varphi_2\}}{\exp\{\varphi_1\}}\, dt] = 0, \\
\frac{1}{\omega}\int_0^\omega f(t)G_3(m(t), \frac{\exp\{\varphi_1\}}{\exp\{\varphi_2\}})\, dt - \widehat{d} - \mu\widehat{r}\exp\{\varphi_2\} = 0,
\end{cases} \tag{2.19}$$

where $\mu \in [0, 1]$ is a parameter. By using similar arguments as above, we can show that any solution $(\varphi_{13}^*, \varphi_{23}^*)^T \in \mathbb{R}^2$ of (2.19) with $\mu \in [0, 1]$ satisfies

$$\begin{aligned}
&l_1^{(3)} := \ln\left\{\frac{\widehat{a}}{\widehat{b}(\frac{d}{f})^l}F\left(k^u, \exp\{m_2^{(3)}\}\right)\left[\left(\frac{d}{f}\right)^l - \frac{\widehat{c}}{\widehat{a}(\frac{c}{a})^u}G_3\left(m^l, \Upsilon\left(m_2^{(3)}\right)\right)\right]\right\} \\
&\quad \leq \varphi_{13}^* \leq \ln\left(\frac{\widehat{a}}{\widehat{b}}\right) = m_1, \\
&l_2^{(3)} := \ln\left(\delta_3^*\right) \leq \varphi_{23}^* \leq \ln\left(\frac{\Phi_3(m_1)}{\widehat{d}}\right) := m_2^{(3)},
\end{aligned} \tag{2.20}$$

where $\delta_3^* > 0$ satisfies

$$G_3\left(m^u, \frac{\exp\{l_1^{(3)}\}}{\delta_3^*}\right)\widehat{f} > \widehat{d} + \widehat{r}\delta_3^*.$$

Note that $M_1, M_2^{(3)}, L_1^{(3)}, L_2^{(3)}, m_1, m_2^{(3)}, l_1^{(3)}$, and $l_2^{(3)}$ are independent of μ. Define

$$\Omega^{(3)} = \{\varphi \in X \mid \|\varphi\| < U^{(3)}\},$$

where

$$U^{(3)} = 1 + \max\{|l_1^{(3)}|, |L_1^{(3)}|, |m_1|, |M_1|\} + \max\{|l_2^{(3)}|, |L_2^{(3)}|, |m_2^{(3)}|, |M_2^{(3)}|\}.$$

Clearly, $\Omega^{(3)}$ satisfies conditions (i) and (ii) in Lemma 2.1. To compute the Brouwer degree, let us consider the homotopy

$$\Xi_\mu^{(3)}((\varphi_1,\varphi_2)^T) = \mu QN((\varphi_1,\varphi_2)^T) + (1-\mu)\Psi^{(3)}((\varphi_1,\varphi_2)^T), \quad \mu \in [0,1],$$

where

$$\Psi^{(3)}((\varphi_1,\varphi_2)^T) = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega a(t)F(k(t),\exp\{\varphi_2\})\,dt - \widehat{b}\exp\{\varphi_1\} \\ \frac{1}{\omega}\int_0^\omega f(t)G_3(m(t),\frac{\exp\{\varphi_1\}}{\exp\{\varphi_2\}})\,dt - \widehat{d} \end{bmatrix}.$$

From (2.20), it follows that $0 \notin \Xi_\mu^{(3)}(\partial\Omega^{(3)} \cap \mathbb{R}^2)$ for $\mu \in [0,1]$. In addition, from conditions (H), (H2), and (H3), one can easily show that $\Psi^{(3)}((\varphi_1,\varphi_2)^T) = 0$ has a unique solution $(\widetilde{\varphi_{13}^*}, \widetilde{\varphi_{23}^*})^T$ in $\mathbb{R}^2$ if $\Pi_3 > \widehat{d}$. Let $e_{13} = \exp\{\widetilde{\varphi_{13}^*}\} > 0$, $e_{23} = \exp\{\widetilde{\varphi_{23}^*}\} > 0$. From conditions (H) and (H3), we have

$$\varsigma_1^{(3)} := \frac{1}{\omega}\int_0^\omega a(t)\frac{\partial F(k(t),e_{23})e_{23}}{\partial y}\,dt \le 0,$$
$$\varsigma_2^{(3)} := \frac{1}{\omega}\int_0^\omega f(t)\frac{\partial G_3(m(t),\frac{e_{13}}{e_{23}})}{\partial z}\frac{e_{13}}{e_{23}}\,dt > 0.$$

A direct calculation produces

$$\begin{aligned}\deg\{\Psi^{(3)}, \partial\Omega^{(3)} \cap \operatorname{Ker} L, 0\} &= \operatorname{sign}\begin{vmatrix} -\widehat{b}e_{13} & \varsigma_1^{(3)} \\ \varsigma_2^{(3)} & -\varsigma_2^{(3)} \end{vmatrix} \\ &= \operatorname{sign}\{\varsigma_2^{(3)}(\widehat{b}e_{13} - \varsigma_1^{(3)})\} \\ &= 1 \neq 0.\end{aligned}$$

Note that $J = I$, by the invariance property of homotopy, we have

$$\begin{aligned}\deg\{JQN, \partial\Omega^{(3)} \cap \operatorname{Ker} L, 0\} &= \deg\{QN, \partial\Omega^{(3)} \cap \operatorname{Ker} L, 0\} \\ &= \deg\{\Psi^{(3)}, \partial\Omega^{(3)} \cap \operatorname{Ker} L, 0\} \neq 0.\end{aligned}$$

By Lemma 2.1, if condition (A3) holds, then model (2.1) admits at least one ω-periodic solution. □

3 Some remarks

Our results improve and extend some of the previous results. Some of the remarks below will compare our results with some of the previous results.

If we choose

$$F(k,y)=\frac{1}{1+ky},\qquad G(t,x,y)\equiv G_1(t,x,y)=\frac{x}{\alpha_1(t)+\alpha_2(t)x+\alpha_3(t)y+\alpha_4(t)xy},$$

then model (1.2) becomes the following periodic predator–prey model with Crowley–Martin functional response:

$$\begin{cases}\dot{x}(t)=x(t)[\frac{a(t)}{1+k(t)y(t)}-b(t)x(t)\\ \qquad -\frac{c(t)y(t)}{\alpha_1(t)+\alpha_2(t)x(t)+\alpha_3(t)y(t)+\alpha_4(t)x(t)y(t)}],\\ \dot{y}(t)=y(t)[-d(t)-r(t)y(t)+\frac{f(t)x(t)}{\alpha_1(t)+\alpha_2(t)x(t)+\alpha_3(t)y(t)+\alpha_4(t)x(t)y(t)}].\end{cases}\tag{A}$$

For model (A), an application of our main results is as follows.

Theorem 3.1 *Assume that the following condition*

$$(H_A^{(1)})\quad \widehat{c}>0,\qquad \frac{1}{\omega}\int_0^\omega \frac{f(t)(\frac{a}{b})^l}{\alpha_1(t)+\alpha_2(t)(\frac{a}{b})^l}\,dt>\widehat{d}$$

holds, then model (A) *has at least one positive ω-periodic solution.*

A direct corollary of Theorem 3.1 is given below.

Corollary 3.1 *Assume that the following condition*

$$(H_A^{(2)})\quad \widehat{c}>0,\qquad \left(\frac{a}{b}\right)^l(\widehat{f}-\widehat{d}\alpha_2^u)>\alpha_1^u\widehat{d}$$

holds, then model (A) *has at least one positive ω-periodic solution.*

Remark 3.1 Recently, Tripathi et al. [14] proved that model (A) has at least one positive ω-periodic solution under the following condition:

$$(H_A^{(3)})\quad \begin{cases} k(t)\equiv 0,\qquad c^l>0,\qquad \alpha_2^l>0,\qquad \alpha_4^l>0,\qquad r^l>0,\\ d^l<\frac{f^u(\frac{a^u}{b^l})}{\alpha_1^l+\alpha_2^l(\frac{a^u}{b^l})}:=D_1,\qquad a^l>\frac{c^u(\frac{D_1-d^l}{r^l})}{\alpha_1^l+\alpha_3^l(\frac{D_1-d^l}{r^l})}:=D_2,\\ \frac{f^l(\frac{a^l-D_2}{b^u})}{\alpha_1^u+\alpha_2^u(\frac{a^l-D_2}{b^u})+\alpha_3^u(\frac{D_1-d^l}{r^l})+\alpha_4^u(\frac{a^l-D_2}{b^u})(\frac{D_1-d^l}{r^l})}>d^u.\end{cases}$$

Clearly, our condition $(H_A^{(2)})$ of Corollary 3.1 is weaker than condition $(H_A^{(3)})$. Thus, our Corollary 3.1 improves Theorem 8 in [14].

Remark 3.2 Li and She [10] proved that model (A) has at least one positive ω-periodic solution under the following condition:

$$(H_A^{(4)})\quad \begin{cases} k(t)\equiv 0, & c^l>0, \quad \alpha_2^l>0, \quad \alpha_4(t)\equiv 0, \quad r^l>0, \\ \alpha_3^l a^l > c^u, & (\frac{a^l}{b^u}-\frac{c^u}{b^u\alpha_3^l})(f^l-d^u\alpha_2^u)>\alpha_1^u d^u. \end{cases}$$

Clearly, our condition $(H_A^{(2)})$ of Corollary 3.1 is weaker than condition $(H_A^{(4)})$. Thus, our Corollary 3.1 improves Theorem 4.2 in [10].

Remark 3.3 Fan and Kuang [7] proved that model (A) has at least one positive ω-periodic solution under the following condition:

$$(H_A^{(5)})\quad \begin{cases} k(t)\equiv 0, & c^l>0, \quad \alpha_2^l>0, \quad \alpha_4(t)\equiv 0, \quad r(t)\equiv 0, \\ \alpha_3^l a^l > c^u, & (\frac{a^l}{b^u}-\frac{c^u}{b^u\alpha_3^l})(f^l-d^u\alpha_2^u)>\alpha_1^u d^u. \end{cases}$$

In [7], the authors only assumed that the function $\alpha_1(t)$ is nonnegative. If $\alpha_1^l>0$, then our condition $(H_A^{(2)})$ of Corollary 3.1 is weaker than condition $(H_A^{(5)})$. Thus, our Corollary 3.1 extends Theorem 3.1 in [7].

Remark 3.4 For model (A), or some of its special cases, some scholars have obtained some plentiful results of the existence of positive periodic solutions [4, 7–9, 14]. Note that our Theorem 3.1 and Corollary 3.1 do not limit the size of the period ω. Compared with some results in [4, 7, 9, 14] (see Theorem 3.2 in Fan and Kuang [7], Theorem 3.1 in Bohner et al. [4], Theorems 1 and 2 in Fazly and Hesaaraki [9], and Theorem 9 in Tripathi [14]), we obtain different results.

If we choose

$$F(k,y)\equiv 1,\qquad G(t,x,y)\equiv G_3\left(m(t),\frac{x}{y}\right)=\frac{\frac{x}{y}}{m(t)+\frac{x}{y}}=\frac{x}{m(t)y+x},$$

then model (1.2) becomes the following periodic ratio-dependent type predator–prey model with Michaelis–Menten type functional response:

$$\begin{cases} \dot{x}(t)=x(t)[a(t)-b(t)x(t)-\frac{c(t)y(t)}{m(t)y(t)+x(t)}], \\ \dot{y}(t)=y(t)[-d(t)-r(t)y(t)+\frac{f(t)x(t)}{m(t)y(t)+x(t)}]. \end{cases}\tag{B}$$

For model (B), an application of our main results is as follows.

Theorem 3.2 *Assume that the following condition*

$$(H_B^{(1)})\quad \widehat{c}>0,\qquad \widehat{f}>\widehat{d},\qquad \left(\frac{c}{a}\right)^u\left[1-\left(\frac{d}{f}\right)^l\right]<m^l$$

holds, then model (B) has at least one positive ω-periodic solution.

Remark 3.5 Li and She [3] proved that model (B) has at least one positive ω-periodic solution under the condition

$$(H_B^{(2)}) \quad c^l > 0, \qquad r^l > 0, \qquad f^l > d^u, \qquad a^l > \frac{c^u}{m^l}.$$

Clearly, our condition $(H_B^{(1)})$ of Theorem 3.2 is weaker than condition $(H_B^{(2)})$. Thus, our Theorem 3.2 improves Theorem 3.1 in [3].

Remark 3.6 For model (B), if $r(t) \equiv 0$, then we would like to mention the following two facts.

(i) If model (B) degenerates into the corresponding autonomous system (the case of $a(t) \equiv a > 0$, $b(t) \equiv b > 0$, $c(t) \equiv c > 0$, $m(t) \equiv m > 0$, $d(t) \equiv d > 0$, and $f(t) \equiv f > 0$), then our condition $(H_B^{(2)})$ can be reduced to the sufficient and necessary condition $(f > d, \frac{c}{a}(1 - \frac{d}{f}) < m)$ for the existence of the positive equilibrium of model (B) (see [29]).

(ii) Fan, Wang, and Zou [2] proved that model (B) has at least one positive ω-periodic solution under the condition

$$(H_B^{(3)}) \quad c^l > 0, \qquad \widehat{f} > \widehat{d}, \qquad \widehat{a} > \widehat{\left(\frac{c}{m}\right)}.$$

As can be seen from (i), we give a different result compared to Theorem 3.5 in [2]. Thus, our Theorem 3.2 extends Theorem 3.5 in [2].

4 Conclusions and numerical simulations

In this paper, the existence of periodic solutions of a class of periodic predator–prey model with fear effect and general functional responses is investigated by means of the continuation theorem of coincidence degree theory [24]. The general functional responses can cover the Holling types II and III functional response, the Beddington–DeAngelis functional response, the Crowley–Martin functional response, the ratio-dependent type with Michaelis–Menten type functional response, etc. The most crucial aspect of using the coincidence theorem is to find a bounded open set Ω that satisfies the conditions of the theorem, which requires estimating the upper and lower bounds on the a priori solution of the corresponding operator equation ($L\varphi = \mu N\varphi$, $0 < \mu < 1$). This paper mainly obtains three classes of sufficient conditions for the existence of positive periodic solutions of model (1.2). Our main results improve or extend some of the known results (see Remarks 3.1–3.6 in Sect. 3).

At the end of the paper, we present numerical simulations to illustrate our theoretical results. We fix the following parameters: $a(t) = 4 + 0.5\sin(t)$, $b(t) = 1 + 0.25\cos(t)$, $c(t) = 1 + 0.2\cos(t)$, $d(t) = 0.6 + 0.5\cos(t)$, $r(t) = 3(1 + 0.5\cos(t))$, $f(t) = 2(1 + 0.5\cos(t))$.

(i) Let us further choose $k(t) = 3(1 + 0.5\cos(t))$, $\alpha_1(t) = 0.6(1 + 0.45\sin(t))$, $\alpha_2(t) = 0.3 + 0.2\sin(t)$, $\alpha_3(t) = 0.5 + 0.1\sin(t)$, $\alpha_4(t) = 2(1 + 0.15\sin(t))$. Then we have $\omega = 2\pi$, $\widehat{c} = 1 > 0$,

$$\frac{1}{\omega}\int_0^\omega \frac{f(t)(\frac{a}{b})^l}{\alpha_1(t) + \alpha_2(t)(\frac{a}{b})^l}\,dt \approx 4.968889 > \widehat{d} = 0.6.$$

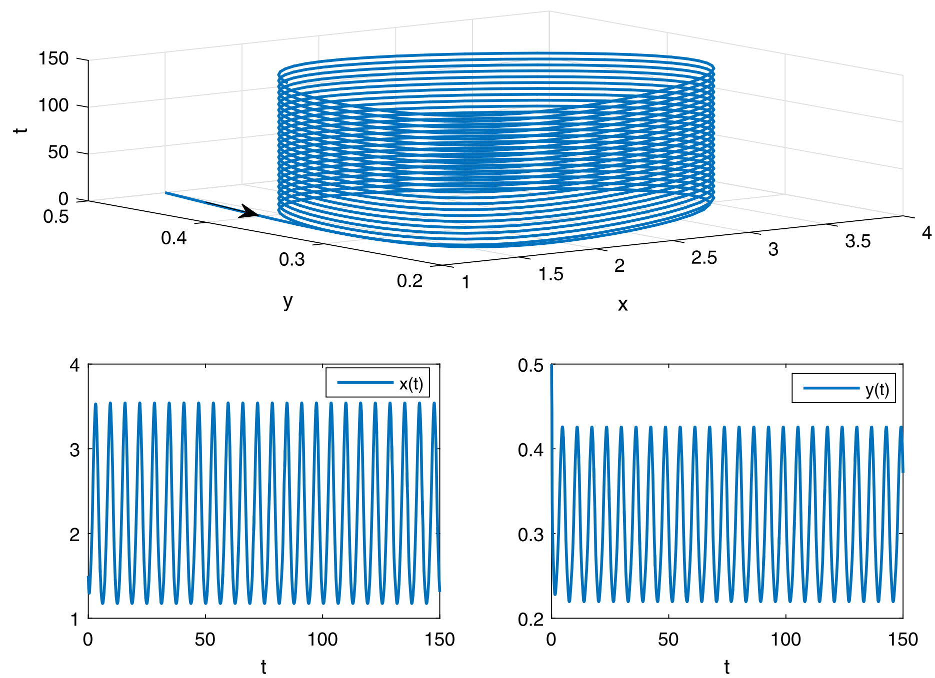

Figure 1 The phase trajectory and solution curves of model (A) with the initial value $(1.5, 0.5)^T$

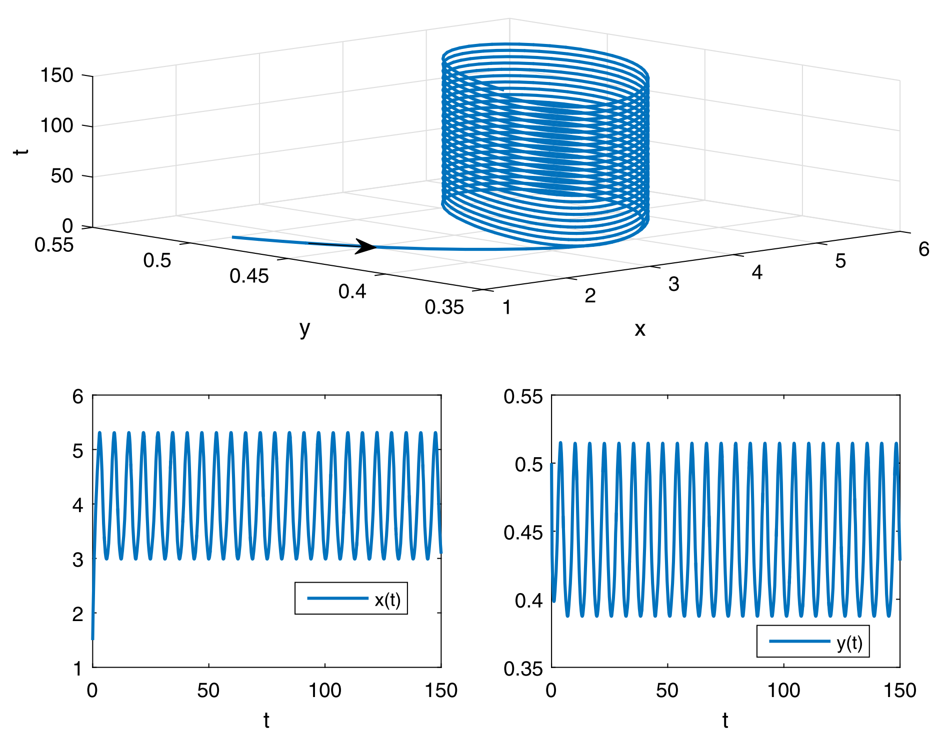

Figure 2 The phase trajectory and solution curves of model (B) with the initial value $(1.5, 0.5)^T$

From Theorem 3.1, it follows that model (A) has at least one positive 2π-periodic solution (see Fig. 1).

(ii) Let us further choose $m(t) = 0.5 + 0.1\sin(t)$. Then we have $\omega = 2\pi$, $\widehat{c} = 1 > 0$,

$$\widehat{f} = 2 > \widehat{d} = 0.6, \qquad \left(\frac{c}{a}\right)^{u}\left[1 - \left(\frac{d}{f}\right)^{l}\right] \approx 0.282176 < m^{l} = 0.4.$$

From Theorem 3.2, it follows that model (B) has at least one positive 2π-periodic solution (see Fig. 2).

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The authors declare that they have no competing interests.

Author contributions
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