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RESEARCH

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# The Lax pair structure for the spin Benjamin–Ono equation

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À la mémoire de Jean Ginibre

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**Abstract**

We prove that the recently introduced spin Benjamin–Ono equation admits a Lax pair and deduce a family of conservation laws that allow proving global wellposedness in all Sobolev spaces $H^k$ for every integer $k \geq 2$. We also infer an additional family of matrix-valued conservation laws of which the previous family is just the traces.

**MSC:** Primary 37K15; secondary 47B35

**Keywords:** Lax pair; Benjamin–Ono; Spin systems

## 1 Introduction

In a recent paper [1], Berntson, Langmann, and Lenells have introduced the following spin generalization of the Benjamin–Ono equation on the line $\mathbb{R}$ or on the torus $\mathbb{T}$,

$$\partial_t U + \{U, \partial_x U\} + H\partial_x^2 U - i[U, H\partial_x U] = 0, \quad x \in X,$$

where $X$ denotes $\mathbb{R}$ or $\mathbb{T}$, the unknown $U$ is valued into $d \times d$ matrices, and $H$ denotes the scalar Hilbert transform on $X$; in fact, the authors chose the normalization $H = i\mathrm{sign}(D)$ so that $H\partial_x = -|D|$, where $|D|$ denotes the Fourier multiplier associated to the symbol $|k|$. Notice that in front of the commutator term on the right-hand side, we take a different sign from the one used in [1]. However, passing to the other sign by applying the complex conjugation is easy. Consequently, the above equation reads

$$\partial_t U = \partial_x\big(|D|U - U^2\big) - i\big[U, |D|U\big]. \tag{1}$$

The purpose of this note is to prove that equation (1) enjoys a Lax pair structure and to infer the first consequences on the corresponding dynamics.

## 2 The Lax pair structure

Let us first introduce some more notation. Given operators $A, B$, we denote

$$\{A, B\} := AB + BA, \qquad [A, B] := AB - BA$$

and $A^*$ denote the adjoint of $A$. We consider the Hilbert space $\mathscr{H} := L^2_+(X, \mathbb{C}^{d\times d})$ made of $L^2$ functions on $X$ with Fourier transforms supported in nonnegative modes, and valued into $d \times d$ matrices, endowed with the inner product $\langle A|B\rangle = \int_X \operatorname{tr}(AB^*)\,dx$. We denote by $\Pi_{\geq 0}$ the orthogonal projector from $L^2(X, \mathbb{C}^{d\times d})$ onto $\mathscr{H}$. According to the study of the integrability of the scalar Benjamin–Ono equation [2], given $U \in L^2(X, \mathbb{C}^{d\times d})$ valued into $\mathbb{C}^{d\times d}$, we define on $\mathscr{H}$ the unbounded operator

$$L_U := D - T_U, \qquad D := \frac{1}{i}\partial_x,$$

where $\operatorname{dom}(L_U) := \{F \in \mathscr{H} : DF \in \mathscr{H}\}$, and $T_U$ is the Toeplitz operator of symbol $U$ defined by $T_U(F) := \Pi_{\geq 0}(UF)$. It is easy to check that $L_U$ is self-adjoint if $U$ is valued in Hermitian matrices. However, we do not need the latter property for establishing the Lax pair structure. If $U$ is smooth enough (say belonging to the Sobolev space $H^2$), we define the following bounded operator,

$$B_U := i\big(T_{|D|U} - T_U^2\big),$$

which is anti-self-adjoint if $U$ is valued in Hermitian matrices. Our main result is the following.

**Theorem 1** *Let $I$ be a time interval and $U$ be a continuous function on $I$ valued into $H^2(X, \mathbb{C}^{d\times d})$ such that $\partial_t U$ is continuous valued into $L^2(X, \mathbb{C}^{d\times d})$. Then $U$ is a solution of (1) on $I$ if and only if*

$$\partial_t L_U = [B_U, L_U].$$

*Proof* Obviously, $\partial_t L_U = -T_{\partial_t U}$. Since $T_G = 0$ implies classically $G = 0$, the claim is equivalent to the identity

$$-T_{\partial_x(|D|U - U^2) - i[U,|D|U]} = [B_U, L_U].$$

We have

$$\begin{aligned} -T_{\partial_x(|D|U - U^2) - i[U,|D|U]} &= [iT_{|D|U}, D] + T_{U\partial_x U + \partial_x U U} + iT_{[U,|D|U]} \\ &= [B_U, D] + T_{U\partial_x U + \partial_x U U} - T_U T_{\partial_x U} - T_{\partial_x U} T_U + iT_{[U,|D|U]} \\ &= [B_U, L_U] + T_{\{U,\partial_x U\}} - \{T_U, T_{\partial_x U}\} + iT_{[U,|D|U]} - i[T_U, T_{|D|U}] \end{aligned}$$

So, we have to check that

$$T_{\{U,\partial_x U\}} - \{T_U, T_{\partial_x U}\} + iT_{[U,|D|U]} - i[T_U, T_{|D|U}] = 0. \tag{2}$$

We need the following lemma, where we denote $\Pi_{<0} := Id - \Pi_{\geq 0}$.

**Lemma 1** *Let $A, B \in L^\infty(X, \mathbb{C}^{d\times d})$. Then, for every $F \in \mathscr{H}$,*

$$(T_{AB} - T_A T_B)F = \Pi_{\geq 0}\big(\Pi_{\geq 0}(A)\Pi_{<0}\big(\Pi_{<0}(B)F\big)\big).$$

Let us prove Lemma 1. Write

$$T_{AB}F = \Pi_{\geq 0}(ABF) = \Pi_{\geq 0}\big(A\Pi_{\geq 0}(BF)\big) + \Pi_{\geq 0}\big(A\Pi_{<0}(BF)\big) = T_A T_B F + \Pi_{\geq 0}\big(A\Pi_{<0}(BF)\big)$$

so that observing that the ranges of $\Pi_{\geq 0}$ and of $\Pi_{<0}$ are stable through the multiplication,

$$(T_{AB} - T_A T_B)F = \Pi_{\geq 0}\big(A\Pi_{<0}(BF)\big) = \Pi_{\geq 0}\big(\Pi_{\geq 0}(A)\Pi_{<0}\big(\Pi_{<0}(B)F\big)\big).$$

This completes the proof of Lemma 1. Let us apply Lemma 1 to $A = U$, $B = |D|U$. We get

$$\begin{aligned} i(T_{U|D|U} - T_U T_{|D|U})F &= \Pi_{\geq 0}\big(\Pi_{\geq 0}(U)\Pi_{<0}\big(\Pi_{<0}\big(i|D|U\big)F\big)\big) \\ &= -\Pi_{\geq 0}\big(\Pi_{\geq 0}(U)\Pi_{<0}\big(\Pi_{<0}(\partial_x U)F\big)\big), \end{aligned}$$

and similarly

$$\begin{aligned} i(T_{|D|UU} - T_{|D|U} T_U)F &= \Pi_{\geq 0}\big(\Pi_{\geq 0}\big(i|D|U\big)\Pi_{<0}\big(\Pi_{<0}(U)F\big)\big) \\ &= \Pi_{\geq 0}\big(\Pi_{\geq 0}(\partial_x U)\Pi_{<0}\big(\Pi_{<0}(U)F\big)\big) \end{aligned}$$

so that

$$\begin{aligned} \big(iT_{[U,|D|U]} - i[T_U, T_{|D|U}]\big)F &= -\Pi_{\geq 0}\big(\Pi_{\geq 0}(U)\Pi_{<0}\big(\Pi_{<0}(\partial_x U)F\big)\big) \\ &\quad - \Pi_{\geq 0}\big(\Pi_{\geq 0}(\partial_x U)\Pi_{<0}\big(\Pi_{<0}(U)F\big)\big) \\ &= -T_{\{U,\partial_x U\}}(F) + \{T_U, T_{\partial_x U}\}(F), \end{aligned}$$

using again Lemma 1. Hence, we have proved identity (2). □

## 3 Conservation laws and global wellposedness

The following is an application of Theorem 1.

**Corollary 1** *Assume that $U_0$ belongs to the Sobolev space $H^2(X, \mathbb{C}^{d\times d})$ and is valued into Hermitian matrices. Then equation (1) has a unique solution $U$, depending continuously on $t \in \mathbb{R}$, valued into Hermitian matrices of the Sobolev space $H^2(X)$, and such that $U(0) = U_0$. Furthermore, the following quantities are conservation laws,*

$$\mathscr{E}_k(U) = \big\langle L_U^k(\Pi_{\geq 0}U)|\Pi_{\geq 0}U\big\rangle, \quad k = 0, 1, 2 \ldots.$$

*In particular, the norm of $U(t)$ in the Sobolev space $H^2(X)$ is uniformly bounded for $t \in \mathbb{R}$.*

*Proof* The local wellposedness in the Sobolev space $H^2$ follows from an easy adaptation of Kato's iterative scheme—see, e.g., Kato [3] for hyperbolic systems. Global wellposedness will follow if we show that conservation laws control the $H^2$ norm. Set $U_+ := \Pi_{\geq 0}U$, $U_- := \Pi_{<0}U$. Applying $\Pi_{\geq 0}$ to both sides of (1), we get

$$\partial_t U_+ = -i\partial_x^2 U_+ - 2T_U \partial_x U_+ - 2T_{\partial_x U_-} U_+ = iL_U^2(U_+) + B_U(U_+).$$

Therefore, from Theorem 1,

$$\begin{aligned}\frac{d}{dt}\langle L_u^k(U_+)|U_+\rangle &= \langle [B_U, L_U^k]U_+ \big| U_+\rangle + \langle L_U^k\big(iL_U^2(U_+) + B_U(U_+)\big)\big|U_+\rangle \\ &\quad + \langle L_U^k(U_+)|iL_U^2(U_+) + B_U(U_+)\rangle \\ &= 0,\end{aligned}$$

since $B_U$ and $iL_U^2$ are anti-self-ajoint.

Now observe that $\mathscr{E}_0(U) = \|U_+\|_{L^2}^2$. Since $U$ is Hermitian, we have

$$U = \begin{cases} U_+ + U_+^* & \text{if } X = \mathbb{R}, \\ U_+ + U_+^* - \langle U_+\rangle & \text{if } X = \mathbb{T}, \end{cases}$$

where $\langle F\rangle$ denotes the mean value of a function $F$ on $\mathbb{T}$. We infer that $\mathscr{E}_0(U)$ controls the $L^2$ norm of $U$. Let us come to $\mathscr{E}_1(U)$. In view of the Gagliardo–Nirenberg inequality,

$$\begin{aligned}\mathscr{E}_1(U) &= \langle DU_+\big|U_+\rangle - \langle T_U(U_+)\big|U_+\rangle \ge \langle DU_+|U_+\rangle - O\big(\|U_+\|_{L^3}^3\big) \\ &\ge \langle DU_+|U_+\rangle - O\big(\langle DU_+|U_+\rangle^{1/2}\|U_+\|_{L^2}^2\big) - O\big(\|U_+\|_{L^2}^3\big).\end{aligned}$$

Consequently, $\mathscr{E}_0(U)$ and $\mathscr{E}_1(U)$ control $\|U_+\|_{L^2}^2 + \langle DU_+|U_+\rangle$, which is the square of the $H^{1/2}$ norm of $U_+$, since $U_+$ only has nonnegative Fourier modes. Therefore, the $H^{1/2}$ norm of $U$ is controlled by $\mathscr{E}_0(U)$ and $\mathscr{E}_1(U)$.

Since $\mathscr{E}_2(U)$ is the square of $L^2$ norm of $L_U(U_+)$ and the $L^2$ norm of $T_U(U_+)$ is controlled by the $H^{1/2}$ norm of $U$ by the Sobolev estimate, we infer that $\mathscr{E}_0(U)$, $\mathscr{E}_1(U)$, and $\mathscr{E}_2(U)$ control the $L^2$ norms of $U$ and of $\partial_x U$, namely the Sobolev $H^1$ norm of $U$.

Finally, $\mathscr{E}_4(U)$ is the square if the $L^2$ norm of $L_U^2(U_+)$. Since $L_U(U_+)$ is already controlled in $L^2$ and $U$ is controlled in $L^\infty$ by the Sobolev inclusion $H^1 \subset L^\infty$, we infer that the $H^1$ norm of $L_U(U_+)$ is controlled. But $H^1$ is an algebra, so the $H^1$ norm of $T_U(U_+)$ is also controlled. Finally, we infer that $\{\mathscr{E}_n(U), n \le 4\}$ control the $H^1$ norms of $U_+$ and $\partial_x U_+$, namely the $H^2$ norm of $U_+$, and finally of $U$. □

*Remarks.*

(1) If the initial datum $U$ belongs to the Sobolev space $H^k$ for an integer $k > 2$, a similar argument shows that the $H^k$ norm of $U$ is controlled by the collection $\{\mathscr{E}_n(U), 0 \le n \le 2k\}$.

(2) In [1], the evolution of multi-solitons for (1) is derived through a pole ansatz, and the question of keeping the poles away from the real line—or from the unit circle in the case $X = \mathbb{T}$—is left open. Since Corollary 1 implies that the $L^\infty$ norm of the solution stays bounded as $t$ varies, this implies a positive answer to this question, as far as the poles do not collide. In fact, we strongly suspect that such a collision does not affect the structure of the pole ansatz because it is likely that multisolitons have a characterization in terms of the spectrum of $L_U$, as it has in the scalar case [2].

Let us say a few more about conservation laws. The conservation laws $\mathscr{E}_k$ can be explicitly computed in terms of $U$. For simplicity, we focus on $\mathscr{E}_0$ and $\mathscr{E}_1$. In case $X = \mathbb{R}$, we have exactly

$$\mathscr{E}_0(U) = \frac{1}{2}\int_{\mathbb{R}} \operatorname{tr}\big(U^2\big)\,dx,$$

and

$$\begin{aligned}\mathscr{E}_1(U) &= \big\langle DU_+\big|U_+\big\rangle - \big\langle T_U(U_+)\big|U_+\big\rangle\\ &= \int_{\mathbb{R}} \operatorname{tr}\left(\frac{1}{2}U|D|U - \frac{1}{3}U^3\right)dx,\end{aligned}$$

so we recover the Hamiltonian function derived in [1].

In case $X = \mathbb{T}$, the above formulae must be slightly modified due the zero Fourier mode. This leads us to a *bigger set of conservation laws*. Indeed, every constant matrix $V \in \mathbb{C}^{d\times d}$ is a special element of $\mathscr{H}$, and we observe that $B_U(V) = -iL_U^2(V)$. Arguing exactly as in the proof of Corollary 1, we infer that, for every integer $\ell \geq 1$, for every pair of constant matrices $V, W$, the quantity $\langle L_U^\ell(V)|W\rangle$ is a conservation law. Since $V, W$ are arbitrary, this means that, if $\mathbf{1}$ denotes the identity matrix, all the matrix-valued functionals

$$\mathscr{M}_{\ell-2}(U) := \int_{\mathbb{T}} L_U^\ell(\mathbf{1})\,dx$$

for $\ell \geq 1$ are conservation laws. If the measure of $\mathbb{T}$ is normalised to 1, we have for instance

$$\begin{aligned}&\mathscr{M}_{-1}(U) = -\langle U_+\rangle = -\langle U\rangle,\\ &\mathscr{M}_0(U) = \frac{1}{2}\big\langle U^2 - iUHU\big\rangle + \frac{1}{2}\langle U\rangle^2.\end{aligned}$$

Then one can check that

$$\begin{aligned}&\mathscr{E}_0(U) = \frac{1}{2}\operatorname{tr}\big(\langle U^2\rangle\big) + \frac{1}{2}\operatorname{tr}\big(\langle U\rangle^2\big),\\ &\mathscr{E}_1(U) = \operatorname{tr}\left\langle\frac{1}{2}U|D|U - \frac{1}{3}U^3\right\rangle - \frac{5}{3}\operatorname{tr}\big[\langle U\rangle^3\big] - \operatorname{tr}\big[\mathscr{M}_0(U)\langle U\rangle\big].\end{aligned}$$

Observe again that the first term on the right-hand side of the expression of $\mathscr{E}_1(U)$ is the opposite of the Hamiltonian function in [1].

In the case $X = \mathbb{R}$, all the matrix valued expressions $\mathscr{M}_k(U)$ make sense if $k \geq 0$ and are again conservation laws. For instance,

$$\mathscr{M}_0(U) = \frac{1}{2}\int_{\mathbb{R}} \big(U^2 - iUHU\big)\,dx.$$

Finally, notice that in both cases $X = \mathbb{T}$ and $X = \mathbb{T}$, we have

$$\mathscr{E}_k(U) = \operatorname{tr}\mathscr{M}_k(U)$$

for every $k \geq 0$.

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## Publisher's Note