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Newton's second law as limit of variational problems

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Abstract

We show that the solution of Cauchy problem for the classical ODE $m\mathbf{y}'' = \mathbf{f}$ can be obtained as the limit of minimizers of exponentially weighted convex variational integrals. This complements the known results about weighted inertia-energy approach to Lagrangian mechanics and hyperbolic equations.

MSC: 49J45; 70H30

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1 Introduction and statement of the result

Let $\mathbf{f} \in L^\infty(\mathbb{R}^+;\mathbb{R}^N)$, $\mathbf{u}_0 \in \mathbb{R}^N$, $\mathbf{v}_0 \in \mathbb{R}^N$, and $m > 0$. Let us consider the Cauchy problem

$$\begin{cases} m\mathbf{y}'' = \mathbf{f}, & t > 0, \\ \mathbf{y}(0) = \mathbf{u}_0, & \mathbf{y}'(0) = \mathbf{v}_0, \end{cases} \tag{1.1}$$

governing the motion of a material point of mass m subject to the force field $\mathbf{f}$. Our goal is to show that the solution to (1.1) is the limit as $h \to +\infty$ of the minimizers of the following functionals defined on trajectories $\mathbf{y} : \mathbb{R}^+ \to \mathbb{R}^N$:

$$\frac{m}{2h^2}\int_0^{+\infty} \left|\mathbf{y}''(t)\right|^2 e^{-ht}\,dt - \int_0^{+\infty} \mathbf{f}_h(t)\cdot\mathbf{y}(t)e^{-ht}\,dt, \quad h \in \mathbb{N},$$

subject to the same initial conditions, where $(\mathbf{f}_h)_{h\in\mathbb{N}} \subset L^\infty(\mathbb{R}^+;\mathbb{R}^N)$ is a sequence such that $\mathbf{f}_h \rightharpoonup \mathbf{f}$ in $w^* - L^\infty(\mathbb{R}^+;\mathbb{R}^N)$ as $h \to +\infty$. More precisely, letting

$$\mathcal{A} := \left\{\mathbf{v} \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^N) : \int_0^{+\infty} \left|\mathbf{v}''(t)\right|^2 e^{-t}\,dt < +\infty\right\}$$

and $\mathbf{f}_h \in L^\infty(\mathbb{R}^+;\mathbb{R}^N)$ for every $h \in \mathbb{N}$, we may define the rescaled energy functional (see also Lemma 2.3)

$$\mathcal{J}_h(\mathbf{u}) := \begin{cases} \frac{m}{2}\int_0^{+\infty} |\mathbf{u}''(t)|^2 e^{-t}\,dt - h^{-2}\int_0^{+\infty} \mathbf{f}_h(h^{-1}t)\cdot \mathbf{u}(t)e^{-t}\,dt \\ \qquad \text{if } \mathbf{u} \in \mathcal{A}, \\ +\infty \quad \text{otherwise in } W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^N). \end{cases}$$

We will prove the following result.

Theorem 1.1 *For every $h \in \mathbb{N}$, there exists a unique solution $\overline{\mathbf{u}}_h$ to the problem*

$$\min\bigl\{\mathcal{J}_h(\mathbf{u}) : \mathbf{u} \in \mathcal{A}, \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}'(0) = h^{-1}\mathbf{v}_0\bigr\}.$$

Moreover, if $\mathbf{f}_h \rightharpoonup \mathbf{f}$ in $w^* - L^\infty(\mathbb{R}^+;\mathbb{R}^N)$ as $h \to +\infty$, then by setting $\overline{\mathbf{y}}_h(t) := \overline{\mathbf{u}}_h(ht)$ we have $\overline{\mathbf{y}}_h \rightharpoonup \overline{\mathbf{y}}$ in $w^* - W^{2,\infty}((0,T);\mathbb{R}^N)$ for every $T > 0$, where $\overline{\mathbf{y}}$ is the unique solution on $\mathbb{R}^+$ of problem (1.1).

A variational approach based on the minimization of *weighted inertia-energy* (WIE) functionals can be used for approximating large classes of initial value problems of the second order. An example is the nonhomogeneous wave equation

$$w_{tt} = \Delta w + g \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N.$$

Indeed, it has been shown in [8] that given $g \in L^2_{\mathrm{loc}}((0,+\infty);L^2(\mathbb{R}^N))$, $\alpha \in H^1(\mathbb{R}^N)$, and $\beta \in H^1(\mathbb{R}^N)$, there exists a sequence $(g_h)_{h\in\mathbb{N}}$ converging to g in $L^2((0,T);L^2(\mathbb{R}^N))$ for every $T > 0$ such that the following properties hold. First, the WIE functional

$$\int_0^{+\infty}\int_{\mathbb{R}^N} e^{-t}\left\{\frac{1}{2}\bigl|u_{tt}(t,x)\bigr|^2 + \frac{1}{2}h^{-2}\bigl|\nabla u(t,x)\bigr|^2 - h^{-2}g_h\bigl(h^{-1}t,x\bigr)u(t,x)\right\}dt\,dx$$

has, for every $h \in \mathbb{N}$, a unique minimizer u_h in the class of functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^N)$ such that

$$\begin{cases} \nabla u \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^N), \qquad u'' \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^N), \\ \int_0^{+\infty}\int_{\mathbb{R}^N} e^{-t}\{|u_{tt}|^2 + |\nabla u|^2\}\,dt\,dx < +\infty, \\ u(0,x) = \alpha(x), \qquad u_t(0,x) = h^{-1}\beta(x). \end{cases}$$

Second, by setting $w_h(t,x) := u_h(ht,x)$ the sequence $(w_h)_{h\in\mathbb{N}}$ converges weakly in $H^1((0,T)\times\mathbb{R}^N)$ for every $T > 0$ to a function w that solves in the sense of distributions in $\mathbb{R}^+ \times \mathbb{R}^N$ the initial value problem

$$\begin{cases} w_{tt} = \Delta w + g, \\ w(0,x) = \alpha(x), \qquad w_t(0,x) = \beta(x). \end{cases} \tag{1.2}$$

A similar result holds for other classes of hyperbolic equations as shown in [4, 6, 9]. In particular, it applies to the nonlinear wave equation $w_{tt} = \Delta w - \frac{p}{2}|w|^{p-2}w$, $p \ge 2$, as conjectured by De Giorgi [1] and first proven in [5]; see also [7]. Let us mention that (the scalar version of) Theorem 1.1 is not a direct consequence of the above result from [8], since we should apply the latter to constant-in-space forcing terms g and initial data α, β, and since the approximating sequence $(g_h)_{h\in\mathbb{N}}$ in [8] is not arbitrary but obtained by means of a specific construction, not allowing, for instance, for the choice $g_h \equiv g$ for every h.

Concerning the WIE approach for ODEs, let us mention its application in [3] for providing a variational approach to Lagrangian mechanics by considering an equation of the form

$$m\mathbf{y}'' + \nabla U(\mathbf{y}) = 0, \quad t > 0, \tag{1.3}$$

for given potential energy $U \in C^1(\mathbb{R}^N)$, bounded from below and $m > 0$. The main theorem of [3] proves indeed that solutions to the initial value problem for (1.3) can be approximated by rescaled minimizers, subject to the same initial conditions, of the functionals

$$\mathcal{G}_h(\mathbf{v}) = \int_0^{+\infty} e^{-t}\left\{\frac{m}{2}\left|\mathbf{v}''(t)\right|^2 + h^{-2}U\big(\mathbf{v}(t)\big)\right\} dt, \quad h \in \mathbb{N}.$$

It is worth noticing that also in this case, Theorem 1.1 is not a consequence of the result from [3] since the latter requires that the force field is conservative and independent of t.

We have already observed that in the scalar case, problem (1.1) is a particular case of problem (1.2), obtained by taking constant initial data and letting the forcing term depend only on time. Let us also mention another interpretation of (1.1) from a continuum mechanics point of view. Indeed, Newton's second law (1.1) governs the motion of the center of mass of a body occupying a reference configuration $\Omega \subset \mathbb{R}^N$. In more detail, let ρ be the mass density of the body, and let $\mathbf{u}(t, \mathbf{x})$ be the position of the material point $\mathbf{x}$ at time t. If $\mathbb{T}$ is the Cauchy stress tensor and $\mathbf{b}$ is the body force field acting on Ω, then the equation of motion (see,e.g., [2]) takes the form

$$\rho\mathbf{u}_{tt} = \operatorname{div}\mathbb{T} + \mathbf{b} \quad \text{in } \mathbb{R}^+ \times \Omega. \tag{1.4}$$

Therefore by integrating in Ω both sides of (1.4) we formally get

$$\frac{d^2}{dt^2}\left(\int_\Omega \rho\mathbf{u}\, d\mathbf{x}\right) = \int_\Omega \operatorname{div}\mathbb{T}\, d\mathbf{x} + \int_\Omega \mathbf{b} d\mathbf{x} = \int_{\partial\Omega} \mathbb{T}\cdot\mathbf{n}\, d\mathcal{H}^{N-1} + \int_\Omega \mathbf{b}\, d\mathbf{x} =: \mathbf{f}_\Omega, \quad t > 0,$$

that is,

$$m_\Omega \mathbf{y}'' = \mathbf{f}_\Omega, \quad t > 0,$$

where $\mathbf{f}_\Omega = \mathbf{f}_\Omega(t)$ is the total force acting on the body, accounting for surface and body forces, $m_\Omega = \int_\Omega \rho(\mathbf{x})\, d\mathbf{x}$ is the mass of the body, and

$$\mathbf{y}(t) = m_\Omega^{-1}\int_\Omega \rho(\mathbf{x})\mathbf{u}(t, \mathbf{x})\, d\mathbf{x}$$

is the position at time t of the center of mass of the body during the motion. Therefore Newton's second law (1.1) can be viewed as the average in space of the equation of motion

(1.4). In this perspective, Theorem 1.1 can be seen as a result about the equation of motion in $\mathbb{R}^N$ in the above average sense.

Let us finally stress that the methods described in this paper, here only devoted to the elementary problem (1.1), can be extended to nonlinear problems like $\mathbf{y}'' = \nabla_{\mathbf{y}} G(t,\mathbf{y})$ under suitable assumptions on G and also to hyperbolic problems such as (1.2) allowing us to get further results on these topics. In this perspective, we will develop our analysis in a forthcoming paper.

2 Existence of minimizers

In this section, we provide some preliminary results we are going to use for proving Theorem 1.1. First of all, it is worth noticing that if $\mathbf{u} \in \mathcal{A}$, then $\mathbf{u} \in W^{2,2}((0,T);\mathbb{R}^N)$ for every $T > 0$, hence both $\mathbf{u}(0)$ and $\mathbf{u}'(0)$ are well defined. Moreover, if $\mathbf{u} \in \mathcal{A}$, then by the Cauchy–Schwarz inequality,

$$\left| \int_0^{+\infty} \mathbf{f}\left(h^{-1}t\right) \cdot \mathbf{u}(t) e^{-t}\, dt \right| \leq \|\mathbf{f}\|_\infty \left(\int_0^{+\infty} \left|\mathbf{u}(t)\right|^2 e^{-t}\, dt \right)^{1/2},$$

and the integral in the left-hand side is finite (see Lemma 2.1), so that $\mathcal{J}_h(\mathbf{u})$ is well-defined and finite. In fact, we have the following estimates.

Lemma 2.1 *Let $\mathbf{u} \in \mathcal{A}$. Then $e^{-t/2}\mathbf{u} \in L^2((0,+\infty);\mathbb{R}^N)$, $e^{-t/2}\mathbf{u}' \in L^2((0,+\infty);\mathbb{R}^N)$, and*

$$\int_0^{+\infty} \left|\mathbf{u}'(t)\right|^2 e^{-t}\, dt \leq 2\left|\mathbf{u}'(0)\right|^2 + 4\int_0^{+\infty} \left|\mathbf{u}''(t)\right|^2 e^{-t}\, dt, \tag{2.1}$$

$$\int_0^{+\infty} \left|\mathbf{u}(t)\right|^2 e^{-t}\, dt \leq 2\left|\mathbf{u}(0)\right|^2 + 8\left|\mathbf{u}'(0)\right|^2 + 16\int_0^{+\infty} \left|\mathbf{u}''(t)\right|^2 e^{-t}\, dt. \tag{2.2}$$

Proof We have $\mathbf{u} \in AC([0,T];\mathbb{R}^N)$ and $\mathbf{u}' \in AC([0,T];\mathbb{R}^N)$ for every $T > 0$. Therefore $\frac{d}{dt}|\mathbf{u}(t)|^2 = 2\mathbf{u}(t) \cdot \mathbf{u}'(t)$ and $\frac{d}{dt}|\mathbf{u}'(t)|^2 = 2\mathbf{u}'(t) \cdot \mathbf{u}''(t)$ for a.e. $t > 0$. Moreover, given $T > 0$, we integrate by parts and obtain

$$\begin{aligned}\int_0^T \left|\mathbf{u}'(t)\right|^2 e^{-t}\, dt &= \left[-e^{-t}\left|\mathbf{u}'(t)\right|^2\right]_0^T + 2\int_0^T e^{-t/2}\mathbf{u}'(t) \cdot \mathbf{u}''(t) e^{-t/2}\, dt \\ &\leq \left|\mathbf{u}'(0)\right|^2 + \frac{1}{2}\int_0^T \left|\mathbf{u}'(t)\right|^2 e^{-t}\, dt + 2\int_0^T \left|\mathbf{u}''(t)\right|^2 e^{-t}\, dt,\end{aligned}$$

where we have used the Young inequality. By letting $T \to +\infty$ we get (2.1). The same computation entails

$$\int_0^T \left|\mathbf{u}(t)\right|^2 e^{-t}\, dt \leq \left|\mathbf{u}(0)\right|^2 + \frac{1}{2}\int_0^T \left|\mathbf{u}(t)\right|^2 e^{-t}\, dt + 2\int_0^T \left|\mathbf{u}'(t)\right|^2 e^{-t}\, dt.$$

By letting $T \to +\infty$ and by taking advantage of (2.1) we obtain (2.2). □

The next lemma proves the first statement of Theorem 1.1.

Lemma 2.2 *For every $h \in \mathbb{N}$, there exists a unique solution to the problem*

$$\min\left\{\mathcal{J}_h(\mathbf{u}) : \mathbf{u} \in \mathcal{A}, \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}'(0) = h^{-1}\mathbf{v}_0\right\}. \tag{2.3}$$

Proof We first observe that $\mathcal{J}_h$ is strictly convex and that the minimization set is convex. Therefore if a minimizer exists, then it is necessarily unique, so we are left to prove the existence. If $\mathbf{u} \in \mathcal{A}$ is such that $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}'(0) = h^{-1}\mathbf{v}_0$, then Lemma 2.1 entails

$$\int_0^{+\infty} \left|\mathbf{u}(t)\right|^2 e^{-t}\,dt \le 2|\mathbf{u}_0|^2 + 8h^{-2}|\mathbf{v}_0|^2 + 16\int_0^{+\infty} \left|\mathbf{u}''(t)\right|^2 e^{-t}\,dt \tag{2.4}$$

and

$$\int_0^{+\infty} \left|\mathbf{u}'(t)\right|^2 e^{-t}\,dt \le 2h^{-2}|\mathbf{v}_0|^2 + 4\int_0^{+\infty} \left|\mathbf{u}''(t)\right|^2 e^{-t}\,dt. \tag{2.5}$$

Let $(\mathbf{u}_k)_{k\in\mathbb{N}}$ be a minimizing sequence for problem (2.3). Since $\mathbf{u}_0 + h^{-1}t\mathbf{v}_0$ is admissible for problem (2.3), we have, for any k large enough,

$$\mathcal{J}_h(\mathbf{u}_k) \le \mathcal{J}_h\big(\mathbf{u}_0 + h^{-1}t\mathbf{v}_0\big) + 1,$$

whence by (2.4) and by the Young and Cauchy–Schwarz inequalities, denoting by C various constants only depending on $\|\mathbf{f}_h\|_\infty, h, \mathbf{u}_0, \mathbf{v}_0, m$, we get

$$\begin{aligned}
&\int_0^{+\infty} \left|\mathbf{u}_k''(t)\right|^2 e^{-t}\,dt \\
&\quad\le \frac{2}{m}h^{-2}\int_0^{+\infty} \mathbf{f}_h\big(h^{-1}t\big)\cdot \mathbf{u}_k(t)e^{-t}\,dt \\
&\qquad - \frac{2}{m}h^{-2}\int_0^{+\infty} \mathbf{f}_h\big(h^{-1}t\big)\cdot \big(\mathbf{u}_0 + h^{-1}t\mathbf{v}_0\big)e^{-t}\,dt + \frac{2}{m} \\
&\quad\le \frac{2}{m}\|\mathbf{f}_h\|_\infty h^{-2}\int_0^{+\infty} \left|\mathbf{u}_k(t)\right| e^{-t}\,dt + C \\
&\quad\le \frac{2}{m}\|\mathbf{f}_h\|_\infty h^{-2}\left(\int_0^{+\infty} \left|\mathbf{u}_k(t)\right|^2 e^{-t}\,dt\right)^{\frac{1}{2}} + C \\
&\quad\le \frac{1}{32}\int_0^{+\infty} \left|\mathbf{u}_k(t)\right|^2 e^{-t}\,dt + \frac{32}{m^2}h^{-4}\|\mathbf{f}_h\|_\infty^2 + C \\
&\quad\le \frac{1}{2}\int_0^{+\infty} \left|\mathbf{u}_k''(t)\right|^2 e^{-t}\,dt + C.
\end{aligned} \tag{2.6}$$

By taking into account of (2.4), (2.5), and (2.6) we get that the sequence $(e^{-\frac{t}{2}}\mathbf{u}_k)_{k\in\mathbb{N}}$ is equibounded in $W^{2,2}(\mathbb{R}^+;\mathbb{R}^N)$. So there exists $\mathbf{v} \in W^{2,2}(\mathbb{R}^+;\mathbb{R}^N)$ such that, up to extracting a subsequence, $e^{-\frac{t}{2}}\mathbf{u}_k \rightharpoonup \mathbf{v}$ in $W^{2,2}(\mathbb{R}^+;\mathbb{R}^N)$, and hence $\mathbf{u}_k \rightharpoonup \mathbf{u} := e^{\frac{t}{2}}\mathbf{v}$ in $W^{2,2}((0,T);\mathbb{R}^N)$ for every $T > 0$, and $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}'(0) = h^{-1}\mathbf{v}_0$. Therefore for every $T > 0$, we have

$$\liminf_{k\to+\infty}\int_0^{+\infty} \left|\mathbf{u}_k''(t)\right|^2 e^{-t}\,dt \ge \liminf_{k\to+\infty}\int_0^{T} \left|\mathbf{u}_k''(t)\right|^2 e^{-t}\,dt \ge \int_0^{T} \left|\mathbf{u}''(t)\right|^2 e^{-t}\,dt,$$

and hence

$$\int_0^{+\infty} \left|\mathbf{u}''(t)\right|^2 e^{-t}\,dt = \sup_{T>0}\int_0^{T} \left|\mathbf{u}''(t)\right|^2 e^{-t}\,dt \le \liminf_{k\to+\infty}\int_0^{+\infty} \left|\mathbf{u}_k''(t)\right|^2 e^{-t}\,dt,$$

so eventually we find $\mathbf{u} \in \mathcal{A}$, and since

$$\begin{aligned}\lim_{k\to+\infty} \int_0^{+\infty} h^{-2}\mathbf{f}_h\big(h^{-1}t\big)\cdot \mathbf{u}_k e^{-t}\,dt &= \int_0^{+\infty} h^{-2}\mathbf{f}_h\big(h^{-1}t\big)\cdot \mathbf{v}e^{-t/2}\,dt \\ &= \int_0^{+\infty} h^{-2}\mathbf{f}_h\big(h^{-1}t\big)\cdot \mathbf{u}e^{-t}\,dt,\end{aligned}$$

we get

$$\liminf_{k\to+\infty} \mathcal{J}_h(\mathbf{u}_k) \geq \mathcal{J}_h(\mathbf{u}).$$

We conclude that $\mathbf{u}$ is a solution to (2.3). □

Lemma 2.3 *Let $h \in \mathbb{N}$. If $\overline{\mathbf{u}}_h$ is the unique solution to (2.3), then $\overline{\mathbf{y}}_h(t) := \overline{\mathbf{u}}_h(ht)$ is the unique minimizer of*

$$\mathcal{F}_h(\mathbf{y}) := \begin{cases} \frac{m}{2h^2}\int_0^{+\infty} |\mathbf{y}''(t)|^2 e^{-ht}\,dt - \int_0^{+\infty} \mathbf{f}_h(t)\cdot \mathbf{y}(t)e^{-ht}\,dt & \text{if } \mathbf{y} \in \mathcal{A}_h, \\ +\infty & \text{otherwise in } W^{2,1}_{loc}(\mathbb{R}^+;\mathbb{R}^N) \end{cases}$$

over $\mathcal{A}_h$, where

$$\mathcal{A}_h := \left\{ \mathbf{y} \in W^{2,1}_{\mathrm{loc}}\big(\mathbb{R}^+;\mathbb{R}^N\big) : \int_0^{+\infty} \big|\mathbf{y}''(t)\big|^2 e^{-ht}\,dt < +\infty, \mathbf{y}(0) = \mathbf{u}_0, \mathbf{y}'(0) = \mathbf{v}_0 \right\}.$$

Proof Since $\overline{\mathbf{u}}_h \in \mathcal{A}$ and $\overline{\mathbf{u}}_h(0) = \mathbf{u}_0$, $\overline{\mathbf{u}}_h'(0) = h^{-1}\mathbf{v}_0$, we directly see that $\overline{\mathbf{y}}_h \in \mathcal{A}_h$ and $h^{-1}\mathcal{F}_h(\overline{\mathbf{y}}_h) = \mathcal{J}_h(\overline{\mathbf{u}}_h)$. Moreover, if $\mathbf{y} \in \mathcal{A}_h$, then by setting $\mathbf{u}_h(t) = \mathbf{y}(h^{-1}t)$ we get $\mathbf{u}_h \in \mathcal{A}, \mathbf{u}_h(0) = \mathbf{u}_0, \mathbf{u}_h'(0) = h^{-1}\mathbf{v}_0$, and $h^{-1}\mathcal{F}_h(\mathbf{y}) = \mathcal{J}_h(\mathbf{u}_h)$. Therefore $\mathcal{F}_h(\overline{\mathbf{y}}_h) \leq \mathcal{F}_h(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{A}_h$, and equality holds if and only if $\mathbf{y} = \overline{\mathbf{y}}_h$, as claimed. □

3 Proof of Theorem 1.1

Given $\overline{\mathbf{y}}_h$ minimizing $\mathcal{F}_h$ over $\mathcal{A}_h$, here we prove suitable boundedness estimates for the sequence $(\overline{\mathbf{y}}_h)_{h\in\mathbb{N}}$, which are the main step toward the proof of Theorem 1.1.

Lemma 3.1 *For every $h \in \mathbb{N}$, let $\overline{\mathbf{y}}_h$ be as in* Lemma 2.3. *Then $\overline{\mathbf{y}}_h'' \in L^\infty(\mathbb{R}^+;\mathbb{R}^N)$, and*

$$\big\|\overline{\mathbf{y}}_h''\big\|_\infty \leq m^{-1}\sup_{h\in\mathbb{N}} \|\mathbf{f}_h\|_\infty.$$

Moreover, the sequence $(\overline{\mathbf{y}}_h)_{h\in\mathbb{N}}$ is equibounded in $W^{2,\infty}((0,T);\mathbb{R}^N)$ for every $T > 0$.

Proof Let $h \in \mathbb{N}$, $\boldsymbol{\varphi} \in C_c(\mathbb{R}^+;\mathbb{R}^N)$, and let $\boldsymbol{\xi}$ be the unique solution to

$$\begin{cases} \boldsymbol{\xi}'' = e^t\boldsymbol{\varphi}, & t > 0, \\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}'(0) = 0. \end{cases}$$

By setting $\boldsymbol{\psi}_h(t) := h^{-2}\boldsymbol{\xi}(ht)$ we see that $\boldsymbol{\psi}_h(0) = \boldsymbol{\psi}_h'(0) = 0$ and

$$\int_0^{+\infty} \big|\boldsymbol{\psi}_h''(t)\big|^2 e^{-ht}\,dt = h^{-1}\int_0^{+\infty} \big|\boldsymbol{\varphi}(t)\big|^2 e^t\,dt,$$

and the integral in the right-hand side is finite since $\boldsymbol{\varphi} \in C_c(\mathbb{R}^+;\mathbb{R}^N)$. Thus we get $\overline{\mathbf{y}}_h + \boldsymbol{\psi}_h \in \mathcal{A}_h$. The minimality of $\overline{\mathbf{y}}_h$ entails the validity of the first-order relation

$$mh^{-2}\int_0^{+\infty} \overline{\mathbf{y}}_h''(t)\cdot\boldsymbol{\psi}_h''(t)e^{-ht}\,dt = \int_0^{+\infty} \mathbf{f}_h(t)\cdot\boldsymbol{\psi}_h(t)e^{-ht}\,dt. \tag{3.1}$$

Since $\boldsymbol{\xi}(0) = 0$, using integration by parts, we have, for every $\nu > 0$ and every $\tau > 0$,

$$\begin{aligned}\int_0^{\tau} \left|\boldsymbol{\xi}(t)\right|e^{-t}\,dt &\le \int_0^{\tau} \sqrt{\left|\boldsymbol{\xi}(t)\right|^2+\nu^2}e^{-t}\,dt \\ &= \left[-e^{-t}\sqrt{\left|\boldsymbol{\xi}(t)\right|^2+\nu^2}\right]_0^{\tau} + \int_0^{\tau} \frac{\boldsymbol{\xi}'(t)\cdot\boldsymbol{\xi}(t)}{\sqrt{|\boldsymbol{\xi}(t)|^2+\nu^2}}e^{-t}\,dt \\ &\le \nu + \int_0^{\tau} \left|\boldsymbol{\xi}'(t)\right|e^{-t}\,dt,\end{aligned}$$

Then by the arbitrariness of ν and τ, repeating the same argument taking into account that $\boldsymbol{\xi}'(0) = 0$, we obtain

$$\int_0^{+\infty} \left|\boldsymbol{\xi}(t)\right|e^{-t}\,dt \le \int_0^{+\infty} \left|\boldsymbol{\xi}'(t)\right|e^{-t}\,dt \le \int_0^{+\infty} \left|\boldsymbol{\xi}''(t)\right|e^{-t}\,dt.$$

Therefore

$$\begin{aligned}&\left|\int_0^{+\infty} \mathbf{f}_h(t)\cdot\boldsymbol{\psi}_h(t)e^{-ht}\,dt\right| \\ &\quad = h^{-3}\left|\int_0^{+\infty} \mathbf{f}_h\left(h^{-1}s\right)\cdot\boldsymbol{\xi}(s)e^{-s}\,ds\right| \\ &\quad \le h^{-3}\|\mathbf{f}_h\|_\infty \int_0^{+\infty} e^{-s}\left|\boldsymbol{\xi}''(s)\right|ds = h^{-3}\|\mathbf{f}_h\|_\infty \int_0^{+\infty} \left|\boldsymbol{\varphi}(s)\right|ds.\end{aligned} \tag{3.2}$$

We recall from Lemma 2.3 that $\overline{\mathbf{y}}_h(t) = \overline{\mathbf{u}}_h(ht)$, where $\overline{\mathbf{u}}_h$ is the unique solution to (2.3). Hence, taking into account that

$$\begin{aligned}&h^{-2}\int_0^{+\infty} \overline{\mathbf{y}}_h''(t)\cdot\boldsymbol{\psi}_h''(t)e^{-ht}\,dt \\ &\quad = \int_0^{+\infty} \overline{\mathbf{u}}_h''(ht)\cdot\boldsymbol{\xi}''(ht)e^{-ht}\,dt \\ &\quad = h^{-1}\int_0^{+\infty} \overline{\mathbf{u}}_h''(s)\cdot\boldsymbol{\xi}''(s)e^{-s}\,ds = h^{-1}\int_0^{+\infty} \overline{\mathbf{u}}_h''(s)\cdot\boldsymbol{\varphi}(s)\,ds\end{aligned}$$

and using (3.1) and (3.2), we get

$$\left|\int_0^{+\infty} \overline{\mathbf{u}}_h''(s)\cdot\boldsymbol{\varphi}(s)\,ds\right| \le m^{-1}h^{-2}\|\mathbf{f}_h\|_\infty \int_0^{+\infty} |\boldsymbol{\varphi}|\,ds. \tag{3.3}$$

Since $\boldsymbol{\varphi} \in C_c(\mathbb{R}^+;\mathbb{R}^N)$ is arbitrary and $C_c(\mathbb{R}^+;\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^+;\mathbb{R}^N)$, (3.3) entails

$$\left\|\overline{\mathbf{u}}_h''\right\|_\infty \le \frac{1}{h^2 m}\|\mathbf{f}_h\|_\infty,$$

that is,

$$\left\|\overline{\mathbf{y}}_h''\right\|_\infty \le \frac{1}{m}\|\mathbf{f}_h\|_\infty. \tag{3.4}$$

Eventually, we have, for every $t \in [0, T]$,

$$\overline{\mathbf{y}}_h'(t) = \mathbf{v}_0 + \int_0^t \overline{\mathbf{y}}_h''(s)\,ds \quad \text{and} \quad \overline{\mathbf{y}}_h(t) = \mathbf{u}_0 + t\mathbf{v}_0 + \int_0^t (t-s)\overline{\mathbf{y}}_h''(s)\,ds,$$

and hence (3.4) yields

$$\|\overline{\mathbf{y}}_h\|_{L^\infty(0,T)} \le |\mathbf{u}_0| + T|\mathbf{v}_0| + \frac{T^2}{2m}\|\mathbf{f}_h\|_\infty \tag{3.5}$$

and

$$\left\|\overline{\mathbf{y}}_h'\right\|_{L^\infty(0,T)} \le |\mathbf{v}_0| + \frac{T}{m}\|\mathbf{f}_h\|_\infty. \tag{3.6}$$

Estimates (3.4), (3.5), and (3.6) prove the result, since the sequence $(\mathbf{f}_h)_{h\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^+)$. □

Proof of Theorem 1.1 For every $h \in \mathbb{N}$, let $\overline{\mathbf{y}}_h$ be as in Lemma 2.3. Let $T > 0$, and let $\boldsymbol{\xi} \in C^\infty(\mathbb{R})$ with $\operatorname{spt}\boldsymbol{\xi} \subset (0, T)$. Then setting $\boldsymbol{\varphi}_h(t) := \boldsymbol{\xi}(t)e^{ht}$ and taking into account the first-order minimality condition (3.1), we have

$$\begin{aligned}
&-m\int_0^T \overline{\mathbf{y}}_h'(t)\cdot\left(h^{-2}\boldsymbol{\xi}'''(t) + 2h^{-1}\boldsymbol{\xi}''(t) + \boldsymbol{\xi}'(t)\right)dt \\
&\quad = -mh^{-2}\int_0^T \overline{\mathbf{y}}_h'(t)\cdot\left(\boldsymbol{\varphi}_h''(t)e^{-ht}\right)'\,dt \\
&\quad = mh^{-2}\int_0^T \overline{\mathbf{y}}_h''(t)\cdot\boldsymbol{\varphi}_h''(t)e^{-ht}\,dt = mh^{-2}\int_0^{+\infty} \overline{\mathbf{y}}_h''(t)\cdot\boldsymbol{\varphi}_h''(t)e^{-ht}\,dt \\
&\quad = \int_0^{+\infty} \mathbf{f}_h(t)\cdot\boldsymbol{\varphi}_h(t)e^{-ht}\,dt = \int_0^T \mathbf{f}_h(t)\cdot\boldsymbol{\xi}(t)\,dt.
\end{aligned} \tag{3.7}$$

By Lemma 3.1 there exists $\overline{\mathbf{y}} \in W^{2,\infty}((0,T);\mathbb{R}^N)$ such that, up to subsequences, $\overline{\mathbf{y}}_h \rightharpoonup \overline{\mathbf{x}}$ in $w^* - W^{2,\infty}((0,T);\mathbb{R}^N)$. Therefore we get $\overline{\mathbf{x}}(0) = \mathbf{u}_0$ and $\overline{\mathbf{x}}'(0) = \mathbf{v}_0$. Taking into account (3.7) and the $w^* - L^\infty(\mathbb{R}^+)$ convergence of $\mathbf{f}_h$ to $\mathbf{f}$, in the limit as $h \to +\infty$, we obtain

$$-m\int_0^T \overline{\mathbf{y}}'(t)\cdot\boldsymbol{\xi}'(t)\,dt = \int_0^T \mathbf{f}(t)\cdot\boldsymbol{\xi}(t)\,dt.$$

The latter holds for every $\boldsymbol{\xi} \in C^\infty(\mathbb{R})$ with $\operatorname{spt}\boldsymbol{\xi} \subset (0, T)$, and therefore $\overline{\mathbf{x}}$ is the unique solution of

$$\begin{cases} m\mathbf{y}'' = \mathbf{f}, \\ \mathbf{y}(0) = \mathbf{u}_0, \qquad \mathbf{y}'(0) = \mathbf{v}_0, \end{cases}$$

on $[0, T]$. Hence the whole sequence $(\overline{\mathbf{y}}_h)_{h\in\mathbb{N}}$ is such that $\overline{\mathbf{y}}_h \rightharpoonup \overline{\mathbf{x}}$ in $w^* - W^{2,\infty}((0,T);\mathbb{R}^N)$. Since the Cauchy problem (1.1) has a unique solution $\overline{\mathbf{y}}$ on $\mathbb{R}^+$ and since T is arbitrary, we conclude that $\overline{\mathbf{y}}_h \rightharpoonup \overline{\mathbf{y}}$ in $w^* - W^{2,\infty}((0,T);\mathbb{R}^N)$ as $h \to +\infty$ for every $T > 0$, thus proving the theorem. □

4 Conclusions

The paper contains some new ideas concerning the approximation of the solution of the equation of motion of a body with the minimizers of a sequence of variational problems even in the presence of environmental forces. The method can be extended to hyperbolic equations.

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